QMI Lesson 9: Higher Order Derivatives, Implicit Differentiation, and Differentials

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First Order:	$f'(x)$	$\frac{df}{dx}$	$D_x^1 f(x)$
Second Order:	$f''(x)$	$\frac{d^2f}{dx^2}$	$D_x^2 f(x)$
Third Order:	$f'''(x)$	$\frac{d^3f}{dx^3}$	$D_x^3 f(x)$
n^{th} Order:	$f^{(n)}(x)$	$\frac{d^nf}{dx^n}$	$D_x^n f(x)$

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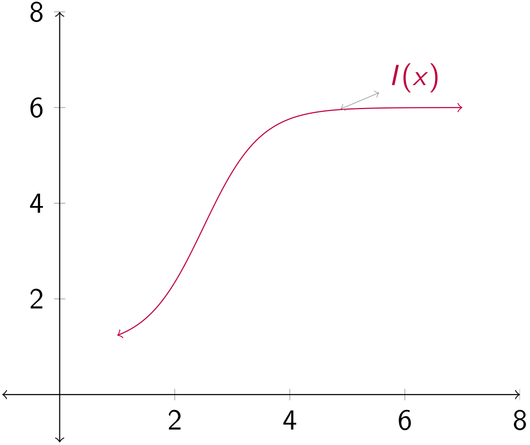
Graph: The CPI and Inflation



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Simply use the Chain Rule!

Implicit Differentiation: Steps

If you wanted to find $\frac{dy}{dx}$ with $y^3x^2 + 6x^2 = y + 12$, you must

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1. Differentiate both sides of the equation with respect to x, using the Chain Rule where necessary, remembering that y is really $y(x)$, a function and not a variable!
2. Solve for y', i.e. for $\frac{dy}{dx}$.

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$$\begin{aligned} y^3(2x) + 3y^2\frac{dy}{dx}x^2 + 12x = \frac{dy}{dx} &\implies \\ 2y^3x + 12x = \frac{dy}{dx} - 3y^2\frac{dy}{dx}x^2 &\implies \\ 2y^3x + 12x = \frac{dy}{dx}(1 - 3y^2x^2) &\implies \\ \frac{2y^3x + 12x}{1 - 3y^2x^2} &= \frac{dy}{dx} \end{aligned}$$

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Note: It's not always necessary to find an explicit expression for $\frac{dy}{dx}$.

Marginal Rate of Technical Substitution

The chief economist of a nation estimates the output of the country as $Q(x, y) = 10x^{\frac{3}{4}}y^{\frac{1}{4}}$, where x is the amount of money spent on labor and y is the amount spent on capital, all measured in billions of dollars.

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Well, $Q(625, 81) = 3750$. So, we need to solve $Q(626, 81 + \delta y) = 3750$.

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Well, $Q(625, 81) = 3750$. So, we need to solve $Q(626, 81 + \delta y) = 3750$. This yields that $\delta y = -0.38756$. But we can also approximate this change using the derivative.

Marginal Rate of Technical Substitution

We get the implicit form

$$\begin{aligned}
3750 = 10x^{\frac{3}{4}}y^{\frac{1}{4}} &\implies \\
x^{\frac{3}{4}}y^{\frac{1}{4}} = 375 &\implies \\
\frac{dy}{dx}\frac{1}{4}y^{-\frac{3}{4}}x^{\frac{3}{4}} + \frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}} = 0 &\implies \\
\frac{dy}{dx} = (4x^{-\frac{3}{4}}y^{\frac{3}{4}})(-\frac{3}{4}x^{-\frac{1}{4}}y^{\frac{1}{4}}) = -3\frac{y}{x}
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So, at (625,81), we have $\frac{dy}{dx} = -0.3888$, which approximates the exact answer. The negative of this quantity is called the marginal rate of technical substitution. Generally speaking, it measures the rate at which a producer is technically capable of reducing one input (capital) in favor of another (labor) while maintaining the same output.

