CONDITIONAL WELL-POSEDNESS FOR AN ELLIPTIC INVERSE PROBLEM

IAN KNOWLES AND MARY A. LARUSSA

Abstract. For the elliptic equation 

\[ -\nabla \cdot (p(x) \nabla v) + \lambda q(x)v = f, \quad x \in \Omega \subset \mathbb{R}^n, \]

the problem of determining when one or more of the coefficient functions \(p, q,\) and \(f\) depends continuously on the solution functions \(v = v_{p,q,f,\lambda}\), is considered.

1. Introduction and Scientific Context. According to Hadamard [5, 6], a problem, defined as the obtaining of a solution from some given data, is well-posed if three requirements are met:
   (1) a solution exists;
   (2) that solution is unique, and
   (3) depends continuously on the data.

Otherwise the problem is ill-posed. In the context of computational inverse problems it is commonly believed, at least in the wider scientific community, that all inverse problems are ill-posed. This has led to the general perception in these communities that the methods, such as Tychonov regularization, that have been specifically developed for truly ill-posed problems, have to be applied to all inverse problems. One of the aims of this article is to show that in some situations there are alternatives that should be considered.

Specifically, consider the elliptic differential equation

\[ Lv = -\nabla \cdot (p(x) \nabla v) + \lambda q(x)v = f(x), \quad x \in \Omega, \]  

(1.1)

where \(\lambda > 0\), and \(\Omega\) is a bounded domain in \(\mathbb{R}^n\) with a \(C^{2,1}\)-boundary, \(n \geq 1\), and

\[ f \in C^{0,1}(\bar{\Omega}), \quad q \in L^\infty(\Omega), \quad \text{and} \quad p \in C^{0,1}(\bar{\Omega}), \]  

(1.2)

are real, and \(q\) and \(p\) satisfy

\[ q(x) \geq 0, \quad p(x) \geq \nu > 0, \quad x \in \Omega, \]  

(1.3)

for some constant \(\nu\). It is known [11, §30,V] that, under these conditions, for any \(\phi \in C^{1,1}(\bar{\Omega})\) the Dirichlet problem formed from (1.1) and the boundary condition

\[ v = \phi \text{ on } \partial \Omega, \]  

(1.4)

has a unique solution \(v = v_{p,q,f,\lambda}\) in the Hölder space \(C^{1,1}(\bar{\Omega})\). In the proofs that follow, we work with solutions \(u\) in a Sobolev space context, specifically the Sobolev space \(H^1(\Omega) = \mathcal{W}^{1,2}(\Omega)\), which contains \(C^{1,1}(\bar{\Omega})\). Having the solution \(v \in C^{1,1}(\bar{\Omega})\) provides sufficient smoothness on the boundary for integration by parts, which is an important tool in the proofs that follow. Also, note that since \(p\) and \(f\) are both Lipschitz continuous (i.e. in \(C^{0,1}(\bar{\Omega})\)), they are both bounded, and therefore in \(L^2(\Omega)\).

We are concerned here with determining precise conditions under which inverse problems related to (1.1) are well-posed: specifically, when are one or more of the coefficients \(p > 0, q \geq 0, f\) continuously dependent on the solutions \(v_{p,q,f,\lambda}\) for one or more values of \(\lambda\)?

*Department of Mathematics, University of Alabama at Birmingham, Birmingham, Alabama, 35294
Interest in these particular elliptic inverse problems arises naturally from the study of underground aquifer systems (as well as oil reservoir simulations) which are often modeled by the diffusion equation:

\[ q(x) \frac{\partial w}{\partial t} = \nabla \cdot [p(x)\nabla w(x, t)] + R(x, t); \tag{1.5} \]

here \( w \) represents the (known) piezometric head (well pressure), \( p > 0 \) the hydraulic conductivity (or sometimes, for a two dimensional aquifer, the transmissivity), \( R \) the recharge, and the function \( q \geq 0 \) the storativity of the aquifer (see, for example, [1, 2]).

It is well known among hydro-geologists [1, Chapter 8] that the inability to obtain reliable values for the coefficients in (1.5) is a serious impediment to the confident use of such models.

An effective new method for the simultaneous recovery of \( p \), \( q \), and \( R \) (including its time dependence) from a known \( w \) is presented in [8, 9, 14]. This method relies upon first transforming the known solution \( w \) of the parabolic equation (1.5) to solutions of the elliptic equation (1.1) via a finite Laplace transform, and then recovering the coefficients by solving the above elliptic inverse problem. During the course of this work we were struck by the fact that the computations appeared to converge steadily, oftentimes over thousands of iterations, to the correct functions, which were typically quite discontinuous. As this was clearly not the behavior one normally associates with ill-posed problems, we conjectured that these were actually well-posed problems, at least under certain conditions. As existence is evident from the formulation, and uniqueness for these elliptic inverse problems is treated in [7], our main goal here is to present conditions under which these inverse recoveries exhibit continuous dependence on the data. We assume throughout that uniqueness holds for each of the inverse problems considered in the paper.

2. Preliminary Theory. For later use we need a new trace estimate for the co-normal derivatives of \( H^1(\Omega) \) functions. Such an estimate is known for \( H^2(\Omega) \) functions (see for example [4, Theorem 2.4.1.3]), but when \( u \in H^1(\Omega) \), even the definition of the co-normal derivative is initially problematical.

For an open bounded region \( \Omega \subset \mathbb{R}^n \) we define the boundary space \( H^{1/2}(\partial \Omega) \) to be \( H^1(\Omega)/H^1_0(\Omega) \), and denote the dual space \( H^{1/2}(\partial \Omega)^* \) by \( H^{-1/2}(\partial \Omega) \). It is known [12, §A.6] that if we define \( \mathcal{W} \) to be the subspace of all \( w \) in \( H^1(\Omega) \) such that \((I - \Delta)w = 0\) distributionally, then we have, with respect to the usual inner product on \( H^1(\Omega) \), that

\[ H^1(\Omega) = H^1_0(\Omega) \oplus \mathcal{W}. \]

From this one can also see that the trace map \( \gamma : \mathcal{W} \to H^{1/2}(\partial \Omega) \) defined by

\[ \gamma w = \{w\} + H^1_0(\Omega), \]

where

\[ \|\gamma w\|_{H^{1/2}(\partial \Omega)} = \inf \{ \|v\|_{H^1(\Omega)} : v = w + v_0 \in \gamma w \} = \|w\|_{H^1(\Omega)} \]

is a linear isometry of \( \mathcal{W} \) onto \( H^{1/2}(\partial \Omega) \), and thus an isometric isomorphism. For later use, note also that, for any \( u \in H^1(\Omega) \), \( u = w + u_0 \), where \( w \in \mathcal{W} \) and \( u_0 \in H^1_0(\Omega) \), and we have the standard trace inequality

\[ \|\gamma u\|_{H^{1/2}(\partial \Omega)} = \|\gamma w\|_{H^{1/2}(\partial \Omega)} = \|w\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)}. \tag{2.1} \]
We henceforth regard $u|_{\partial \Omega}$ as $\gamma u$.

Now, for $u \in \mathcal{H}^1(\Omega)$ and $p \in \mathcal{L}^\infty(\Omega)$ satisfying (1.3), we can define the continuous linear functional $f = (I - \nabla \cdot p \nabla)u \in \mathcal{H}_{0}^{1}(\Omega)^{*} = \mathcal{H}^{-1}(\Omega)$ by

$$f(\phi) = \int_{\Omega} p \nabla u \cdot \nabla \phi + u \phi \quad (2.2)$$

for all $\phi \in \mathcal{H}_{0}^{1}(\Omega)$. By the Hahn-Banach theorem $f$ can be extended to a linear functional $F^{*}$ that is continuous on $H^1(\Omega) \supset \mathcal{H}_{0}^{1}(\Omega)$ with

$$\|F^{*}\|_{H^{1}(\Omega)^{*}} = \|f\|_{H^{-1}(\Omega)}. \quad (2.3)$$

For any $F$ continuously extending $f$, define the functional $g \in \mathcal{H}^{-1/2}(\partial \Omega)$ by

$$g(\gamma \phi) = -F(\phi) + \int_{\Omega} p \nabla u \cdot \nabla \phi + u \phi, \quad (2.4)$$

for all $\phi \in H^1(\Omega)$. It follows from (2.4) and the Lax-Milgram lemma that, as $g \in \mathcal{H}^{-1/2}(\partial \Omega)$ and $F \in \mathcal{H}^{1}(\Omega)^{*}$, the weak Neumann problem

$$(I - \nabla \cdot p \nabla)u = F \quad (2.5)$$
$$p \partial_{\nu} u = g, \quad (2.6)$$

has $u \in \mathcal{H}^1(\Omega)$ as the unique solution. So, each choice of $F \in \mathcal{H}^{1}(\Omega)^{*}$ extending $f$ gives rise to a corresponding co-normal derivative $g$ defined by (2.4), and vice versa.

