UNIQUENESS FOR AN ELLIPTIC INVERSE PROBLEM

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Abstract. For the elliptic equation

\[ -\nabla \cdot (p(x)\nabla v) + \lambda q(x)v = f, \quad x \in \Omega \subset \mathbb{R}^n, \]

the problem of determining when one or more of the coefficient functions \( p, q, \) and \( f \) are defined uniquely by a knowledge of one or more of the solution functions \( v = v_{p,q,f,\lambda} \) is considered.

Key words. uniqueness, elliptic inverse problem, parameter estimation, aquifer

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1. Introduction. Consider the differential equation

\[ Lv = -\nabla \cdot (p(x)\nabla v) + \lambda q(x)v = f(x), \quad x \in \Omega, \]

where \( \Omega \) is a \( C^2 \) domain in \( \mathbb{R}^n; \) \( \lambda > 0; \)

\[ f \in C^0(\Omega), \quad q \in C^0(\Omega), \quad \text{and} \quad p \in C^2(\Omega) \]

are real; \( q \geq 0; \) \( f \geq 0; \) and \( p \) satisfies

\[ p(x) \geq \nu > 0, \quad x \in \Omega, \]

for some constant \( \nu. \) It is known [8, Theorem 6.14] that, under these conditions, for any \( \phi \in C^2(\bar{\Omega}) \) the Dirichlet problem formed from (1.1) and

\[ v = \phi \text{ on } \partial \Omega \]

has a unique solution \( v_{p,q,f,\lambda} \in C^2(\bar{\Omega}), \) and that the operator \( L = L_{p,q,\lambda} \) defined by (1.1) in \( L^2(\Omega) \) is positive. We are concerned here with the uniqueness question for the corresponding inverse problems: When are (one or more of) the coefficients \( p, q, \) \( f \) uniquely determined by a knowledge of \( v_{p,q,f,\lambda} \) for one or more values of \( \lambda? \)

Interest in these inverse problems arises from the study of underground aquifer systems (as well as oil reservoir simulations) which are often modeled by the diffusion equation

\[ S(x) \frac{\partial w}{\partial t} = \nabla \cdot [p(x)\nabla w(x,t)] + R(x,t). \]

Here \( w \) represents the piezometric head, \( p \) the hydraulic conductivity (or sometimes, for a two-dimensional aquifer, the transmissivity), \( R \geq 0 \) the recharge, and the function \( S \geq 0 \) the storativity of the aquifer (see, for example, [2, 3]). It is well known among hydrogeologists [2, Chapter 8] that the inability to obtain reliable values for the coefficients in (1.5) is a serious impediment to the confident use of such models. Methods that have been employed range from educated guesswork (referred to

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as “trial and error calibration” in the hydrology literature—the method preferred by most practitioners at this time [2, p. 226]) to various attempts at “automatic calibration” (see [1, 4, 5, 6, 7, 13, 14, 17, 22] as part of the extensive literature on these inverse problems).

In the steady-state case we have that \( \frac{\partial w}{\partial t} = 0 \) and \( R = R(x) \); if \( R \) is presumed known, the inverse problem reduces to finding \( p \) from a knowledge of \( w \). In [10] a new approach to this reduced problem was given in terms of finding a unique minimum for a convex functional. This approach was shown to be effective in the presence of mild discontinuities in the coefficients, an important practical consideration in view of the fact that fractures in the porous media are commonly encountered. In [11] these ideas were extended to the general equation (1.1), and we digress briefly to summarize these results. Let a solution \( u \) of (1.1) be given for which \( P, Q, \) and \( F \) are the coefficients corresponding to \( p, q, \) and \( f \), respectively, that we seek to compute. For functions \( p, q, f \) satisfying (1.2) and (1.3), let \( v = u_{p,q,f,\lambda} \) denote the solution of the boundary value problem determined by (1.1) and (1.4) with \( \phi = u|_{\partial \Omega} \). Thus \( u = u_{P,Q,F,\lambda} \). Define

\[
D_G = \{(p, q, f, \lambda) : p, q, f \text{ satisfy } (1.2), (1.3), \lambda > 0, \text{ and } p|_{\Gamma} = P|_{\Gamma}\},
\]

where \( \Gamma \) is a hypersurface in \( \Omega \) transversal to \( \nabla u \). It generally is appropriate to take \( \Gamma \) to be the boundary of the bounded region \( \Omega \), and we assume this to be so. For \((p, q, f, \lambda) \) in \( D_G \) define

\[
G(p, q, f, \lambda) = \int_\Omega p(x)(|\nabla u|^2 - |\nabla u_{p,q,f,\lambda}|^2)
+ \lambda q(x)(u^2 - u_{p,q,f,\lambda}^2) - 2f(x)(u - u_{p,q,f,\lambda}) \, dx.
\]

Some of the properties of \( G \) proved in [11] are collected in Theorem 2.1 below.