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4. Differentiate the equation with respect to t.

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2. Write the *given* values of the variables and their rates of of change with respect to t.
3. Find an equation relating x and y.
4. Differentiate the equation with respect to t.
5. Replace the variables and their derivatives by the numerical data from Step 2 and solve the resulting equation for the required rate of change.

Related Rates

A supplier is willing to make a x thousand solid-state drives for \$p given the demand equation $x^2 - 3xp + p^2 = 5$. How fast is the supply of drives changing when the price per drive is \$11, the quantity supplied is 4000 drives, and the price of the drives is increasing at the rate of \$0.10 per drive each week?

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Thus, plugging in, we get
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Thus, plugging in, we get $2(4)x'(t) - 3(4)(0.1) - 3(11)x'(t) + 2(11)(0.1) = 0$. Solving for $x'(t)$, we get $x'(t) = 0.04$, meaning that the supply is increasing at the rate of 40 drives per week.

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First, though, note that the increment between two points is simply the change between them. So if we have two x values $x_1 = 2.01$ and $x_2 = 2.02$, then the increment (Δx) is just $2.02 - 2.01 = 0.01 = \Delta x$.

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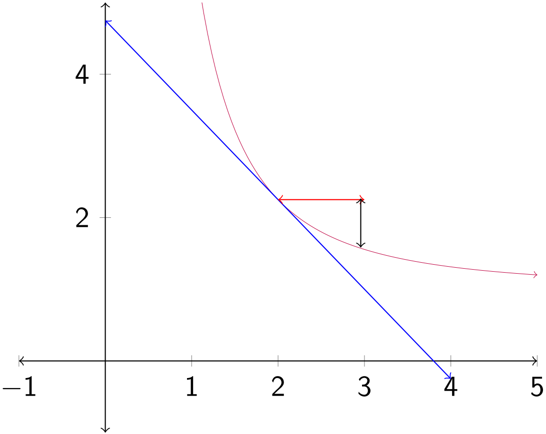
The corresponding y increment (Δy) if $y = f(x)$ is given by $f(x_1 + \Delta x) - f(x_1) = \Delta y$.

Increments: Graph



Here, Δx is given in red and Δy is given in black.

Definition: The Differential

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Definition (Differential of f at x)

The differential dx of the independent variable x is given by $dx = \Delta x$.
The differential dy of the dependent variable y is given by $dy = f'(x)\Delta x = f'(x)dx$.

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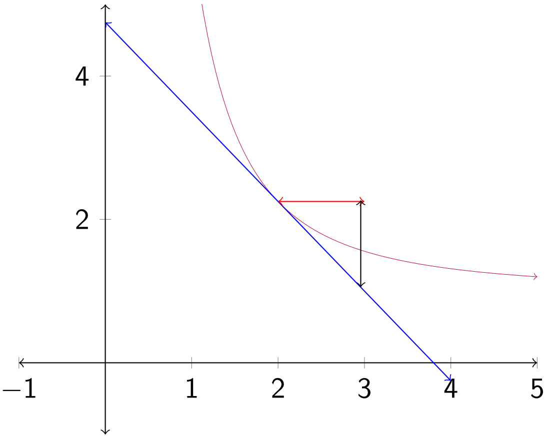

Here, $dx = \Delta x$ is given in red and $dy = f'(x)dx$ is given in black.

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$$\left|\frac{dV}{V}\right| = \left|3 \cdot \frac{ds}{s}\right| \leq 3(0.02) = 0.06.$$

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So $\sqrt[3]{28.5} \approx \sqrt[3]{27} + 0.056 \approx 3.056$. The actual value is 3.0546.

Assignment

Read 4.1. Do problems 18, 26, 30, 42 in 3.5; 6, 26, 32, 46, 48, 52 in 3.6; and 6, 18, 24, 28, 38, 40 in 3.7.