The key point here is that once $F$ (extending $f$) and $g$ are chosen so that (2.4) holds, the $F$ and $g$ uniquely associate with $u$. Notice that for all $F$ extending a given $f$, we have from (2.3) that their norms in $H^1(\Omega)^{*}$ satisfy

$$\|F\|_{H^{1}(\Omega)^{*}} = \sup_{\|\phi\|_{H^{1}(\Omega)} = 1} |F(\phi)| \geq \sup_{\|\phi\|_{H^{1}(\Omega)} = 1} |f(\phi)| = \|f\|_{H^{-1}(\Omega)}. \quad (2.7)$$

Also, by the Riesz representation theorem, if we define an equivalent inner product (and associated norm) on $\mathcal{H}^{1}(\Omega)$ (and $\mathcal{H}_{0}^{1}(\Omega)$) by

$$(v, w)_{1,p} = \int_{\Omega} p \nabla v \cdot \nabla w + vw$$

then, as $\mathcal{H}_{0}^{1}(\Omega)$ is a Hilbert space, there is a unique $w \in \mathcal{H}_{0}^{1}(\Omega)$ such that for all $\phi \in \mathcal{H}_{0}^{1}(\Omega)$,

$$f(\phi) = (w, \phi)_{1,p} \quad (2.8)$$

and $\|f\|_{H^{-1}(\Omega)} = \|w\|_{1,p}$. By the same reasoning, there is also a unique $v^{*} \in \mathcal{H}^{1}(\Omega)$ such that for all $\phi \in \mathcal{H}^{1}(\Omega)$

$$F^{*}(\phi) = (v^{*}, \phi)_{1,p} \quad (2.9)$$

and

$$\|w\|_{1,p} = \|F^{*}\|_{H^{1}(\Omega)^{*}} = \|f\|_{H^{-1}(\Omega)} = \|w\|_{1,p}. \quad (2.10)$$
Setting \( \phi = w \) in (2.9) and (2.8) we have \((v^*, w)_{1,p} = \|w\|_{1,p}^2\). Assuming without loss that \(v^* \neq 0\) (and hence \(w \neq 0\)) and using (2.10) we have

\[
\left( \frac{v^*}{\|v^*\|_{1,p}}, \frac{w}{\|w\|_{1,p}} \right)_{1,p} = 1,
\]

from which it follows that \(v^* = w\). This means that among all the continuous extensions \( F \) of a given \( f \), the Hahn-Banach extension \( F^* \) is uniquely specified by the condition \( \|F^*\|_{H^1(\Omega)^*} = \|f\|_{H^{-1}(\Omega)} \). Notice also that from (2.4) that the unique \( g^* \) associated with \( F^* \) (and \( u \)) via (2.4) satisfies

\[
g^*(\gamma \phi) = (u - v^*, \phi)_{1,p} = (u - w, \phi)_{1,p}
\]

for all \( \phi \in H^1(\Omega) \), so that

\[
\|g^*\|_{H^{-1/2}(\partial \Omega)}^2 = \|u - w\|_{1,p}^2.
\]

Observe that for any \( F \in H^1(\Omega)^* \) and \( g \in H^{-1/2}(\partial \Omega) \) the weak Neumann problem

\[
(I - \nabla \cdot p \nabla)u = F
\]

\[
p \partial_n u = g,
\]

has a unique solution \( u \in H^1(\Omega) \), by the Lax-Milgram lemma. We partition \( H^1(\Omega)^* \times H^{-1/2}(\partial \Omega) \) into equivalence classes in which \((F_1, g_1)\) equivalent to \((F_2, g_2)\) means that they associate with the same \( u \in H^1(\Omega) \). Motivated by (2.3), (2.7) and (2.12), we define the norm of the equivalence class \([[(F, g)]\) generated by \( u \) to be

\[
\|[(F, g)]\| = \|f\|_{H^{-1}(\Omega)} + \|u - w\|_{1,p}^2,
\]

and denote the set of all such equivalence classes by \( B \). Consider a Cauchy sequence \([[(F_n, g_n)]\) in \( B \), where the pair \((F_n, g_n)\) associates with \( u_n \in H^1(\Omega) \). Let \( f_n \) be defined by (2.2) with \( u_n \), replacing \( u \), let \( w_n \) be defined by (2.8) with \( f_n \), replacing \( f \). As \([[(F_n, g_n)]\) is Cauchy, it follows from (2.15) that the sequence \([f_n]\) in \( H^{-1}(\Omega) \) is Cauchy, hence convergent to \( f \in H^{-1}(\Omega) \). Consequently, from (2.8) the sequence \([w_n]\) is Cauchy, hence convergent to \( w \in H_0^1(\Omega) \). As the sequence \([u_n - w_n]\) in \( H^1(\Omega) \) is also Cauchy from (2.15) again, and \( u_n = (u_n - w_n) + w_n \), we see that the sequence \([u_n]\) is Cauchy, and hence convergent to \( u \in H^1(\Omega) \). Define \( F^* \in H^1(\Omega)^* \) by \( F^* = w \) and \( g^* \in H^{-1/2}(\partial \Omega) \) by (2.11). Then \([[(F^*, g^*)]]\) is the equivalence class generated by \( u \). It follows that \([[(F_n, g_n)]\) converges to \([[(F^*, g^*)]]\) in \( B \), and that \( B \) is a Hilbert space.

So, for each \( u \in H^1(\Omega) \) we associate the equivalence class \([[(F, g)]\) in \( B \), and each such class contains a distinguished pair \((F^*, g^*)\), where \( F^* \) is the unique Hahn-Banach extension associated with \( f = (I - \nabla \cdot p \nabla)u \in H^{-1}(\Omega) \), and \( g^* \) is defined by (2.11). For any \( u \in H^1(\Omega) \) we define the co-normal derivative of \( u \) by \( p \partial_n u = g^* \in H^{-1/2}(\partial \Omega) \) and \((I - \nabla \cdot p \nabla)u = F^* \in H^1(\Omega)^* \).

Finally, note that for smooth functions \( u \), such as \( u \in C^{1,1}(\Omega) \) for example, and assuming \( p \in C^{0,1}(\Omega) \), the co-normal derivative \( p \partial_n u \) and the function \((I - \nabla \cdot p \nabla)u \) exist independently, and it follows from (2.13) and (2.14) that there is a unique \( F \) extending \( f \) in this case, and the aforementioned equivalence class associated with \( u \) becomes a singleton set.

We now have
Theorem 2.1. If \( \partial \Omega \) is Lipschitz and \( p \partial_{\nu} w \) denotes the co-normal derivative of \( w \in H^1(\Omega) \) on \( \partial \Omega \) and \( p \in C^{0,1}(\Omega) \) satisfies (1.3) then the mapping

\[
w \mapsto \left[ (I - \nabla \cdot p \nabla) w, p \partial_{\nu} w \right]
\]
is an isomorphism of \( H^1(\Omega) \) onto the Hilbert space \( \mathcal{B} \), where \( \mathcal{B} \) is the Hilbert space of equivalence classes defined above. In particular there exists a constant \( C \) independent of \( w \) such that

\[
\| p \partial_{\nu} w \|_{H^{-1/2}(\partial \Omega)} \leq C \| w \|_{H^1(\Omega)}.
\]

(2.16)

Proof. From the discussion above, we see that this mapping is defined on \( H^1(\Omega) \). By the Lax-Milgram lemma, for any \( F \in H^1(\Omega)^* \) and any \( g \in H^{-1/2}(\partial \Omega) \) the weak Neumann problem

\[
(I - \nabla \cdot p \nabla) v = F \\
p \partial_{\nu} v = g
\]

has a unique weak solution \( v \in H^1(\Omega) \), so that the mapping is onto. As \( \partial \Omega \) is continuous, by [3, V.4.7] the restrictions to \( \Omega \) of functions in \( C_\infty^0(\mathbb{R}^n) \) are dense in \( H^1(\Omega) \). Let \( u \in H^1(\Omega) \). Then there is a sequence \( \{ u_n \} \), of restrictions to \( \Omega \) of \( C_\infty^0(\mathbb{R}^n) \) functions, with limit \( u \) in \( H^1(\Omega) \).

Using Green’s theorem, we have that

\[
\int_{\Omega} u_n^2 + p | \nabla u_n |^2 = \int_{\partial \Omega} p \partial_{\nu} u_n \gamma u_n + \int_{\Omega} (I - \nabla \cdot p \nabla) u_n \cdot u_n \\
= g_n^*(\gamma u_n) + F_n^*(u_n),
\]

(2.17)

where \( (F_n^*, g_n^*) \) is the unique representative in the singleton equivalence class generated by \( u_n \), i.e. in this case

\[
g_n^*(\gamma \phi) = \int_{\partial \Omega} p \partial_{\nu} u_n \gamma \phi
\]

and

\[
F_n^*(\phi) = \int_{\Omega} (I - \nabla \cdot p \nabla) u_n \phi
\]

Following (2.2) define \( f_n \in H^{-1}(\Omega) \) by

\[
f_n(\phi) = (u_n, \phi)_{1,p} = (w_n, \phi)_{1,p}
\]

(2.18)

for some unique \( w_n \in H^1_0(\Omega) \) and all \( \phi \in H^1_0(\Omega) \). Notice that \( \| u_n \|_{1,p} = \| f_n \|_{H^{-1}(\Omega)} \), for all \( n \). As \( F_n^* \) is the Hahn-Banach extension of \( f_n \), from the preceding discussion we have that \( F_n^* = w_n \), together with \( g_n^*(\gamma \phi) = (u_n - w_n, \phi)_{1,p} \) for all \( \phi \in H^1(\Omega) \). From (2.18) we have for all \( m, n \) that

\[
\| f_n - f_m \|_{H^{-1}(\Omega)} = \sup_{\phi \in H^1_0(\Omega), \| \phi \|_{H^1(\Omega)} = 1} | (f_n - f_m)(\phi) | \\
\leq \| (u_n - u_m, \phi)_{1,p} |_{1,p} \| \| \phi \|_{H^1(\Omega)} = 1 \\
= \| u_n - u_m \|_{1,p}.
\]
Consequently, as the sequence \( \{u_n\} \) converges strongly to \( u \) in \( \mathcal{H}^1(\Omega) \), the sequence \( \{f_n\} \) is Cauchy, and hence is convergent to an \( f \in \mathcal{H}^{-1}(\Omega) \). As \( \|w_n - w_m\|_{1,p} = \|f_n - f_m\|_{\mathcal{H}^{-1}(\Omega)} \) for all \( m, n \), we also have that the sequence \( \{w_n\} \) converges to a function \( w \in \mathcal{H}_0^1(\Omega) \). Finally we see that the sequence \( \{g_{n}^*\} \) converges to \( g^* \in \mathcal{H}^{-1/2}(\partial\Omega) \) where 

\[
  g^*(\gamma\phi) = (u - w, \phi)_{1,p} \quad \text{for all} \quad \phi \in \mathcal{H}^1(\Omega),
\]

and \( \{F_n^*\} = \{w_n\} \) converges to \( w = F^* \) where \( \|F^*\|_{\mathcal{H}^1(\Omega)^*} = \|w\|_{1,p} = \|f\|_{\mathcal{H}^{-1}(\Omega)} \).

