Lecture 5:
Chapter 5

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UAB Mathematics

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§5.1 Differences Between Statistics and Probability

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In this chapter, we combine the two methods! We use a statistical approach to estimate parameters used in a probabilistic approach.

§5.2 Probability Distributions

Definition (Random Variable)

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Definition (Continuous vs. Discrete Random Variables)

A **discrete random variable** can take on a value from a finite or countably infinite collection of values whereas a **continuous random variable** can take on a value from a collection of uncountably infinite values.

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- $\sum P(x) = 1$, where we sum over all possible values of x. Sometimes rounding errors will cause these sums to be slightly above or below 1. In the continuous case, this summation becomes an **integral**.

Example

Example (Probability of Having x Females in a Biological Family with Three Children)

Note: The following distribution is **not** realistic. Why?

Number of Females (x)	$P(x)$
0	$\frac{1}{6}$
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You can graph a probability distribution of a discrete random variable as a histogram. What would the probability histogram look like for this distribution?

Mean, Variance, and Standard Deviation

Definition (Mean)

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The mean is the "expected value" of the random variable x. We write this as $E = \mu$.

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Definition (Variance)

$$\sigma^2 = \sum_{i=1}^{n} [(x_i - \mu)^2 \cdot P(x_i)] = \sum_{i=1}^{n} [x_i^2 \cdot P(x_i)] - \mu^2$$

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Definition (Standard Deviation)

$$\sigma = \sqrt{\sum_{i=1}^{n}[x_i^2 \cdot P(x_i)] - \mu^2}$$

Mean, Variance, and Standard Deviation: Example

Let's calculate μ, σ^2, and σ for our previous example.

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Number of Females (x)	$P(x)$	$x \cdot P(x)$
0	$\frac{1}{6}$	0
1	$\frac{1}{3}$	0.333
2	$\frac{1}{3}$	0.666
3	$\frac{1}{6}$	0.5
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3	$\frac{1}{6}$	9	1.5
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Thus, $\sigma = 0.9574$.

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We may also determine if a value is unusually high or low by using the 5% rule:

- A value X is unusually low if $P(x \leq X) \leq 0.05$.
- A value X is unusually high if $P(x \geq X) \leq 0.05$.

Binomial Probability Distribution

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- The trials are independent.
- The results of the trial must have only two possible outcomes - often called "success" and "failure".
- The probability for success must remain the same throughout all trials.

Binomial Probability Distribution: Example

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What's the probability $P(x = 3)$? That would be the probability of having two females: FFF.

So $P(x = 0) = \frac{1}{8}$, $P(x = 1) = \frac{3}{8}$, $P(x = 2) = \frac{3}{8}$, $P(x = 3) = \frac{1}{8}$.

Binomial Probability Distribution: Calculating Probabilities

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Here, $\binom{n}{K}$ (read n choose K) simply denotes $\dfrac{n!}{(n-K)!K!}$.

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$$P(x = 5) = \binom{20}{5}(0.2)^5(0.8)^{20-5} = 15504(0.2)^5(0.8)^{15} \approx 0.175.$$

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In this case, we would say that 10 is an unusually high number of successes.

Example: Binomial(15,0.25)

Use StatCrunch to determine

- $P(x > 4)$

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- $P(x > 4) \approx 0.314$.
- $P(x = 2) \approx 0.156$.
- $P(x \geq 7) \approx 0.057$.
- $P(x \leq 0) \approx 0.013$.

Parameters for Binomial Distributions

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- $\sigma = \sqrt{npq}$

With this information, we could use the range rule of thumb rather than the 5% rule of thumb to determine if a value is typical or not.

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Well, $\mu = 15(0.25) = 3.75$ and $\sigma = \sqrt{15(0.25)(0.75)} \approx 1.677$. So our typical range of values is

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$$[3.75 - 2(1.667), 3.75 + 2(1.667)] = [0.396, 7.104].$$

Because we know the values only take whole numbers, we say that the range of typical values are the whole numbers from 1 to 7.