Lecture 6: Chapter 6

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UAB Mathematics

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§6.1 Continuous Probability Distributions

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Definition (Normal Distribution)

If a continuous random variable has a symmetric, bell-shaped graph and can be described by the equation

$$y = \frac{e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}}{\sigma\sqrt{2\pi}},$$

then we say that the random variable has a **normal distribution**.

§6.1 Normal Distribution

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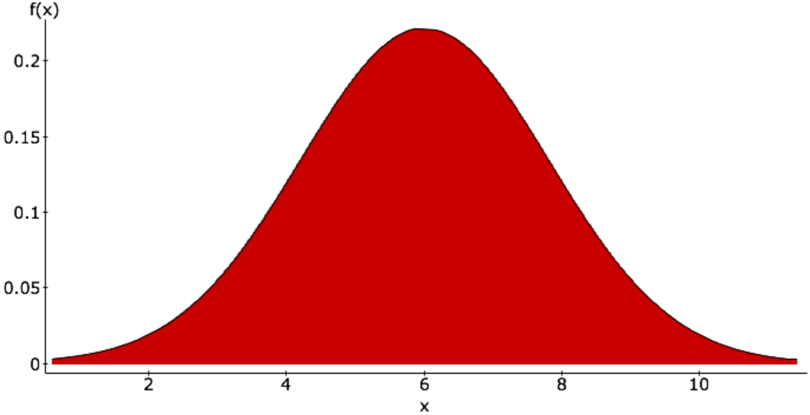

§6.1 Normal Distribution

The area under the curve between points a and b (where $a < b$) on the x-axis represents the probability that the random variable takes values between a and b, i.e. $P(a \leq x \leq b)$. In the image below, $a = -\infty$ and $b = 4$.

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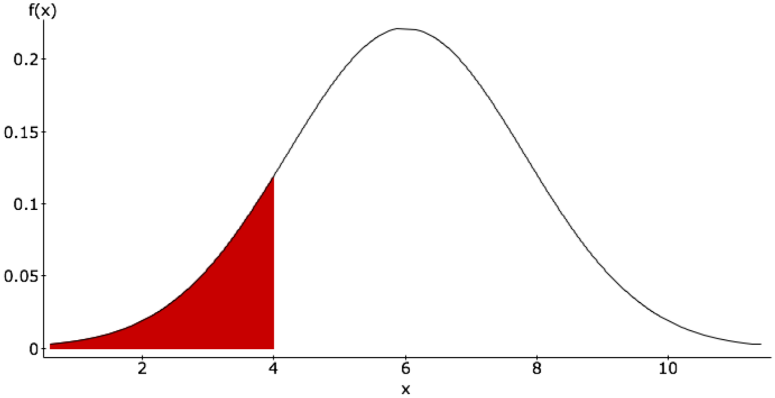

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- calculate the corresponding z-score and look up the value in a table or
- use StatCrunch to calculate the probability, ensuring that you have changed the mean and standard deviation in the normal calculator appropriately.

§6.2 Uniform Distribution

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Definition (Uniform Distribution)

A continuous random variable as a **uniform distribution** if its values are spread evenly over the range of possibilities. The graph of a uniform distribution results in a rectangular shape.

§6.2 Uniform Distribution

Here, the lower bound of values is $a = 1$, and the upper bound is $b = 15$. The height of the graph is $\frac{1}{b-a}$, so the area under the graph between b and a is 1. Then the area between $x_1 = 3$ and $x_2 = 8$ is $\frac{1}{b-a} \cdot (x_2 - x_1) = \frac{5}{14} \approx 0.357$.

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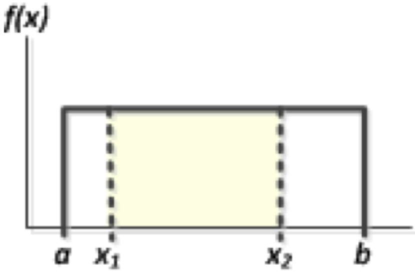

§6.2 What Makes a Density Curve?

In order for a curve to be a density curve, i.e. to describe a continuous probability distribution, the following must occur.

- The total area under the curve must equal 1.
- Every point on the curve must have a vertical height that is 0 or greater, i.e. it must lie above the x-axis.

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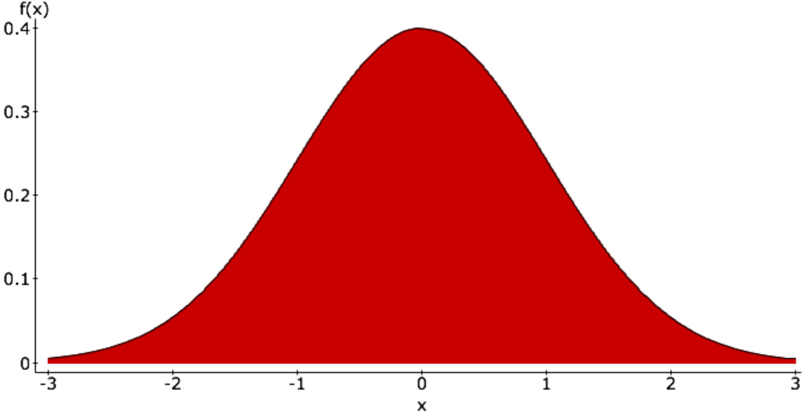

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Once this has been accomplished, you can calculate other probabilities $P(x > z)$, $P(z_1 < x < z_2)$, and $P(z_1 > x > z_2)$.