Consequently, on letting \( n \to \infty \) in (2.17) one obtains

\[
  \int_{\Omega} u^2 + p|\nabla u|^2 = F^*(u) + g^*(\gamma u).
\]

From (2.1)

\[
  \|\gamma u\|_{\mathcal{H}^{1/2}(\partial\Omega)} \leq \|u\|_{\mathcal{H}^1(\Omega)},
\]

so that, if \( \kappa^* = \min(\kappa, 1) \),

\[
  \kappa^* \|u\|_{\mathcal{H}^1(\Omega)}^2 \leq \|g^*\|_{\mathcal{H}^{-1/2}(\partial\Omega)} \|\gamma u\|_{\mathcal{H}^{1/2}(\partial\Omega)} + \|F^*\|_{\mathcal{H}^1(\Omega)^*} \|u\|_{\mathcal{H}^1(\Omega)}
\]

\[
  \leq \|g^*\|_{\mathcal{H}^{-1/2}(\partial\Omega)} \|u\|_{\mathcal{H}^1(\Omega)} + \|F^*\|_{\mathcal{H}^1(\Omega)^*} \|u\|_{\mathcal{H}^1(\Omega)}.
\]

Thus

\[
  \kappa^* \|u\|_{\mathcal{H}^1(\Omega)} \leq \|g^*\|_{\mathcal{H}^{-1/2}(\partial\Omega)} + \|F^*\|_{\mathcal{H}^1(\Omega)^*} = \|((F^*, g^*))\|_B
\]

So the map in question, in addition to being onto, is closed and injective, and has a continuous inverse; its continuity follows by the closed graph theorem. \( \square \)

We conclude this section with a lemma that will find frequent use in the sequel.

**Lemma 2.2.** Let \( \mathcal{H} \) be a Hilbert space and let \( T : \mathcal{D} \subset \mathcal{H} \to \mathcal{H} \) be a densely defined positive linear selfadjoint operator in \( \mathcal{H} \). Assume also that \( T^{-1} : \mathcal{H} \to \mathcal{D} \) is compact. If \( \{g_n\} \) converges to zero weakly in \( \mathcal{H} \), and \( \{Tv_n\} \) is bounded in \( \mathcal{H} \), then \( \{(g_n, v_n)_{\mathcal{H}}\} \) converges to zero.

**Proof.** Let \( y_n = T^{-1}g_n \) for all \( n \). By [13, Theorem 6.3] and the compactness of \( T^{-1} \) and the weak convergence of \( \{g_n\} \), the sequence \( \{y_n\} \) converges strongly to zero in \( \mathcal{H} \). Then

\[
  (g_n, v_n)_{\mathcal{H}} = (Ty_n, v_n)_{\mathcal{H}} = (y_n, Tv_n)_{\mathcal{H}}.
\]

From the foregoing and the Cauchy-Schwarz inequality, this goes to zero as \( n \to \infty \). \( \square \)

**3. The One-coefficient Case.** Consider first that case in which \( p, f, \) and \( \lambda > 0 \) in (1.1) are given, and the coefficient \( q \) is to be recovered from the known solution \( u_{p,q,f,\lambda} \in \mathcal{H}^1(\Omega) \). The recovery method discussed earlier makes use of a gradient descent technique in which a sequence of iterates \( \{q_n\} \) is produced with the property that the corresponding sequence of solutions \( \{u_{p,q_n,f,\lambda}\} \) converges to \( u_{p,q,f,\lambda} \) strongly in \( \mathcal{H}^1(\Omega) \) (c.f. [10, Theorems 5.1 and 5.3]). In the next theorem we connect the convergence of the solutions \( \{u_{p,q_n,f,\lambda}\} \) with the convergence of their inversions \( \{g_n\} \).

**Theorem 3.1.** Assume that \( \lambda > 0 \), \( p \), and \( f \) are given, and for some constant \( M \geq q \geq 0 \),

\[
  0 \leq q_n \leq M, \quad \text{for all} \quad n,
\]

(3.1)
and

\[ p \geq \nu > \frac{\lambda M}{\lambda_0} > 0, \quad (3.2) \]

where \( \lambda_0 \) is the smallest positive eigenvalue of the negative Dirichlet Laplacian on \( \Omega \).

Assume also that \( u = u_{p,q,f,\lambda} \) is a given solution of (1.1) for which

\[ |u_{p,q,f,\lambda}| \geq c > 0 \text{ on } \Omega \quad (3.3) \]

for some constant \( c \). Let \( u_{p,q,f,\lambda} \) denote solutions of (1.1), with \( q \) replaced by \( q_n \), that have the same boundary data as \( u \). Then the sequence \( \{q_n\} \) converges weakly to \( q \) in \( \mathcal{L}^2(\Omega) \) if and only if the sequence \( \{u_{p,q,f,\lambda}\} \) converges to \( u_{p,q,f,\lambda} \) in \( \mathcal{H}^1(\Omega) \).

Proof. Let \( u_n = u_{p,q_n,f,\lambda} \), and assume first that \( u_n \to u \) in \( \mathcal{H}^1(\Omega) \). Subtracting the equations satisfied by \( u_{p,q,f,\lambda} \) and \( u_{p,q_n,f,\lambda} \) gives

\[ \lambda (q_n - q)u = -\nabla \cdot p \nabla (u - u_n) + \lambda q_n (u - u_n), \quad (3.4) \]

where, by (1.2), all the terms above are functions in \( \mathcal{L}^\infty(\Omega) \). Let \( \phi^* \in C^\infty_0(\Omega) \) be chosen arbitrarily. Noting that \( \psi = \phi^*/(\lambda u) \in \mathcal{H}^1_0(\Omega) \) and applying Green’s theorem, we have

\[ \int_\Omega (q_n - q) \phi^* = -\int_\Omega \nabla \cdot p \nabla (u - u_n) \psi + \lambda \int_\Omega q_n (u - u_n) \psi \]

which approaches zero as \( n \to \infty \). Let \( \phi \in \mathcal{L}^2(\Omega) \) and \( \epsilon > 0 \) be given, and choose \( \phi^* \in C^\infty_0(\Omega) \) so that

\[ \|\phi - \phi^*\|_{\mathcal{L}^2(\Omega)} < \frac{\epsilon}{4M.m(\Omega)^{1/2}}, \]

where \( m(\Omega) \) denotes the Lebesgue measure of the bounded open set \( \Omega \). Then choose \( N \) so that for \( n > N \),

\[ \left| \int_\Omega (q - q_n) \phi^* \right| < \frac{\epsilon}{2}. \]

Consequently, for \( n > N \)

\[ \left| \int_\Omega (q - q_n) \phi \right| \leq \left| \int_\Omega (q - q_n) \phi^* \right| + 2M. \int_\Omega |\phi - \phi^*| < \epsilon. \]

As \( \phi \) was chosen arbitrarily, we have the desired weak convergence in \( \mathcal{L}^2(\Omega) \).

Conversely, assume that \( q_n \to q \) weakly in \( \mathcal{L}^2(\Omega) \). After multiplying (3.4) by \( u - u_n \in \mathcal{H}^1_0(\Omega) \) and using Green’s Theorem, we have that

\[ \int_\Omega p|\nabla (u - u_n)|^2 + \lambda q_n (u - u_n)^2 = \lambda \int_\Omega u.(q_n - q). (u - u_n). \quad (3.5) \]

From the definition of \( \lambda_0 \) we know that \( \int_\Omega |\nabla (u - u_n)|^2 \geq \lambda_0 \int_\Omega (u - u_n)^2 \). Using this
inequality and the assumptions that $p \geq \nu > \frac{\lambda M}{\lambda_0}$, $\lambda > 0$, and $q_n \leq M$ we have

\[
\int_{\Omega} p|\nabla (u - u_n)|^2 + \lambda q_n (u - u_n)^2 \\
\geq (\nu - \frac{\lambda M}{\lambda_0}) \int_{\Omega} |\nabla (u - u_n)|^2 + \frac{\lambda M}{\lambda_0} \int_{\Omega} |\nabla (u - u_n)|^2 - \lambda \int_{\Omega} |u - u_n|^2 \\
\geq (\nu - \frac{\lambda M}{\lambda_0}) \int_{\Omega} |\nabla (u - u_n)|^2 \\
\geq \frac{1}{2} (\nu - \frac{\lambda M}{\lambda_0}) \int_{\Omega} |\nabla (u - u_n)|^2 + \frac{1}{2} (\nu - \frac{\lambda M}{\lambda_0}) \lambda \int_{\Omega} |u - u_n|^2 \\
\geq \frac{1}{2} (\nu - \frac{\lambda M}{\lambda_0}) \min(1, \lambda_0) ||u - u_n||^2_{H^1(\Omega)}. \quad (3.6)
\]

Now, from (3.5), (3.6), and $|q_n - q| \leq 2M$, for any $\epsilon > 0$ we have

\[
||u - u_n||^2_{H^1(\Omega)} \leq K_1 \int_{\Omega} u(q_n - q)(u - u_n) \\
\leq K_2(\epsilon ||u - u_n||^2_{L^2} + \frac{1}{\epsilon} ||u||^2_{L^2}) \quad (3.7)
\]

\[
\leq K_2 \epsilon ||u - u_n||^2_{H^1(\Omega)} + \frac{K_2}{\epsilon} ||u||^2_{L^2(\Omega)}, \quad (3.8)
\]

Choose $\epsilon$ so that $K_2 \epsilon = \frac{1}{2}$. It follows that

\[
||u - u_n||^2_{L^2(\Omega)} \leq ||u - u_n||^2_{H^1(\Omega)} \leq K_3,
\]

for all $n$. Hence,

\[
||u_n||^2_{L^2(\Omega)} \leq ||u||^2_{L^2(\Omega)} + ||u - u_n||^2_{L^2(\Omega)} \quad (3.10)
\]

\[
\leq ||u||^2_{L^2(\Omega)} + K_3, \quad (3.11)
\]

and the sequence $\{u_n\}$ is bounded in $L^2(\Omega)$.