The value of such an extended theory can be seen by looking at (1.5) in the case \( R = R(x) \). In this situation, (1.5) can be transformed to (1.1) by (for example) applying a Laplace transform to the time variable. The raw data needed consist of head measurements taken over both space and time, as well as hydraulic conductivity (or transmissivity) values on the spatial boundary; such data in general are readily available and reasonably accurate. In principle, one can then use an appropriate functional (of the type discussed below) to recover \( p, S, \) and \( R \). We note in passing that while there are other methods to obtain \( p \) (mainly from steady-state data on the heads), it has been observed [2, pp. 152, 197] that there are essentially no universally applicable methods for estimating \( R \) and \( S \) and many practitioners use quite rough estimates of these parameters. This leads to instabilities in the model, especially when transient simulations are involved.

The basic idea for the recovery of the three coefficients \( P, Q, \) and \( F \) is to assume that solutions \( u_{P,Q,F,\lambda} \) are known for three (unequal) values of \( \lambda \), say \( \lambda_1, \lambda_2, \lambda_3 \), and then consider, for \((p, q, f) \) in

\[
D_H = \{(p, q, f) : p, q, f \text{ satisfy } (1.2), (1.3), \text{ and } p|_{\Gamma} = P|_{\Gamma}\},
\]

the functional

\[
H(p, q, f) = \sum_{i=1}^{3} G(p, q, f, \lambda_i)
\]
with the intention of using minimization over $D_H$ to obtain the coefficients $P, Q, F$. It is here that the uniqueness becomes important. If one can present conditions on the functions $u_{P,Q,F,\lambda}, 1 \leq i \leq 3$, ensuring that $P, Q, F$ are uniquely determined, not only is the approach validated in an essential way, but as an added bonus it follows from part (i) of Theorem 2.1 that the functional $H$ has a unique global minimum at $(P, Q, F)$ and from (ii) that $H'(p, q, f) = 0$ if and only if $H(p, q, f) = 0$, and thus that $(P, Q, F)$ is the unique stationary point for $H$. Furthermore, as each $L_{p,q,f,\lambda}$ is positive, from part (iii) of the theorem $H''(p, q, f) = 0$ if and only if

$$-\nabla \cdot (h_1 \nabla u_{p,q,f,\lambda}) + \lambda_i h_2 u_{p,q,f,\lambda} - h_3 = 0, \quad 1 \leq i \leq 3.$$  

If $h_1|_{\partial \Omega} = 0$, this means that we have, for $1 \leq i \leq 3$,

$$-\nabla \cdot ((p + h_1) \nabla u_{p,q,f,\lambda}) + \lambda_i (q + h_2) u_{p,q,f,\lambda} - (f + h_3) = 0,$$

where $(p + h_1, q + h_2, f + h_3)$ and $(p, q, f)$ are in $D_H$; uniqueness then dictates that $(p + h_1, q + h_2, f + h_3) = (p, q, f)$ so that $h_1 = h_2 = h_3 = 0$. From this we may infer that $H$ is a strictly convex functional. Similar remarks apply when one wishes to recover only single coefficients or pairs of coefficients.

2. Properties of the functional $G$. Some of the properties of the functional $G$ defined above are summarized in the following theorem.

**Theorem 2.1.**

(i) $G(c) \geq 0$ for all $c = (p, q, f, \lambda)$ in $D_G$, and $G(c) = 0$ if and only if $u = u_c$.

(ii) The first Gâteaux differential for $G$ is given by

$$(2.1) \quad G'(p, q, f, \lambda)[h_1, h_2, h_3] = \int_{\Omega} (| \nabla u|^2 - |\nabla u_c|^2) h_1 + \lambda(u^2 - u_c^2)h_2 - 2(u - u_c)h_3$$

$\forall h_1, h_2 \in \mathcal{L}^\infty(\Omega); h_3 \in \mathcal{L}^2(\Omega); and G'(p, q, f, \lambda) = 0 if and only if u = u_c$.

