Integral mean value theorems and the Ganelius inequality

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Synopsis

The inequality of Ganelius states that, for suitable functions \( f \) and \( g \) on an interval \([a, b]\),

\[
\int_a^b f(x)g(x) \, dx = f(\xi) \int_a^\xi g(x) \, dx
\]

for some number \( \xi \) between \( a \) and \( b \). This result appears to be due originally
to Bonnet [2], with improvements and modifications due to Weierstrass and Du Bois
Reymond [4] (see also [9, \S421]).

The main aim of the present paper is to establish versions of these theorems
valid under considerably weaker assumptions on the function \( f \) and \( \phi \), and to
point out the connection between these mean value theorems and certain integral
inequalities of a type introduced by Ganelius in [6, 7]. The original Ganelius
inequality stated that if \( f \) and \( g \) are functions of bounded variation on an interval
\([a, b]\) with \( f \) non-negative, and \( f \) and \( g \) having no points of discontinuity in
common, then

\[
\int_a^b f(x) \, dg(x) \leq \left( \inf f + \var f \right) \sup_{J \subseteq [a, b]} \int_J dg(x)
\]

where \( \inf f \) and \( \var f \) denote the infimum and total variation, respectively, of \( f \)
over \([a, b]\) and the integrals are of Riemann-Stieltjes type. This result has found
useful application in the theory of ordinary differential equations, and elsewhere
[3; 6; 8, p. 374; 10], although in more recent work (e.g. [5]), its use has largely
been superseded by a very effective method due to Atkinson [1, Lemma 5]. A
short proof of the Ganelius inequality has been given by Wüst [11].

Observe that, on replacing \( g \) by \( -g \) in the inequality, one obtains an equivalent
inequality:

\[
\left( \inf f + \var f \right) \inf_{J \subseteq [a, b]} \int_J dg(x) \leq \int_a^b f(x) \, dg(x).
\]
In this form, it is not hard to see that, under appropriate conditions on \( f \) and \( g \), there exists an interval \( K \subset [a, b] \) with

\[
\int_a^b f(x) \, dg(x) = [\inf f + \text{var} f] \int_K dg(x).
\]

We show that results of this type include all the standard integral mean-value theorems.

### 2. An extension of Ganelius' inequality

The main theorem of the paper is the following improved version of the Ganelius inequality. The proof, which rests on an alternative (unpublished) proof of the original inequality due to D. B. Sears, is given in §3.

**Theorem.** Let \( f \) and \( g \) be functions of bounded variation on \([a, b]\) with no points of discontinuity in common, and assume \( f \) is non-negative. Then,

\[
(i) \quad \int_a^b f \, dg \leq \frac{1}{3} [f(a) + f(b) + \text{var} f] \sup_{J \subset [a, b]} \int_J dg;
\]

(ii) if \( g \) is continuous, then there is a sub-interval \( K \) of \([a, b]\) such that (4) holds as an equality with \( \sup_{J} dg \) replaced by \( \int_K dg \);

(iii) if \( f \) is non-increasing [respectively, non-decreasing], then the sub-intervals \( J, K \) on the right side in (i) and (ii) may be assumed to have left end-point \( a \) [respectively, right end-point \( b \)].

**Remarks**

1. The assumption that \( f \) and \( g \) have no points of discontinuity in common is sufficient to ensure that the Riemann-Stieltjes integral \( \int_a^b f \, dg \) exists. (See [9, p. 546]; cf. also [9, p. 542]).

2. Observe also that in both part (i) of the theorem above and the original inequality, the assumption that \( f \) be non-negative is necessary. This may be seen by considering (4) in the case when \( f \) is negative and \( g \) is continuous and strictly decreasing; here the left side is positive, while the right side is zero.

3. Observe that, on replacing \( f \) by \( f - \inf f \), we obtain

\[
\int_a^b (f - \inf f) \, dg \leq \frac{1}{3} [f(a) + f(b) - 2 \inf f + \text{var} f] \sup_{J \subset [a, b]} \int_J dg.
\]

The theorem remains true when (4) is replaced by (5). In this form, the assumption that \( f \) be non-negative is not needed.

The following results may now be deduced from the theorem.

**Corollary 1** (Ganelius). If \( f \) and \( g \) are of bounded variation on \([a, b]\) with no points of discontinuity in common and \( f \) is non-negative, then (2) holds.