Finally, the linear operator $L = -\nabla \cdot p \nabla + \lambda q$ defined on the dense domain

\[
D = \{u \in H^1_0(\Omega) : L u \in L^2(\Omega)\}
\]

in $L^2(\Omega)$ is selfadjoint and positive, and has a compact inverse operator with domain $L^2(\Omega)$, by [3, Theorem 1.4, p. 306]. From (3.4) we have that the sequence $\{g_n\}$, where

\[
g_n = -\nabla \cdot p \nabla (u - u_n) + \lambda q_n (u - u_n) = \lambda (q_n - q) u,
\]

converges weakly to zero in $L^2(\Omega)$, and from a calculation, $L(u - u_n) = \lambda (q_n - q) u_n$, so that $\{L(u - u_n)\}$ is bounded in $L^2(\Omega)$. Finally, from (3.6) we have

\[
||u - u_n||^2_{H^1(\Omega)} \leq K_4 \int_{\Omega} p|\nabla (u - u_n)|^2 + \lambda q_n |u - u_n|^2 \\
= K_4 (g_n, u - u_n)_{L^2(\Omega)},
\]

which approaches zero as $n \to \infty$ by Lemma 2.2. It follows that $u_n \rightharpoonup u$ in $H^1(\Omega)$, as promised. $\square$
Remarks. 1. The inverse problem may be written as \( q = F(u) \), where the solution \( u = u_{p,q,f,\lambda} \),

\[
A_{p,f,\lambda} = \{ v \in H^1(\Omega) : v \text{ satisfies (1.1) for some } q \in L^\infty(\Omega) \},
\]
and \( F : A_{p,f,\lambda} \to L^\infty(\Omega) \subset L^2(\Omega) \). While the set \( A_{p,f,\lambda} \) is not a linear space, it can be given a metric topology via the metric \( d \) defined by

\[
d(v, w) = \| v - w \|_{H^1(\Omega)}, \quad v, w \in A_{p,f,\lambda}.
\]

If the range of \( F \) is considered to be a subset of the space \( L^2(\Omega) \), then we can infer from the above theorem that the inverse problem of obtaining \( q \) from \( u \) is well-posed in the sense that the continuity required of part (3) of the definition of well-posedness is taken in the sense of the above metric topology on \( A_{p,f,\lambda} \) and the weak topology of \( L^2(\Omega) \) on the range of \( F \).

2. A typical recovery of a discontinuous coefficient \( q \) from a known solution \( u \) (using the method of [8]) is illustrated in the figure below. Notice that while the location and height of the discontinuity is well recovered here, the pointwise errors ("jaggies") are quite noticeable. This is in line with the weak convergence proven above. In fact if, for any set \( \Omega' \subset \Omega \) one chooses \( \phi \in L^2(\Omega) \) to be the characteristic function of that set, then we have that

\[
\int_{\Omega'} (q_n - q)(x) \, dx = \int_{\Omega} (q_n - q)(x)\phi(x) \, dx \to 0
\]
as \( n \to \infty \). This indicates that observed errors in the recovery, \((q - q_n)(x)\), are small on average over any subset \( \Omega' \subset \Omega \), rather than necessarily small in a pointwise sense at a given \( x \). Notice also that while the \( q_n \) approximate \( q \) only in a weak sense, the corresponding functions \( u_n \) are strongly convergent to \( u \) in \( H^1(\Omega) \). This shows that the modeling with the inversely recovered \( q \) is effective in the sense that the the data, \( u \), is faithfully reproduced when one solves equation (1.1) using the recovered \( q \). Similar comments apply to the remaining theorems in the paper.

3. The condition (3.3) is satisfied naturally in practice. Our solutions

\[
u_{p,q,f,\lambda}(x,\lambda) = \int_0^1 w(x,t)e^{\lambda t} \, dt, \quad x \in \Omega,
\]
are finite Laplace transforms of piezometric head values \( w(x,t) \). Here, one normally knows that \( w(x,t) \geq \beta > 0 \) uniformly in \( x \) and \( t \), and one can easily infer (3.3) from this.
We consider next the situation in which \( p, q, \) and \( \lambda > 0 \) and \( u_{p,q,f,\lambda} \) are given and the continuous recovery of \( f \) is desired.

**Theorem 3.2.** Assume that \( \lambda > 0, p \) and \( q \) satisfy (1.3), and that there exist constants \( M \) and \( K \) such that, \( 0 \leq q \leq M, \| f \|_{L^2(\Omega)} \leq K; \) \( || f_n ||_{L^2(\Omega)} \leq K, \) for all \( n, (3.12) \) and \( \nu > \lambda M \), \( \lambda \), \( \lambda_0 \) is the smallest positive eigenvalue of the negative Dirichlet Laplacian on \( \Omega. \) Assume also that \( \lambda > 0 \) and \( \nu > \lambda M \), \( \lambda \), \( \lambda_0 \) is the smallest positive eigenvalue of the negative Dirichlet Laplacian on \( \Omega. \) Assume also that \( u = u_{p,q,f,\lambda} \) is a given solution of (1.1). Let \( u_{p,q,f,n,\lambda} \) denote solutions of (1.1), with \( f \) replaced by \( f_n \), that have the same boundary data as \( u_{p,q,f,\lambda} \). Then the sequence \( \{ f_n \} \) converges weakly to \( f \) in \( L^2(\Omega) \) if and only if the sequence \( \{ u_{p,q,f,n,\lambda} \} \) converges to \( u_{p,q,f,\lambda} \) in \( H^1(\Omega) \).

**Proof.** Let \( u_n = u_{p,q,f,n,\lambda} \), and assume first that \( u_n \rightarrow u \) in \( H^1(\Omega) \). Subtracting the equations satisfied by \( u_{p,q,f,\lambda} \) and \( u_{p,q,f,n,\lambda} \) we have
\[
 f - f_n = -\nabla \cdot p \nabla (u - u_n) + \lambda q (u - u_n). \tag{3.14}
\]
For \( \psi \in C^\infty_0(\Omega) \) we have
\[
 \int_\Omega (f - f_n)\psi = - \int_\Omega \nabla \cdot p \nabla (u - u_n)\psi + \lambda \int_\Omega q (u - u_n)\psi \\
= \int_\Omega p \nabla (u - u_n) \cdot \nabla \psi + \lambda \int_\Omega q (u - u_n)\psi,
\]
which approaches zero as \( n \rightarrow \infty \). Thus, we have the desired weak convergence in \( L^2(\Omega) \) by a similar argument to that used in the previous theorem.

Conversely, assume that \( f_n \rightarrow f \) weakly in \( L^2(\Omega) \). From (3.14) we have that the sequence \( \{ g_n \} \), where
\[
 g_n = L (u - u_n) = -\nabla \cdot p \nabla (u - u_n) + \lambda q (u - u_n) = f - f_n,
\]
converges weakly to zero in \( L^2(\Omega) \). The sequence \( \{ L (u - u_n) \} \) is also bounded in \( L^2(\Omega) \). Thus, by Lemma 2.2 it follows as before that \( u_n \rightarrow u \) in \( H^1(\Omega) \).

If \( q, f, \) and \( \lambda > 0 \) and \( u_{p,q,f,\lambda} \) are given then we have the following theorem on the continuous recovery of \( p. \)

**Theorem 3.3.** Assume that \( \lambda > 0, q, \) and \( f \) are given, and for some constants \( K \) and \( M, |f| \leq K, 0 < \nu \leq p \leq K, 0 \leq q \leq M, \) and and
\[
 \nu > \frac{\lambda M}{\lambda_0}, \tag{3.15}
\]
where \( \lambda_0 \) is the smallest positive eigenvalue of the negative Dirichlet Laplacian on \( \Omega. \) Assume also that \( u = u_{p,q,f,\lambda} \) is a given solution of (1.1) with the property that for each \( \psi \in L^2(\Omega) \) there is a \( \phi \in H^1(\Omega) \) solving the first order equation
\[
 \nabla \phi \cdot \nabla u_{p,q,f,\lambda} = \psi. \tag{3.16}
\]
Let $u_{p_n,q,f,\lambda}$ denote solutions of (1.1), with $p$ replaced by $p_n$, that have the same boundary data as $u$, and assume that $0 < \nu \leq p_n \leq K$, for all $n$, and

$$(p - p_n)|_{\partial \Omega} = 0, \quad \text{for all } n.$$  \hfill (3.17)

If the sequence $\{u_{p_n,q,f,\lambda}\}$ converges to $u_{p,q,f,\lambda}$ in $H^1(\Omega)$ then the sequence $\{p_n\}$ converges weakly to $p$ in $L^2(\Omega)$.

Proof. Let $u_n = u_{p_n,q,f,\lambda}$, and assume that $u_n \to u$ in $H^1(\Omega)$. Subtracting the equations satisfied by $u_{p,q,f,\lambda}$ and $u_{p_n,q,f,\lambda}$ we have

$$\nabla \cdot (p - p_n)\nabla u = -\nabla \cdot p_n \nabla (u - u_n) + \lambda q(u - u_n).$$ \hfill (3.18)

For an arbitrary $\psi \in L^2(\Omega)$ let $\phi \in H^1(\Omega)$ satisfy (3.16). We have, via Green’s theorem, (3.17), and (3.18),

$$\int_{\Omega} (p - p_n)\psi = \int_{\Omega} (p - p_n)\nabla u \cdot \nabla \phi$$

$$\quad = -\int_{\Omega} \phi \nabla \cdot (p - p_n)\nabla u$$

$$\quad = \int_{\Omega} \phi \nabla \cdot p_n \nabla (u - u_n) - \int_{\Omega} \lambda q_n(u - u_n) \phi$$

$$\quad = \int_{\partial \Omega} \phi p_n \partial_n (u - u_n) - \int_{\Omega} p_n \nabla (u - u_n) \cdot \nabla \phi + \int_{\Omega} \lambda q_n(u - u_n) \phi$$

$$\quad = \int_{\partial \Omega} \phi p \partial_n (u - u_n) - \int_{\Omega} p_n \nabla (u - u_n) \cdot \nabla \phi + \int_{\Omega} \lambda q_n(u - u_n) \phi.$$

The second and third terms on the right above approach zero as $n \to \infty$ from the convergence to zero in $H^1(\Omega)$ of the sequence $\{u - u_n\}$ and the Cauchy-Schwarz inequality. As for the first term we have, by the trace inequalities (2.1) and (2.16),

$$|\int_{\partial \Omega} \phi p \partial_n (u - u_n)| \leq \|\phi\|_{H^{1/2}(\partial \Omega)} \|p \partial_n (u - u_n)\|_{H^{-1/2}(\partial \Omega)}$$

$$\quad \leq C \|\phi\|_{H^1(\Omega)} \|u - u_n\|_{H^1(\Omega)},$$

and this term also tends to zero. As $\psi$ was chosen arbitrarily, we have the desired weak convergence in $L^2(\Omega)$.