(iii) The second Gâteaux differential of $G$ is given by

$$(2.2) \quad G''(p, q, f, \lambda)[h, k] = 2 \left( L_{p,q,\lambda}^{-1} (e(h)), e(k) \right),$$

where $h = (h_1, h_2, h_3)$, $k = (k_1, k_2, k_3)$, and $h_1, h_2, k_1, k_2 \in \mathcal{L}^\infty(\Omega), h_3, k_3 \in \mathcal{L}^2(\Omega)$, and

$$e(h) = -\nabla \cdot (h_1 \nabla u_{p,q,f,\lambda}) + \lambda h_2 u_{p,q,f,\lambda} - h_3.$$

$(\cdot, \cdot)$ denotes the usual inner product in $\mathcal{L}^2(\Omega)$.

**Proof.** In the following discussion it will be convenient to set $c = (p, q, f)$. Observe that

$$G(c) = \int_{\Omega} p|\nabla(u - u_c)|^2 + 2p \nabla u_c \cdot \nabla(u - u_c) + \lambda q(u^2 - u_c^2) - 2f(u - u_c)$$

$$= \int_{\Omega} p|\nabla(u - u_c)|^2 - 2(u - u_c) \nabla \cdot (p \nabla u_c) + \lambda q(u^2 - u_c^2) - 2f(u - u_c)$$

after integration by parts

$$= \int_{\Omega} p|\nabla(u - u_c)|^2 - 2(u - u_c) (\lambda qu_c - f) + \lambda q(u^2 - u_c^2) - 2f(u - u_c)$$

from (1.1)

$$= \int_{\Omega} p(x)|\nabla(u - u_c)|^2 + \lambda q(x)(u - u_c)^2 \, dx.$$
Part (i) of the theorem now follows. Also, if and only if \( G'(c) = 0 \) we have that

\[
|\nabla u|^2 - |\nabla u_c|^2 = 0,
\]
\[
u^2 - u_c^2 = 0,
\]
\[
u - u_c = 0,
\]
and this is true if and only if \( G(c) = 0 \); thus the last part of part (ii) follows from part (i).

Before proving the remaining statements in the theorem we first note that for \( c \) and \( h \) as above (and fixed)

\[
\lim_{\epsilon \to 0} u_{c+\epsilon h} = u_c
\]

in \( W^{1,2}(\Omega) \) and for any function \( \eta \in \mathcal{L}^\infty(\Omega) \)

\[
||\nabla \cdot (\eta \nabla u_{c+\epsilon h})||_{W^{1,2}(\Omega)} \leq K,
\]

where the constant \( K \) does not depend on \( \epsilon \). These observations are obtained by differencing the equations

\[
-\nabla \cdot (p \nabla u_c) + \lambda qu_c = f,
\]
\[
-\nabla \cdot ((p + \epsilon h_1) \nabla u_{c+\epsilon h}) + \lambda (q + \epsilon h_2) u_{c+\epsilon h} = f + \epsilon h_3
\]
to obtain

\[
L_{p,q}(u_{c+\epsilon h} - u_c) = \epsilon (\nabla \cdot (h_1 \nabla u_{c+\epsilon h}) - \lambda h_2 u_{c+\epsilon h} + h_3)
\]

and using integration by parts (see [11] for further details).

Now from (1.6) and some algebra

\[
(G(c+\epsilon) - G(c))/\epsilon
\]
\[
= \int \Omega \left[ \lambda h_2 \right] (u^2 - u_c^2) - 2h_3(u - u_c)
\]
\[
+ \epsilon^{-1} \int \Omega \left[ p(\nabla u_c)^2 - |\nabla u_{c+\epsilon h}|^2 \right]
\]
\[
+ \lambda q(u^2 - u_c^2) - 2f(u - u_c).
\]

By (2.3) it is sufficient to show that the second integral expression above tends to zero as \( \epsilon \to 0 \). But

\[
\epsilon^{-1} \int \Omega p(\nabla u_c)^2 - |\nabla u_{c+\epsilon h}|^2
\]
\[
= \epsilon^{-1} \int \Omega p \nabla (u_c - u_{c+\epsilon h}) \cdot \nabla (u_c + u_{c+\epsilon h})
\]
\[
= \epsilon^{-1} \int \Omega (u_{c+\epsilon h} - u_c) \nabla (p \nabla (u_c + u_{c+\epsilon h}))
\]
\[
after an integration by parts
\]
\[
= \epsilon^{-1} \int \Omega (u_{c+\epsilon h} - u_c) \left\{ \lambda q u_c - f \right\} + \lambda q(u_{c+\epsilon h}^2 - u_c^2)
\]
\[
+ \lambda (q + \epsilon h_2) u_{c+\epsilon h} - (f + \epsilon h_3) - \epsilon \nabla \cdot (h_1 \nabla u_{c+\epsilon h})
\]
\[
= \int \Omega (u_{c+\epsilon h} - u_c) \left\{ -\nabla \cdot (h_1 \nabla u_{c+\epsilon h}) + \lambda h_2 u_{c+\epsilon h} - h_3 \right\}
\]
\[
+ \epsilon^{-1} \int \Omega \lambda q(u_{c+\epsilon h}^2 - u_c^2) - 2f(u_{c+\epsilon h} - u_c).
\]
Consequently, the second integral expression in (2.7) equals