**Proof.** It is enough to show that

\[
0 \leq f(a) + f(b) - 2 \inf f \leq \text{var} f.
\]
But, for each $\varepsilon > 0$, one can find a number $c_\varepsilon$ such that $f(c_\varepsilon) < \varepsilon + \inf_{x \in [a,b]} f$. Thus

$$f(a) + f(b) - 2 \inf_{x \in [a,b]} f(x) \leq [f(a) - f(c_\varepsilon)] + [f(b) - f(c_\varepsilon)] + 2\varepsilon$$

$$\leq \var f + 2\varepsilon$$

for all $\varepsilon > 0$; the result now follows easily. $\square$

In the same way that (ii) above follows from (i), we have from Corollary 1,

**Corollary 2.** Let $f \equiv 0$ and $g$ be of bounded variation on $[a, b]$ with no points of discontinuity in common. Then there exists an interval $K \subseteq [a, b]$ with

$$\int_a^b f \, dg = [\inf f + \var f] \int_K \, dg.$$ 

Finally, we have

**Corollary 3 (Second Mean-Value Theorem).** Assume that $f$ is bounded and monotonic, and $\phi$ is integrable on $[a, b]$. Let $A$, $B$ be constants such that $A \leq f(a^-) \leq f(b^-) \leq B$ if $f$ is non-decreasing and $A \leq f(a^+) \leq f(b^-) \leq B$ if $f$ is non-increasing. Then there exists a value $\xi \in [a, b]$ such that

$$\int_a^b f \phi = A \int_a^\xi \phi + B \int_\xi^b \phi.$$  

**Proof.** We assume that $f$ is non-increasing; the proof for $f$ non-decreasing is similar. Define $h$ on $[c, d] \supseteq [a, b]$ by $h(x) = f(x)$, $a \leq x \leq b$, $h(x) = A$, $c \leq x < a$, $h(x) = B$, $b < x \leq d$. Thus $h$ is non-increasing. Extend $\phi$ to $[c, d]$ by defining it to be zero outside $[a, b]$. We now apply the theorem using $h - \inf h$, $g = \int \phi$ over $[c, d]$. By observing that $\inf_{[c, d]} h = B$ and $\var h = A - B$, we thus infer the existence of a number $\xi$ such that

$$\int_a^b (f - B) \phi = (A - B) \int_a^\xi \phi;$$

the result now follows by rearranging this identity. $\square$

Observe that one can obtain the original mean-value theorem (1) by setting $B = 0$ and choosing $A = f(a)$.

### 3. Proof of the Theorem

Before proceeding, we need the following simple

**Lemma.** If $g$ is continuous on $[a, b]$, then there exist intervals $K$ and $L$ in $[a, b]$ with

$$\int_K \, dg = \sup_{J \subseteq [a, b]} \int_J \, dg,$$

$$\int_L \, dg = \inf_{J \subseteq [a, b]} \int_J \, dg.$$
Proof. In measure-theoretic terms, this says that signed measure $m(J) = \int_J dg$ attains maximum and minimum values over sub-intervals of the compact interval $[a, b]$. More directly, for $a \leq \alpha \leq \beta \leq b$, set

$$F(\alpha, \beta) = \int_\alpha^\beta dg.$$ 

As $g$ is continuous on $[a, b]$, $F$ is continuous on the closed triangle $\{(\alpha, \beta): a \leq \alpha \leq \beta \leq b\}$ in the plane. The result then follows from the fact that continuous functions on compact subsets of $\mathbb{R}^2$ attain their maximum and minimum values. \qed

To establish parts (i) and (iii), we consider first the case when $f$ is a step function. Let $f$ take the value $c_r$ on $E(r) = (x_{r-1}, x_r)$ (one or both end-points may not be included), where $\{x_r: 0 \leq r \leq n\}$ is a partition of $[a, b]$. Set $I_r = \int_{E(r)} dg$. We now show that there exist numbers $c'_r$ and integrals $I'_r$ such that

$$\int_a^b f dg = \sum_{r=1}^n c_r I_r = \sum_{r=1}^n c'_r I'_r, \quad \text{ (6)}$$

where $c'_r \geq 0$ $(1 \leq r \leq n)$ and

$$\sum_{r=1}^n c'_r \leq \frac{1}{2} \left[ c_1 + c_n + \sum_{r=1}^{n-1} |c_{r+1} - c_r| \right]. \quad \text{ (7)}$$