As a partial converse of the previous theorem, we also have

**Theorem 3.4.** Assume that $\lambda > 0$, $q$, and $f$ are given, and for some constants $K$ and $M$, $|f| \leq K$, $0 < \nu \leq p \leq K$, $0 \leq q \leq M$, and $\nu > \lambda M/\lambda_0$, where $\lambda_0$ is the smallest positive eigenvalue of the negative Dirichlet Laplacian on $\Omega$. Assume also that $u = u_{p,q,f,\lambda}$ is a given solution of (1.1), and let $u_{p_n,q,f,\lambda}$ denote solutions of (1.1), with $p$ replaced by $p_n$, that have the same boundary data as $u$, and assume that $0 < \nu \leq p_n \leq K$ and $|\nabla p_n| \leq K$ for all $n$, and that (3.17) holds. If

$$\int_{\Omega} (p - p_n)\nabla u_{p_n,q,f,\lambda} \cdot \nabla \phi \to 0$$ \hfill (3.19)

for all $\phi \in H^1(\Omega)$, then $\{u_{p_n,q,f,\lambda}\}$ converges to $u_{p,q,f,\lambda}$ in $H^1(\Omega)$.

Proof. Assume that (3.19) holds. From (3.18) we have that the sequence $\{g_n\}$, where

$$g_n = -\nabla \cdot p_n \nabla (u - u_n) + \lambda q(u - u_n) = \nabla \cdot (p - p_n)\nabla u,$$
converges weakly to zero in $L^2(\Omega)$. Now
\[
L(u - u_n) = -\nabla \cdot p \nabla (u - u_n) + \lambda q (u - u_n)
= p \Delta u_n + \nabla p \cdot \nabla u_n - \nabla \cdot p \nabla u + \lambda q (u - u_n),
\]
and from the differential equation satisfied by $u_n$
\[
\Delta u_n = (-\nabla p_n \cdot \nabla u_n + \lambda q u_n - f)/p_n.
\]
So, from the assumptions on $p_n$ and $|\nabla p_n|$ and the boundedness of $\{u_n\}$ in $H^1(\Omega)$ (proved as before), the sequence $\{L(u - u_n)\}$ is bounded in $L^2(\Omega)$. By Lemma 2.2 it follows that $(g_n, u - u_n)_{L^2(\Omega)} \to 0$ and thus that $u_n \to u$ in $H^1(\Omega)$ as required.

**Remarks.**

1. By the method of characteristics, the first order linear partial differential equation (3.16) will have solutions $\phi$ when the flow lines of the vector field $\nabla u_{p,q,f,\lambda}$ are such that any point in $\Omega$ exits to the boundary along a flow line. Given that the function $\nabla u_{p,q,f,\lambda}$ forms part of the data of the inverse problem, one can in principle at least check this approximately via a computer plot of the flow field of $\nabla u_{p,q,f,\lambda}$.

2. If we forego the condition (3.16), combining the remaining conditions of the last two theorems one may also prove, using the same method, that $u_n - u \to 0$ in $H^1_0(\Omega)$ if and only if for all $\phi \in H^1(\Omega)$
\[
\int_{\Omega} (p - p_n) \nabla u \cdot \nabla \phi \to 0.
\]

One may consider this last statement as a kind of weighted weak convergence of the sequence $\{p_n\}$.

3. One should note that uniqueness for the inverse problem is guaranteed if one specifies the coefficient $p$ on the boundary [7]. Because of this, the descent methods mentioned above all enforce the same boundary data on the iterates that converge to $p$ by using Neuberger gradients in the descent processes. The condition (3.17) in the theorem above is thus automatically true for these inverse methods.

4. **The Two-coefficient Case.**

4.1. **Recovering $q$ and $f$ with $p$ known.** Consider first the case in which $p$ is known and $q$ and $f$ are to be recovered from known solutions $u_1 = u_{p,q,f,\lambda_1}$ and $u_2 = u_{p,q,f,\lambda_2}$, where $\lambda_1 \neq \lambda_2$.

**Theorem 4.1.** With $p$ and $\lambda_2 > \lambda_1 > 0$ given, assume that for some constants $K$ and $M$, $0 \leq q \leq M$, $|f| \leq K$,
\[
|f_n| \leq K, \quad 0 \leq q_n \leq M, \quad \text{for all } n,
\]
and $\nu > \lambda_2 M/\lambda_0$, where $\lambda_0$ is the smallest positive eigenvalue of the negative Dirichlet Laplacian on $\Omega$. Assume also that $u_i = u_{p,q,f,\lambda_i}$, $i = 1, 2$, are given solutions of (1.1) for which
\[
|\lambda_1 u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2}| \geq c > 0,
\]
for some constant $c$. Let $u_{p,q_n,f_n,\lambda_1}$ denote solutions of (1.1), with $q$ replaced by $q_n$ and $f$ replaced by $f_n$, that have the same boundary data as $u_i$. Then the sequences
\(\{q_n\}\) and \(\{f_n\}\) converge weakly to \(q\) and \(f\), respectively, in \(L^2(\Omega)\) if and only if for \(i = 1, 2\) the sequences \(\{u_{p,q_n,f_n,\lambda_i}\}\) converge to \(u_{p,q,f,\lambda_i}\) in \(H^1(\Omega)\).

**Proof.** Assume that for \(i = 1, 2\) the sequences \(\{u_{p,q_n,f_n,\lambda_i}\}\) converge to \(u_{p,q,f,\lambda_i}\) in \(H^1(\Omega)\). We have

\[
-\nabla \cdot p \nabla u_{p,q,f,\lambda_1} + \lambda_1 q u_{p,q,f,\lambda_1} = f
\]

(4.3)

and

\[
-\nabla \cdot p \nabla u_{p,q,f,\lambda_2} + \lambda_2 q u_{p,q,f,\lambda_2} = f
\]

(4.4)

Subtracting these we get

\[
-\nabla \cdot p \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) + q(\lambda_1 u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2}) = 0
\]

(4.7)

\[
-\nabla \cdot p \nabla (u_{p,q_n,f_n,\lambda_1} - u_{p,q_n,f_n,\lambda_2}) + q_n(\lambda_1 u_{p,q_n,f_n,\lambda_1} - \lambda_2 u_{p,q_n,f_n,\lambda_2}) = 0. \tag{4.8}
\]

If we set \(v_{i,n} = u_{p,q,f,\lambda_i} - u_{p,q_n,f_n,\lambda_i}\), subtracting the last two equations then gives

\[
-\nabla \cdot p \nabla [v_{1,n} - v_{2,n}] + q_n[\lambda_1 v_{1,n} - \lambda_2 v_{2,n}]
= (q_n - q)(\lambda_1 u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2}) \tag{4.9}
\]

As \(v_{i,n} \to 0\) in \(H^1(\Omega)\) for \(i = 1, 2\) the proof that the sequence \(\{q_n\}\) converges weakly to \(q\) in \(L^2(\Omega)\) now follows by the corresponding argument in the one coefficient case, using (4.2). Subtracting (4.3) and (4.5) gives

\[
-\nabla \cdot p \nabla v_{1,n} + \lambda_1((q_n) u_{p,q,f,\lambda_1} + q_n v_{1,n}] = f - f_n. \tag{4.10}
\]

If we set

\[
g_n = -\nabla \cdot p \nabla v_{1,n} + \lambda_1 q_n v_{1,n}
\]

(4.12)

and we have also that \(\{f_n\}\) converges weakly to \(f\) in \(L^2(\Omega)\).

Conversely, assume that the sequences \(\{q_n\}\) and \(\{f_n\}\) converge weakly to \(q\) and \(f\), respectively, in \(L^2(\Omega)\). On rearranging (4.10) we find that

\[
-\nabla \cdot p \nabla v_{1,n} + \lambda_1 q_n v_{1,n} = \lambda_1(q_n - q) u_{p,q,f,\lambda_1} + (f - f_n). \tag{4.11}
\]

From (4.11) we have that

\[
L v_{1,n} = -\nabla \cdot p \nabla v_{1,n} + \lambda_1 q_n v_{1,n}
= \lambda_1(q_n - q) u_{p,q_n,f_n,\lambda_1} + (f - f_n)
\]

form a bounded set in \(L^2(\Omega)\). It follows from Lemma 2.2 that \(\{u_{p,q_n,f_n,\lambda_1}\}\) converges to \(u_{p,q,f,\lambda_1}\) in \(H^1(\Omega)\). The convergence of the sequence \(\{u_{p,q_n,f_n,\lambda_2}\}\) is handled similarly.

**Remark.** We may write

\[
\lambda_1 u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2} = \int_0^1 (\lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t}) w(x,t) dt,
\]
where the \( w(x,t) \) values are piezometric head measurements. If we note that the equation \( \lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t} = 0 \) has a unique root
\[
t_0 = \frac{\ln(\frac{\lambda_2}{\lambda_1})}{\lambda_1(\frac{\lambda_2}{\lambda_1} - 1)}
\]
then if we choose \( \lambda_1, \lambda_2 \) so that \( t_0 > 1 \) then the factor \( \lambda_1 e^{-\lambda_1 t} - \lambda_2 e^{-\lambda_2 t} \neq 0 \) for \( t \in [0,1] \). As the head measurements \( w(x,t) \) are usually uniformly bounded below by a positive number, one can see that the restriction (4.2) is quite natural here.

### 4.2. Recovering \( p \) and \( f \) with \( q \) known

Consider next the case in which \( q \) is known and \( p \) and \( f \) are to be recovered from known solutions \( u_1 = u_{p,q,f,\lambda_1} \) and \( u_2 = u_{p,q,f,\lambda_2} \), \( \lambda_1 \neq \lambda_2 \).

