Lecture 5:
Chapter 5

C C Moxley

UAB Mathematics

15 June 15

§5.1 Differences Between Statistics and Probability

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In Chapter 4, we determined expected chances of certain outcomes using a probabilistic approach.

In this chapter, we combine the two methods! We use a statistical approach to estimate parameters used in a probabilistic approach.

§5.2 Probability Distributions

Definition (Random Variable)

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Definition (Continuous vs. Discrete Random Variables)

A **discrete random variable** can take on a value from a finite or countably infinite collection of values whereas a **continuous random variable** can take on a value from a collection of uncountably infinite values.

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Example

Example (Probability of Having x Females in a Biological Family with Three Children)

Note: The following distribution is **not** realistic. Why?

Number of Females (x)	$P(x)$
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You can graph a probability distribution of a discrete random variable as a histogram. What would the probability histogram look like for this distribution?

Mean, Variance, and Standard Deviation

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The mean is the "expected value" of the random variable x. We write this as $E = \mu$.

Mean, Variance, and Standard Deviation

Definition (Variance)

$$\sigma^2 = \sum_{i=1}^{n}[(x_i - \mu)^2 \cdot P(x_i)] = \sum_{i=1}^{n}[x_i^2 \cdot P(x_i)] - \mu^2$$

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Definition (Standard Deviation)

$$\sigma = \sqrt{\sum_{i=1}^{n} [x_i^2 \cdot P(x_i)] - \mu^2}$$

Mean, Variance, and Standard Deviation: Example

Let's calculate μ, σ^2, and σ for our previous example.

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1	$\frac{1}{3}$	0.333
2	$\frac{1}{3}$	0.666
3	$\frac{1}{6}$	0.5
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Thus, $\sigma = 0.9574$.

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We may also determine if a value is unusually high or low by using the 5% rule:

- A value X is unusually low if $P(x \leq X) \leq 0.05$.
- A value X is unusually high if $P(x \geq X) \leq 0.05$.

Binomial Probability Distribution

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- The procedure has a fixed number of trials n.
- The trials are independent.
- The results of the trial must have only two possible outcomes - often called "success" and "failure".
- The probability for success must remain the same throughout all trials.

Binomial Probability Distribution: Example

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What's the probability $P(x = 3)$? That would be the probability of having two females: FFF.

So $P(x = 0) = \frac{1}{8}$, $P(x = 1) = \frac{3}{8}$, $P(x = 2) = \frac{3}{8}$, $P(x = 3) = \frac{1}{8}$.

Binomial Probability Distribution: Calculating Probabilites

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Here, $\binom{n}{K}$ (read n choose K) simply denotes $\dfrac{n!}{(n-K)!K!}$.

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$$P(x = 5) = \binom{20}{5}(0.2)^5(0.8)^{20-5} = 15504(0.2)^5(0.8)^{15} \approx 0.175.$$

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In this case, we would say that 10 is an unusually high number of successes.

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Use StatCrunch to determine

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- $P(x > 4) \approx 0.314$.
- $P(x = 2) \approx 0.156$.
- $P(x \geq 7) \approx 0.057$.
- $P(x \leq 0) \approx 0.013$.

Parameters for Binomial Distributions

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With this information, we could use the range rule of thumb rather than the 5% rule of thumb to determine if a value is typical or not.

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$$[3.75 - 2(1.667), 3.75 + 2(1.667)] = [0.396, 7.104].$$

Because we know the values only take whole numbers, we say that the range of typical values are the whole numbers from 1 to 7.