If $f$ is non-increasing this can be achieved, with equality in (7), by setting $c'_r = c_r - c_{r+1}$, $r = 1, \ldots, n-1$, $c'_n = c_n$, and $I'_r = \sum_{s=1}^{r-1} I_s$. Similarly, if $f$ is non-decreasing, then set $c'_r = c_{r+1} - c_r$, $r = 2, \ldots, n$, $c'_1 = c_1$, and $I'_r = \sum_{s=r+1}^{n} I_s$. In the general case, we use induction on $n$. The case $n = 2$ is covered above. Suppose the result is true for all $f \geq 0$ with at most $n$ values. For a function $f$ with $n + 1$ values, set $c_k = \min \{c_r: 1 \leq r \leq n + 1\}$. Then,

$$\sum_{r=1}^{k-1} (c_r - c_k) I_r = \sum_{r=1}^{k-1} c'_r I'_r, \quad \text{ (8)}$$

where

$$\sum_{r=1}^{k-1} c'_r \leq \frac{1}{2} \left[ c_1 + c_k - 2c_k + \sum_{r=1}^{k-2} |c_{r+1} - c_r| \right] \quad \text{ (9)}$$

and

$$\sum_{r=k+1}^{n+1} (c_r - c_k) I_r = \sum_{r=k+1}^{n+1} c'_r I'_r, \quad \text{ (10)}$$

where

$$\sum_{r=k+1}^{n+1} c'_r \leq \frac{1}{2} \left[ c_{k+1} + c_{n+1} - 2c_k + \sum_{r=k+1}^{n} |c_{r+1} - c_r| \right]. \quad \text{ (11)}$$

Thus, setting $c'_k = c_k$ and $I'_k = I_1 + \ldots + I_{n+1}$, we have from (8–11),

$$\sum_{r=1}^{n+1} c_r I_r = \sum_{r=1}^{n+1} c'_r I'_r,$$
where
\[
\sum_{r=1}^{n+1} c'_r \leq \frac{1}{2} \left[ c_1 + c_{n+1} + \sum_{r=1}^{n} |c_{r+1} - c_r| \right],
\]
as required. Part (i), for step functions \( f \geq 0 \), now follows from (6) and (7). Also, if \( f \) is non-increasing, then \( \sum c'_r I'_r \leq (\sum c'_r) I \), where \( I = \max I'_r \) is an integral over an interval with left end-point \( a \). Similar remarks apply also when \( f \) is non-decreasing, except that \( I \) is now an interval with right end-point \( b \).

Now let \( f \) be a non-negative function of bounded variation on \([a, b]\) and write it as \( f = P - N \) where the functions \( P \) and \( N \) are non-negative and increasing on \([a, b]\). We approximate to \( P \) and \( N \) as step functions as follows. First, given \( \varepsilon > 0 \), divide the range of \( P \) into sub-intervals of length not larger than \( \varepsilon \) and define a new function \( P_\varepsilon \) to be equal to the infimum of \( P \) over each interval \( K \) whose range under \( K \) is one of the above sub-intervals. If the range of \( N \) is of length not larger than \( \varepsilon \) in \( K \), define \( N_\varepsilon \) as inf \( N \) (over \( K \)). Otherwise, split up the range \( N(K) \) into sub-intervals with length not larger than \( \varepsilon \) and define \( N_\varepsilon \) in each of these sub-intervals to be the infimum of \( N \) in that sub-interval. Then replace \( K \) in \( P_\varepsilon \) by as many sub-intervals as necessary (it is not necessary to redefine \( P_\varepsilon \) ). Define \( f_\varepsilon = P_\varepsilon - N_\varepsilon \). It is clear that \( f_\varepsilon \to f \) uniformly over \([a, b]\) as \( \varepsilon \to 0 \). Parts (i) and (iii) now follow by replacing \( f \) in (4) by \( f_\varepsilon \) and letting \( \varepsilon \) tend to zero.

Finally, to establish part (ii) of the theorem, observe that \( f(a) + f(b) + \var v f = 0 \) if and only if \( f = 0 \). To see this, first note that as \( f \geq 0 \), \( \var v f = -(f(a) + f(b)) \leq 0 \), and hence \( \var v f = 0 \), i.e. \( f \) is a constant. This constant must be zero as \( f(a) = -f(b) \). Consequently, as the result is trivially true for \( f = 0 \), we can assume without loss of generality that \( \frac{1}{2}[f(a) + f(b) + \var v f] \neq 0 \). If we replace \( g \) by \(-g\) in (4), we obtain the inequality,
\[
\frac{1}{2}[f(a) + f(b) + \var v f] \inf_{f'=0} \int_a^b dg \leq \int_a^b f dg.
\] 
\[(12)\]
On setting \( Q = 2\int_a^b f dg [f(a) + f(b) + \var v f]^{-1} \) and using (4) and (12) and the lemma, we have
\[
\int_L dg \leq Q \leq \int_K dg.
\] 
\[(13)\]
Assume first that \( Q \geq 0 \), and set \( K = [c, d] \) and \( H(\alpha) = \int_c^\alpha dg(x) \). Then \( H(c) = 0 \), and from (13) it is clear that \( H(c) \leq Q \leq H(d) \). As \( H \) is continuous on \([c, d]\), the required result follows by the intermediate value theorem. On the other hand, if \( Q < 0 \), one can use the left inequality in (13) in a similar fashion. This completes the proof.

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References


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