**Theorem 4.2.** With \( q \) and \( \lambda_2 > \lambda_1 > 0 \) given, assume that for some constants \( K \) and \( M \), \( 0 < \nu \leq p \leq K, |f| \leq K, 0 \leq q \leq M \) and
\[
0 < \nu \leq p_n \leq K, \quad |f_n| \leq K, \quad \text{for all } n, \quad (4.13)
\]
and \( \nu > \lambda_2 M/\lambda_0 \), where \( \lambda_0 \) is the smallest positive eigenvalue of the negative Laplace operator on \( \Omega \). Assume also that each \( p_n \) satisfies (1.3) and that for all \( n \)
\[
(p - p_n)|_{\partial \Omega} = 0, \quad (4.14)
\]
and \( u_i = u_{p,q,f,\lambda_i}, \ i = 1,2 \), are given solutions of (1.1) with \( |u_{p,q,f,\lambda_1}| \geq c_i > 0 \), for some constants \( c_i \), \( i = 1,2 \), and for any \( \psi \in L^2(\Omega) \), there is a solution \( \phi \in H^1(\Omega) \) of the first order equation
\[
\nabla \phi \cdot \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) = \psi. \quad (4.15)
\]

Let \( u_{p_n,q,f_n,\lambda_i} \) denote solutions of (1.1), with \( p \) replaced by \( p_n \) and \( f \) replaced by \( f_n \), that have the same boundary data as \( u_{p,q,f,\lambda_i} \). If, for \( i = 1,2 \) the sequences \( \{u_{p_n,q,f_n,\lambda_i}\} \) converge, respectively, to \( u_{p,q,f,\lambda_i} \) in \( H^1(\Omega) \), then the sequences \( \{p_n\} \) and \( \{f_n\} \) converge weakly to \( p \) and \( f \), respectively, in \( L^2(\Omega) \).

**Proof.** Assume that for \( i = 1,2 \) the sequences \( \{u_{p_n,q,f_n,\lambda_i}\} \) converge to \( u_{p,q,f,\lambda_i} \) in \( H^1(\Omega) \). We have
\[
\begin{align*}
-\nabla \cdot p \nabla u_{p,q,f,\lambda_1} + \lambda_1 q u_{p,q,f,\lambda_1} &= f \\
-\nabla \cdot p \nabla u_{p,q,f,\lambda_2} + \lambda_2 q u_{p,q,f,\lambda_2} &= f
\end{align*}
\]
and
\[
\begin{align*}
-\nabla \cdot p_n \nabla u_{p_n,q,f_n,\lambda_1} + \lambda_1 q u_{p_n,q,f_n,\lambda_1} &= f_n \\
-\nabla \cdot p_n \nabla u_{p_n,q,f_n,\lambda_2} + \lambda_2 q u_{p_n,q,f_n,\lambda_2} &= f_n
\end{align*}
\]
Subtracting these we get
\[
\begin{align*}
-\nabla \cdot p \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) + q(\lambda_1 u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2}) &= 0 \quad (4.20)
\end{align*}
\]
If we set \( v_{i,n} = u_{p,q,f,\lambda_i} - u_{p_n,q,f_n,\lambda_i}, \ i = 1,2 \), subtracting the last two equations then gives
\[
-\nabla \cdot (p_n - p) \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) = -\nabla \cdot p_n \nabla (v_{1,n} - v_{2,n}) + q[\lambda_1 v_{1,n} - \lambda_2 v_{2,n}] \quad (4.22)
\]
Consequently, multiplying by $\phi \in \mathcal{H}^1(\Omega)$ and applying Green’s theorem, using (4.14) and the trace estimate (2.16) on the boundary term on the right, gives

$$\int_{\Omega} (p_n - p) \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) \cdot \nabla \phi \to 0$$

for all $\phi \in \mathcal{H}^1(\Omega)$. It follows from (4.15) that $\{ p_n \}$ converges weakly in $\mathcal{L}^2(\Omega)$. If we now subtract (4.16) and (4.18) we see that

$$f - f_n = -\nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda_1} - \nabla \cdot p_n \nabla v_{1,n} + \lambda_1 q v_{1,n}.$$  

It follows that the sequence $\{ f_n \}$ converges weakly to $f$ in $\mathcal{L}^2(\Omega)$. □

We also have the following partial converse.

**Theorem 4.3.** With $q$ and $\lambda_2 > \lambda_1 > 0$ given, assume that for some constants $K$ and $M$, $0 < \nu \leq p \leq K, |f| \leq K, 0 \leq q \leq M$ and

$$0 < \nu \leq p_n \leq K, |\nabla p_n| \leq K, |f_n| \leq K, \text{ for all } n,$$

and $\nu > \lambda_2 M/\lambda_0$, where $\lambda_0$ is the smallest positive eigenvalue of the negative Laplace operator on $\Omega$. Assume also that each $p_n$ satisfies (1.3) and that for all $n$ (4.14) holds and $u_i = u_{p,q,f,\lambda_i}$, $i = 1, 2$, are given solutions of (1.1) with $|u_{p,q,f,\lambda_i}| \geq c_i > 0$, for some constants $c_i$, $i = 1, 2$. Let $u_{p_n,q,f,n,\lambda_i}$ denote solutions of (1.1), with $p$ replaced by $p_n$ and $f$ replaced by $f_n$, that have the same boundary data as $u_{p,q,f,\lambda_i}$. If the sequence $\{ f_n \}$ converges weakly to $f$ in $\mathcal{L}^2(\Omega)$ and for $i = 1, 2$

$$\int_{\Omega} (p - p_n) \nabla u_{p,q,f,\lambda_i} \cdot \nabla \phi \to 0$$

for all $\phi \in \mathcal{H}^1(\Omega)$ then for $i = 1, 2$ the sequences $\{ u_{p_n,q,f,n,\lambda_i} \}$ converge, respectively, to $u_{p,q,f,\lambda_i}$ in $\mathcal{H}^1(\Omega)$.

**Proof.** Assume that the sequence $\{ f_n \}$ converges weakly to $f$ in $\mathcal{L}^2(\Omega)$ and for $i = 1, 2$ and every $\phi \in \mathcal{H}^1(\Omega)$

$$\int_{\Omega} (p - p_n) \nabla u_{p,q,f,\lambda_i} \cdot \nabla \phi \to 0.$$  

A further subtraction gives

$$g_n = -\nabla \cdot p_n \nabla v_{1,n} + \lambda_1 q v_{1,n} = f - f_n + \nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda_i},$$

so that the sequence $\{ g_n \}$ converges weakly to zero in $\mathcal{L}^2(\Omega)$. Also,

$$Lv_{1,n} = -\nabla \cdot p \nabla v_{1,n} + \lambda_1 q v_{1,n}
= f - \lambda_1 q u_{p,q,f,\lambda_1} + \nabla \cdot p \nabla u_{p_n,q,f,n,\lambda_1} + \lambda_1 q v_{1,n}
= f - \lambda_1 q u_{p,q,f,\lambda_1} + \nabla \cdot p \nabla u_{p_n,q,f,n,\lambda_1} + p \Delta u_{p_n,q,f,n,\lambda_1} + \lambda_1 q v_{1,n},$$

and from the equation for $u_{p_n,q,f,n,\lambda_1}$ we have

$$\Delta u_{p_n,q,f,n,\lambda_1} = (-\nabla p_n \cdot \nabla u_{p_n,q,f,n,\lambda_1} + \lambda_1 q u_{p_n,q,f,n,\lambda_1} - f_n)/p_n.$$  

So, from the boundedness of $\{ u_{p_n,q,f,n,\lambda_1} \}$ in $\mathcal{H}^1(\Omega)$ (proven as before) and the assumptions on $p_n$ we see that the sequence $\{ Lv_{1,n} \}$ is bounded in $\mathcal{L}^2(\Omega)$. It follows from Lemma 2.2 that $\{ u_{p_n,q,f,n,\lambda_1} \}$ converges to $u_{p,q,f,\lambda_1}$ in $\mathcal{H}^1(\Omega)$; the convergence of $\{ u_{p_n,q,f,n,\lambda_2} \}$ to $u_{p,q,f,\lambda_2}$ is similar. □
4.3. Recovering $p$ and $q$ with $f$ known. Consider next the case in which $f$ is known and $p$ and $q$ are to be recovered from known solutions $u_1 = u_{p,q,f,\lambda_1}$ and $u_2 = u_{p,q,f,\lambda_2}$, $\lambda_1 \neq \lambda_2$.

**THEOREM 4.4.** With $f$ and $\lambda_2 > \lambda_1 > 0$ given, assume that for some constants $K$ and $M$, $0 < \nu \leq p \leq K$, $0 \leq q \leq M$,

$$0 < \nu \leq p_n \leq K, \quad 0 \leq q_n \leq M, \quad \text{for all } n,$$

and $\nu > \lambda_2 M/\lambda_0$, where $\lambda_0$ is the smallest positive eigenvalue of the negative Dirichlet Laplacian on $\Omega$. Assume also that each $p_n$ satisfies (1.3) and that for all $n$

$$(p - p_n)|_{\partial \Omega} = 0,$$ (4.27)

and $u_i = u_{p,q,f,\lambda_i}$, $i = 1, 2$, are given solutions of (1.1) with $|u_{p,q,f,\lambda_i}| \geq c_i > 0$ and with the property that for any $\psi \in L^2(\Omega)$ the first order equation

$$(\lambda_2 u_{p,q,f,\lambda_2} \nabla u_{p,q,f,\lambda_2} - \lambda_1 u_{p,q,f,\lambda_1} \nabla u_{p,q,f,\lambda_1}) \cdot \nabla \phi + (\lambda_1 - \lambda_1) \phi(\lambda_2 u_{p,q,f,\lambda_2} \cdot \nabla u_{p,q,f,\lambda_1}) = \psi$$ (4.28)

has a solution $\phi \in H^1(\Omega)$. Let $u_{p_n,q_n,f,\lambda_i}$ denote solutions of (1.1), with $p$ replaced by $p_n$ and $q$ replaced by $q_n$, that have the same boundary data as $u_{p,q,f,\lambda_i}$. If for $i = 1, 2$ the sequences $\{u_{p_n,q_n,f,\lambda_1}\}$ converge to $u_{p,q,f,\lambda_1}$ in $H^1(\Omega)$ then the sequences $\{p_n\}$ and $\{q_n\}$ converge weakly to $p$ and $q$, respectively, in $L^2(\Omega)$.