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Let's draw the pictures for this to understand why it makes sense.

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§6.2 Critical Values

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§6.3 Applications of the Normal Distribution

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where x is the original random variable and z is a standard normal random variable.

§6.3 Example

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This area is recorded in the next slide.

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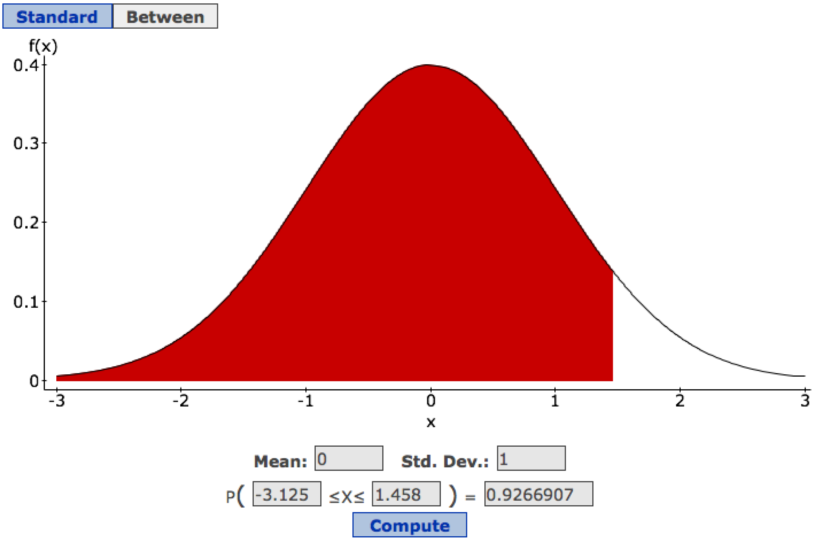
Standard
Between
f(x)
0.4
0.3
0.2
0.1
0
-3
-2
-1
0
1
2
3
x
Mean: 0
Std. Dev.: 1
P(-3.125 ≤X≤ 1.458) = 0.9266907
Compute

§6.3 Example

A local gym offers three running groups for their members - a fast group, and medium-paced group, and a slow group. If gym-going runners have a normally distributed pace with an average of 7.5 minutes per mile with a standard deviation of 1.25 minutes per mile, what should be the two cut-off times for the three groups?

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We need about one third of the runners in each group, so we need to find X_1 so that $P(x < X_1) = 0.33333$ and X_2 so that $P(x < X_2) = 0.66666$.

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This means that the cut-off times should be $\overline{X_1} = 7.5 + (-0.431)(1.25) = 6.96$ and $\overline{X_2} = 7.5 + (0.431)(1.25) = 8.04$.

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This means that the cut-off times should be $\overline{X_1} = 7.5 + (-0.431)(1.25) = 6.96$ and $\overline{X_2} = 7.5 + (0.431)(1.25) = 8.04$. Draw these areas.

§6.4 Sampling Distributions and Estimators

Definition (Samling Distribution of a Statistic)

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A sampling distribution could be made for any statistic - e.g. a mean, variance, standard deviation, mean, median, etc.

§6.4 Example

Construct the sampling distribution for the mean number of children from population of 5 families with the following number of children where the sample size is 2.

Family	Number of Children
A	1
B	0
C	2
D	2
E	1

§6.4 Example

We need to find all the ways of choosing a pair from the 5 families. They are: AB(1), AC(3), AD(3), AE(2), BC(2), BD(2), BE(1), CD(4), CE(3), and DE(3). Each of these has a $\frac{1}{10}$ chance of being the sample. So we have

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Notice, the mean of the sample mean given this probability distribution is 1.2, which matches the population mean! This makes mean an unbiased estimator!

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We call these unbiased estimators. These do not:

- median
- range
- standard deviation

We call these biased estimators.

§6.5 The Central Limit Theorem

Theorem (Central Limit Theorem)

For all samples of the same size n with $n > 30$, the sampling distribution of $\overline{x}$ can be approximated by a normal distribution with mean μ and standard deviation $\dfrac{\sigma}{\sqrt{n}}$.

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Note: You can use the CLT when the samples are coming from a normally distributed population even when the sample size is smaller than 30.

§6.5 Example

We believe the mean weight of a population of 2000 men is 160lbs and that the standard deviation for these weights is 30lbs. We take a sample of 36 of these men and find that their average weight is 171lbs. Does this agree with our assumption that the average weight of the population of men was 160lbs with a standard deviation of 30lbs?

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No! Why?