Lesson 6:
Chapter 4 Sections 4-5

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BSC Mathematics

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§4.4 Means and Variances of Random Variables

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Knowledge about a probability distribution/random variable is always more extensive than knowledge about a sample! Naturally, then, the ideas of measures of center and spread from Chapter 1 should carry over to random variables and their probability distributions. Indeed, they do. We'll discuss how to calculate them in the **discrete** case. The continuous case involves integral calculus and is beyond the scope of the course.

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$P(X)$	0.5	0.4	0.1

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If you played the game 1000 times, how many times would you expect to win \$0? \$5? \$50? How much would you when on average for these 1000 games? How much should the casino charge to play this game so that it breaks even in the long run?

§4.4 Means and Variances of Random Variables

Definition (mean of a probability distribution)

The **mean of a probability distribution of a discrete random variable** is the weighted average of its values where the weights are given by the probability assigned to a particular value.

$$\mu_X = p_1 x_1 + p_2 x_2 + \ldots = \sum_i p_i x_i$$

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We often call the mean of a probability distribution its **expected value**.

Example (equally likely values of the random variable)

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Example (Minitab example)

Use Minitab to confirm that the mean of this probability distribution is $\mu_X = 4.27588$.

X	2.1	5	3.78	7	7.7	7.6
$P(X)$	0.134	0.234	0.511	0.053	0.051	0.017

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Theorem (Law of Large Numbers)

You can estimate the mean μ of a population by the mean of a sample of that population $\bar{x}$ by taking successively larger samples. The sample means $\bar{x}$ will approach μ and, as they approach μ, they will ***stay that close****.*

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Note: The statement for the Law of Large Numbers given here is closely related to the Central Limit Theorem. A similar statement can be made about population and sample proportions.

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- The samples must be **large**, obviously.
- The samples must be good — think SRSs. Or if the samples are a number of trails of a random process, the samples must be independent or **independent in the long run**.
- Even though a mean is sensitive to outliers, this variability dies off as the sample becomes large. Try this: Say the average weight of 1000 tires was 45lbs and that the 1001st tire sampled had a weight far from the average. Say it weighed 200lbs. What's the new average?

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Theorem (rules for means of linear transformations of random variables)

1. *If X is a random variable and a and b are constants, then*

$$\mu_{a+bX} = b\mu_X + a.$$

2. *If X and Y are random variables, then*

$$\mu_{X+Y} = \mu_X + \mu_Y.$$

3. *If X and Y are random variables, then*

$$\mu_{X-Y} = \mu_X - \mu_Y.$$

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$$11 = 2\mu_Y + 1 \implies 10 = 2\mu_Y \implies \mu_Y = 5.$$

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Definition (variance of a finite, discrete random variable)

Suppose that X is a **discrete random variable** whose distribution is given below.

Value of X	x_1	x_2	x_3	$\dots$	x_k
$P(X)$	p_1	p_2	p_3	$\dots$	p_k

The **variance** of this random variable is given by

$$\sigma_X^2 = \sum (x_i - \mu_x)^2 p_i.$$

And the **standard deviation** σ_X is the square root of the variance.

Example (variance and standard deviation of a probability distribution)

Find the variance of the probability distribution below.

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P(X)	0.1	0.2	0.4	0.3

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$\sigma_X^2 = 0.89$.

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Example (variance and standard deviation of a probability distribution)

Use Minitab to confirm the variance of the probability distribution below is 8.0356.

X	0	1	2	3	4	5	6	7	8
P(X)	0.08	0.1	0.2	0.1	0.02	0.05	0.15	0.05	0.25

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$$\sigma^2_{a+bX} = b^2\sigma^2_X.$$

2. *If X and Y are* **independent** *random variables, then*

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y.$$

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Example (variances of linear transformations of random variables)

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- If $X = 2Y + 1$ and the variance of X is 4, find the variance of Y.

$$4 = 2^2\sigma_Y^2 \implies 1 = \sigma_Y^2.$$

We have a general rule for the variance of a linear transformation of two (not necessarily independent) random variables.

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Theorem (rules for variance of the sum/difference of two RVs)

4. *If X and Y are random variables, then*

$$\sigma^2_{X+Y} = \sigma^2_X + \sigma^2_Y + 2\rho\sigma_X\sigma_Y.$$

5. *If X and Y are random variables, then*

$$\sigma^2_{X-Y} = \sigma^2_X + \sigma^2_Y - 2\rho\sigma_X\sigma_Y.$$

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In the above formulas, ρ is the linear correlation coefficient between X and Y.

Example (variance of linear transformation of random variables)

If X and Y have variances 2 and 3 respectively and have a linear correlation coefficient of -0.25, then what's the variance of $X - Y$?

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$$\sigma^2_{X-Y} = 2 + 3 - 2(-0.25)(\sqrt{2})(\sqrt{3}) = 6.225.$$

§4.5 General Probability Rules

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Theorem (Probability Rules)

1. $0 \leq P(A) \leq 1$
2. $P(S) = 1$
3. $P(A \text{ or } B) = P(A) + P(B)$ *if A and B are disjoint.*
4. $P(A^c) = 1 - P(A)$
5. $P(A \text{ and } B) = P(A)P(B)$ *if A and B are independent.*

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We'll extend these rules.

§4.5 General Probability Rules

Theorem (Extra Probability Rules)

6. *$P(A \text{ or } B \text{ or } C) = P(A) + P(B) + P(C)$ if A, B, and C are pairwise disjoint.*
7. $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
8. $P(A \text{ and } B) = P(A)P(B|A)$

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7. $P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$
8. $P(A \text{ and } B) = P(A)P(B|A)$

Notice, this last identity can be solved for $P(B|A)$.

Example (examples using new rules)

In a class of 20 students, 12 have glasses, 8 have contacts, and 6 have both contacts and glasses.

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Notice: $P(A|B) \neq P(B|A)$.

Theorem (Bayesian Formula)

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We can demonstrate this fact with a simple drawing!

Example (Bayesian Formula)

Suppose that we have a probability distribution given in the table below. Let's first talk about how to read this table.

	0-9	10-19	20-29	30-39
Vaccinated	0.14	0.14	0.16	0.15
Unvaccinated	0.01	0.05	0.06	0.07
Unsure	0.02	0.05	0.05	0.10

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What's the probability that someone is unvaccinated, given that they are between the ages of 20 and 29?

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Example (use a Venn diagram)

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Find the probability that C happens but A does not happen.

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Find the probability that none of the three events happen. 0.1.
Find the probability that neither A nor B happens.

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Find the probability that C happens but A does not happen. 0.4.
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Find the probability that neither A nor B happens. 0.3