**Proof.** Assume that for $i = 1, 2$ the sequences $\{u_{p_n,q_n,f,\lambda_i}\}$ converge to $u_{p,q,f,\lambda_i}$ in $H^1(\Omega)$. We have

$$-\nabla \cdot p \nabla u_{p,q,f,\lambda_1} + \lambda_1 q u_{p,q,f,\lambda_1} = f$$ (4.29)

$$-\nabla \cdot p \nabla u_{p,q,f,\lambda_2} + \lambda_2 q u_{p,q,f,\lambda_2} = f$$ (4.30)

and

$$-\nabla \cdot p_n \nabla u_{p_n,q_n,f,\lambda_1} + \lambda_1 q_n u_{p_n,q_n,f,\lambda_1} = f$$ (4.31)

$$-\nabla \cdot p_n \nabla u_{p_n,q_n,f,\lambda_2} + \lambda_2 q_n u_{p_n,q_n,f,\lambda_2} = f.$$ (4.32)

Combining these we get

$$-\lambda_2 u_{p,q,f,\lambda_2} \nabla \cdot p \nabla u_{p,q,f,\lambda_1} + \lambda_1 u_{p,q,f,\lambda_1} \nabla \cdot p \nabla u_{p,q,f,\lambda_2} = (\lambda_2 u_{p,q,f,\lambda_2} - \lambda_1 u_{p,q,f,\lambda_1}) f$$ (4.33)

$$-\lambda_2 u_{p_n,q_n,f,\lambda_2} \nabla \cdot p_n \nabla u_{p_n,q_n,f,\lambda_1} + \lambda_1 u_{p_n,q_n,f,\lambda_1} \nabla \cdot p_n \nabla u_{p_n,q_n,f,\lambda_2} = (\lambda_2 u_{p_n,q_n,f,\lambda_2} - \lambda_1 u_{p_n,q_n,f,\lambda_1}) f.$$ (4.34)

If we set $v_{i,n} = u_{p,q,f,\lambda_i} - u_{p_n,q_n,f,\lambda_i}$, $i = 1, 2$, combining the last two equations then gives

$$-\lambda_2 u_{p,q,f,\lambda_2} \nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda_1} + \lambda_1 u_{p,q,f,\lambda_1} \nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda_2}$$

$$= (\lambda_2 u_{p,q,f,\lambda_2} - \lambda_1 u_{p,q,f,\lambda_1}) f$$ (4.35)
Here
\[
\lambda_2 u_{p,q,f,\lambda_2} \nabla \cdot p_n \nabla u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2} \nabla \cdot p_n \nabla u_{p,q,f,\lambda_1} = \lambda_2 (v_{2,n} \nabla \cdot p_n \nabla u_{p,q,f,\lambda_1} + u_{p,q,f,\lambda_2} \nabla \cdot p_n \nabla v_{1,n}), \tag{4.36}
\]
and
\[
\lambda_1 u_{p,q,f,\lambda_1} \nabla \cdot p_n \nabla u_{p,q,f,\lambda_2} - \lambda_1 u_{p,q,f,\lambda_1} \nabla \cdot p_n \nabla u_{p,q,f,\lambda_2} = \lambda_1 (v_{1,n} \nabla \cdot p_n \nabla u_{p,q,f,\lambda_2} + u_{p,q,f,\lambda_1} \nabla \cdot p_n \nabla v_{2,n}), \tag{4.37}
\]
and
\[
[(\lambda_2 u_{p,q,f,\lambda_2} - \lambda_1 u_{p,q,f,\lambda_1}) - (\lambda_2 u_{p,q,f,\lambda_2} - \lambda_1 u_{p,q,f,\lambda_1})] f
= (\lambda_2 v_{2,n} - \lambda_1 v_{1,n}) f. \tag{4.38}
\]

So, it now follows that
\[
\int_{\Omega} (p - p_n) [\nabla u_{p,q,f,\lambda_1} \cdot \nabla (\lambda_2 u_{p,q,f,\lambda_2} \phi) - \nabla u_{p,q,f,\lambda_2} \cdot \nabla (\lambda_1 u_{p,q,f,\lambda_1} \phi)] \to 0
\]
and hence that
\[
\int_{\Omega} (p - p_n) [(\lambda_2 u_{p,q,f,\lambda_2} \nabla u_{p,q,f,\lambda_1} - \lambda_1 u_{p,q,f,\lambda_1} \nabla u_{p,q,f,\lambda_2}) \cdot \nabla \phi
+ (\lambda_2 - \lambda_1) \phi \nabla u_{p,q,f,\lambda_2} \cdot \nabla u_{p,q,f,\lambda_1}] \to 0,
\]
for all \( \phi \) in \( H^1(\Omega) \). It follows from (4.28) that the sequence \( \{p_n\} \) converges weakly to \( p \) in \( L^2(\Omega) \).

Finally, subtracting (4.29) and (4.31) we have
\[
\lambda_1 (q_n - q) u_{p,q,f,\lambda_1} = \nabla \cdot p_n \nabla v_{1,n} - \lambda_1 q_n v_{1,n} + \nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda_1}, \tag{4.39}
\]
The weak convergence of \( \{q_n\} \) to \( q \) now follows from the weak convergence of \( \{p_n\} \) proved above by following the method used at the same stage in the proof of Theorem 3.1.

We also have the following partial converse.

**Theorem 4.5.** With \( f \) and \( \lambda_2 > \lambda_1 > 0 \) given, assume that and for some constants \( K \) and \( M \), \( 0 < \nu \leq p \leq K, 0 \leq q \leq M \),
\[
0 < \nu \leq p_n \leq K, \quad |\nabla p_n| \leq K, \quad 0 \leq q_n \leq M, \quad \text{for all } n, \tag{4.40}
\]
and \( \nu > \lambda_2 M/\lambda_0 \), where \( \lambda_0 \) is the smallest positive eigenvalue of the negative Dirichlet Laplacian on \( \Omega \). Assume also that each \( p_n \) satisfies (1.3) and that for all \( n \) (4.27) holds and \( u_i = u_{p,q,f,\lambda_i}, i = 1, 2 \), are given solutions of (1.1) with \( |u_{p,q,f,\lambda_i}| \geq c_i > 0 \). Let \( u_{p,q,f,\lambda_i} \) denote solutions of (1.1), with \( p \) replaced by \( p_n \) and \( q \) replaced by \( q_n \), that have the same boundary data as \( u_{p,q,f,\lambda_i} \). If \( \{q_n\} \) converges weakly to \( q \) in \( L^2(\Omega) \) and for \( i = 1, 2 \)
\[
\int_{\Omega} (p - p_n) \nabla u_{p,q,f,\lambda_i} \cdot \nabla \phi \to 0 \tag{4.41}
\]
for all \( \phi \in H^1(\Omega) \), then for \( i = 1, 2 \) the sequences \( \{u_{p,q,f,\lambda_i}\} \) converge to \( u_{p,q,f,\lambda_i} \) in \( H^1(\Omega) \).
\[ g_n = -\nabla \cdot p_n \nabla v_{1,n} + \lambda_1 q_n v_{1,n} = \lambda_1 (q - q_n) u_{p,q,f,\lambda_1} + \nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda_1}, \quad (4.42) \]

so that the sequence \( \{g_n\} \) converges weakly to zero in \( L^2(\Omega) \). Also,

\[ L v_{1,n} = -\nabla \cdot p \nabla v_{1,n} + \lambda_1 q v_{1,n} \]
\[ = f - \lambda_1 q u_{p,q,f,\lambda_1} + \nabla \cdot p \nabla u_{p,q,n,f,\lambda_1} + \lambda_1 q v_{1,n} \]
\[ = f - \lambda_1 q u_{p,q,f,\lambda_1} + \nabla p \cdot \nabla u_{p,q,n,f,\lambda_1} + p \Delta u_{p,q,n,f,\lambda_1} + \lambda_1 q v_{1,n}, \]

where, from the differential equation for \( u_{p,q,n,f,\lambda_1} \),

\[ \Delta u_{p,q,n,f,\lambda_1} = (\nabla \cdot \nabla u_{p,q,n,f,\lambda_1} + \lambda_1 q_n u_{p,q,n,f,\lambda_1} - f) / p_n. \]

From the boundedness of \( \{u_{p,q,n,f,\lambda_1}\} \) in \( H^1(\Omega) \) (proven as before) and the assumptions on \( p_n \) and \( q_n \), the sequence \( \{v_{1,n}\} \) is bounded in \( L^2(\Omega) \). So, by Lemma 2.2 again, \( \{u_{p,q,n,f,\lambda_1}\} \) converges to \( u_{p,q,f,\lambda_1} \) in \( H^1(\Omega) \); the convergence of \( \{u_{p,q,n,f,\lambda_2}\} \) to \( u_{p,q,f,\lambda_2} \) is similar. \( \square \)

5. The Three-coefficient Case. In practice, the most relevant case is the one in which \( p, q, \) and \( f \) are recovered simultaneously.

**Theorem 5.1.** Assume that for some positive numbers \( \lambda_i, \ 1 \leq i \leq 3 \), with \( \lambda_3 > \lambda_2 > \lambda_1 \), solutions \( u_i = u_{p,q,f,\lambda_i} \) of the equation (1.1) are given with the property that for any \( \psi \in L^2(\Omega) \) the first order equation

\[ \nabla \phi \cdot [w_{13} \cdot \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) - w_{12} \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_3})] \]
\[ + [\nabla w_{13} \cdot \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) - \nabla w_{12} \cdot \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_3})], \phi = \psi, \quad (5.1) \]

where for \( i = 2, 3, \ w_{1i} = \lambda_i u_{p,q,f,\lambda_1} - \lambda_i u_{p,q,f,\lambda_i} \), has a solution \( \phi \in H^1(\Omega) \). Assume that \( |w_{1i}| \geq c > 0 \) for some constant \( c \) and \( i = 2, 3 \), and that for some constants \( K \) and \( M \), \( 0 < \nu \leq p \leq K, |f| \leq K, 0 \leq q \leq M \) and

\[ 0 < \nu \leq p_n \leq K, \quad |f_n| \leq K, \quad 0 \leq q_n \leq M \quad \text{for all} \ n, \quad (5.2) \]

and \( \nu > \lambda_3 M / \lambda_0 \), where \( \lambda_0 \) is the smallest positive eigenvalue of the negative Dirichlet Laplacian on \( \Omega \). Assume also that each \( p_n \) satisfies (1.3) and that for all \( n \)

\[ (p - p_n)|_{\partial \Omega} = 0. \]

Let \( u_{p_n,q_n,f_n,\lambda_i} \) denote solutions of (1.1), with \( p \) replaced by \( p_n \), \( q \) replaced by \( q_n \), and \( f \) replaced by \( f_n \) that have the same boundary data as \( u_{p,q,f,\lambda_i} \). If, for \( 1 \leq i \leq 3 \) the sequences \( \{u_{p_n,q_n,f_n,\lambda_i}\} \) converge to \( u_{p,q,f,\lambda_i} \) in \( H^1(\Omega) \) then the sequences \( \{p_n\}, \{q_n\}, \) and \( \{f_n\} \) converge weakly to \( p, q, \) and \( f \), respectively, in \( L^2(\Omega) \).

**Proof.** Assume that for \( 1 \leq i \leq 3 \) the sequences \( \{u_{p_n,q_n,f,\lambda_i}\} \) converge to \( u_{p,q,f,\lambda_i} \) in \( H^1(\Omega) \). We have

\[ -\nabla \cdot p \nabla u_{p,q,f,\lambda_1} + \lambda_1 q u_{p,q,f,\lambda_1} = f \quad (5.3) \]
\[ -\nabla \cdot p \nabla u_{p,q,f,\lambda_2} + \lambda_2 q u_{p,q,f,\lambda_2} = f \quad (5.4) \]
\[ -\nabla \cdot p \nabla u_{p,q,f,\lambda_3} + \lambda_3 q u_{p,q,f,\lambda_3} = f \quad (5.5) \]
It now follows that
\[ -\nabla \cdot p_n \nabla u_{p_n,q_n,f_n,\lambda_1} + \lambda_1 g_n u_{p_n,q_n,f_n,\lambda_1} = f_n \] (5.6)
\[ -\nabla \cdot p_n \nabla u_{p_n,q_n,f_n,\lambda_2} + \lambda_2 g_n u_{p_n,q_n,f_n,\lambda_2} = f_n \] (5.7)
\[ -\nabla \cdot p_n \nabla u_{p_n,q_n,f_n,\lambda_3} + \lambda_3 g_n u_{p_n,q_n,f_n,\lambda_3} = f_n. \] (5.8)
Subtracting gives
\[ -\nabla \cdot p \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) + q(\lambda_1 u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2}) = 0 \] (5.9)
\[ -\nabla \cdot p \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_3}) + q(\lambda_1 u_{p,q,f,\lambda_1} - \lambda_3 u_{p,q,f,\lambda_3}) = 0 \] (5.10)
and
\[ -\nabla \cdot p_n \nabla (u_{p_n,q_n,f_n,\lambda_1} - u_{p_n,q_n,f_n,\lambda_2}) + q_n(\lambda_1 u_{p_n,q_n,f_n,\lambda_1} - \lambda_2 u_{p_n,q_n,f_n,\lambda_2}) = 0 \] (5.11)
\[ -\nabla \cdot p_n \nabla (u_{p_n,q_n,f_n,\lambda_1} - u_{p_n,q_n,f_n,\lambda_3}) + q_n(\lambda_1 u_{p_n,q_n,f_n,\lambda_1} - \lambda_3 u_{p_n,q_n,f_n,\lambda_3}) = 0. \] (5.12)
If we set \( v_{i,n} = u_{p,q,f,\lambda_i} - u_{p_n,q_n,f_n,\lambda_i} \), \( 1 \leq i \leq 3 \), subtracting again then gives
\[ -\nabla \cdot (p - p_n) \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) - \nabla \cdot p_n \nabla (v_{1,n} - v_{2,n}) \]
\[ = (q_n - q)(\lambda_1 u_{p,q,f,\lambda_1} - \lambda_2 u_{p,q,f,\lambda_2}) + q_n[\lambda_2 v_{2,n} - \lambda_1 v_{1,n}] \] (5.13)
and
\[ -\nabla \cdot (p - p_n) \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_3}) - \nabla \cdot p_n \nabla (v_{1,n} - v_{3,n}) \]
\[ = (q_n - q)(\lambda_1 u_{p,q,f,\lambda_1} - \lambda_3 u_{p,q,f,\lambda_3}) + q_n[\lambda_3 v_{3,n} - \lambda_1 v_{1,n}] \]. (5.14)
Finally, eliminating the terms in \( q - q_n \) in the last two equations gives
\[ -w_{13} \nabla \cdot (p - p_n) \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) + w_{12} \nabla \cdot (p - p_n) \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_3}) \]
\[ = [w_{13} \nabla \cdot p_n \nabla (v_{1,n} - v_{2,n}) - w_{12} \nabla \cdot p_n \nabla (v_{1,n} - v_{1,3})] + q_n[w_{12}(\lambda_1 v_{1,n} - \lambda_3 v_{3,n}) - w_{13}(\lambda_1 v_{1,n} - \lambda_2 v_{2,n})]. \]
It now follows that
\[ \int_{\Omega} (p - p_n) \{ \nabla \phi \cdot [w_{13} \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) - w_{12} \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_3})]\} \]
\[ + \nabla w_{13} \cdot \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_2}) - \nabla w_{12} \cdot \nabla (u_{p,q,f,\lambda_1} - u_{p,q,f,\lambda_3})] \phi \to 0 \] (5.15)
for all \( \phi \in \mathcal{H}^1(\Omega) \). From (5.1) we have that the sequence \( \{p_n\} \) converges weakly to \( p \) in \( L^2(\Omega) \). The weak convergence of the sequence \( \{q_n\} \) to \( q \) is then a consequence of (5.13). Finally, subtracting (5.3) and (5.6) gives
\[ f - f_n = -\nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda_1} + \lambda_1 (q - q_n) u_{p,q,f,\lambda_1} - \nabla \cdot (p_n) \nabla v_{1,n} + \lambda_1 q_n v_{1,n}, \] (5.16)
from which follows the weak convergence of the sequence \( \{f_n\} \) to \( f \).

The following partial converse also holds.

**Theorem 5.2.** Assume that for some positive numbers \( \lambda_i, 1 \leq i \leq 3 \), with \( \lambda_3 > \lambda_2 > \lambda_1 \), solutions \( u_i = u_{p,q,f,\lambda_i} \) of the equation (1.1) are given. Assume that
\[ |w_{1,i}| \geq c > 0 \text{ for some constant } c \text{ and } i = 2, 3, \text{ and that for some constants } K \text{ and } M, \ 0 < \nu \leq p \leq K, |f| \leq K, \ 0 \leq q \leq M \text{ and } \]
\[ 0 < \nu \leq p_n \leq K, \ |\nabla p_n| \leq K, \ |f_n| \leq K, \ 0 \leq q_n \leq M, \text{ for all } n, \quad (5.17) \]
and \( \nu > \lambda M/\lambda_0 \), where \( \lambda_0 \) is the smallest positive eigenvalue of the negative Dirichlet Laplacian on \( \Omega \). Assume also that each \( p_n \) satisfies (1.3) and that for all \( n \)
\[ (p - p_n)|_{\partial \Omega} = 0. \]
Let \( u_{p_n,q_n,f_n,\lambda} \) denote solutions of (1.1), with \( p \) replaced by \( p_n \), \( q \) replaced by \( q_n \), and \( f \) replaced by \( f_n \), that have the same boundary data as \( u_{p,q,f,\lambda} \). If \( \{q_n\} \) converges weakly to \( q \) and \( \{f_n\} \) converges weakly to \( f \) in \( L^2(\Omega) \), and for \( i = 1, 2, 3 \)
\[ \int_{\Omega} (p - p_n) \nabla u_{p,q,f,\lambda_i} \cdot \nabla \phi \to 0 \quad (5.18) \]
for all \( \phi \in H^1(\Omega) \), then for \( i = 1, 2, 3 \) the sequence \( \{u_{p_n,q_n,f_n,\lambda_i}\} \) converges to \( u_{p,q,f,\lambda_i} \) in \( H^1(\Omega) \).

Proof. Set
\[ g_n = -\nabla \cdot p_n \nabla v_{1,n} + \lambda_1 q_n v_{1,n} = \lambda_1 (q_n - q) u_{p,q,f,\lambda} + \nabla \cdot (p - p_n) \nabla u_{p,q,f,\lambda} + f - f_n, \quad (5.19) \]
so that the sequence \( \{g_n\} \) converges weakly to zero in \( L^2(\Omega) \). Also,
\[ L v_{1,n} = -\nabla \cdot p \nabla v_{1,n} + \lambda_1 q v_{1,n} \]
\[ = f - \lambda_1 q u_{p,a,q_n,f_n,\lambda} + \nabla \cdot p \nabla u_{p,q_n,f_n,\lambda} + \lambda_1 q v_{1,n} \]
\[ = p \Delta u_{p_n,q_n,f_n,\lambda} + \nabla p \cdot \nabla u_{p_n,q_n,f_n,\lambda} - \lambda_1 q u_{p_n,q_n,f_n,\lambda} + \lambda_1 q v_{1,n} + f \]
and the sequence \( \{L v_{1,n}\} \) is bounded in \( L^2(\Omega) \). So by Lemma 2.2 \( \{u_{p_n,q_n,f_n,\lambda}\} \) converges to \( u_{p,q,f,\lambda} \) in \( H^1(\Omega) \); the convergence of \( \{u_{p_n,q_n,f_n,\lambda}\} \) to \( u_{p,q,f,\lambda}, \) \( i = 2, 3, \) is again similar. \( \square \)

Acknowledgment. The authors are grateful to the referees for their careful reading of the original manuscript and catching several errors in the process, and for other helpful suggestions on improving the readability of the paper.

REFERENCES