\[
(2.8) \quad \int_\Omega (u_{c+\epsilon h} - u_c) \{-\nabla \cdot (h_1 \nabla u_{c+\epsilon h}) + \lambda h_2 u_{c+\epsilon h} - h_3\}
\]

and this tends to zero as \( \epsilon \to 0 \) by (2.3) and (2.4); part (ii) is thus established.

Finally, the second Gâteaux differential is given by

\[
G''(c)[h, k] = \lim_{\epsilon \to 0} \frac{G'(c + \epsilon h)[k] - G'(c)[k]}{\epsilon}.
\]

From (2.1) and some algebra

\[
(G'(c + \epsilon h)[k] - G'(c)[k]) / \epsilon
\]

\[
= \epsilon^{-1} \int_\Omega \left( (|\nabla u_\epsilon|^2 - |\nabla u_{c+\epsilon h}|^2) k_1 + \lambda (u_\epsilon^2 - u_{c+\epsilon h}^2) k_2 - 2(u_\epsilon - u_{c+\epsilon h}) k_3 \right)
\]

\[
= \epsilon^{-1} \int_\Omega k_1 \nabla (u_\epsilon - u_{c+\epsilon h}) \cdot \nabla (u_\epsilon + u_{c+\epsilon h})
\]

\[
+ \lambda (u_\epsilon^2 - u_{c+\epsilon h}^2) k_2 - 2(u_\epsilon - u_{c+\epsilon h}) k_3
\]

\[
= \epsilon^{-1} \int_\Omega (u_{c+\epsilon h} - u_\epsilon) \{\nabla \cdot (k_1 \nabla (u_\epsilon + u_{c+\epsilon h})) - \lambda (u_\epsilon + u_{c+\epsilon h}) k_2 + 2k_3\}
\]

after an integration by parts

\[
= \int_\Omega p \cdot q \{ -\nabla \cdot (h_1 \nabla u_{c+\epsilon h}) + \lambda h_2 u_{c+\epsilon h} - h_3\} \{ -\nabla \cdot (k_1 \nabla (u_\epsilon + u_{c+\epsilon h}))
\]

\[
+ \lambda (u_\epsilon + u_{c+\epsilon h}) k_2 - 2k_3\}
\]

by (2.6)

\[
= \int_\Omega p \cdot q \{ -\nabla \cdot (h_1 \nabla u_{c+\epsilon h}) + \lambda h_2 u_{c+\epsilon h} - h_3\} \{ -\nabla \cdot (k_1 \nabla (u_\epsilon + u_{c+\epsilon h})) + \lambda u_\epsilon k_2 - k_3\}
\]

\[
+ \int_\Omega p \cdot q \{ -\nabla \cdot (h_1 \nabla (u_{c+\epsilon h} - u_\epsilon)) + \lambda h_2 (u_{c+\epsilon h} - u_\epsilon)\}
\]

\[
\times \{ -\nabla \cdot (k_1 \nabla (u_\epsilon + u_{c+\epsilon h})) + \lambda (u_\epsilon + u_{c+\epsilon h}) k_2 - 2k_3\}
\]

\[
+ \int_\Omega p \cdot q \{ -\nabla \cdot (h_1 \nabla u_\epsilon) + \lambda h_2 u_\epsilon - h_3\}
\]

\[
\times \{ -\nabla \cdot (k_1 \nabla (u_{c+\epsilon h} - u_\epsilon)) + \lambda (u_{c+\epsilon h} - u_\epsilon) k_2\}.
\]

It remains to show that the second and third integrals in (2.9) tend to zero as \( \epsilon \to 0 \).

As the operator \( L_{p,q}^{-1} \) is self-adjoint, if we set

\[
w_\epsilon = -\nabla \cdot (k_1 \nabla (u_\epsilon + u_{c+\epsilon h})) + \lambda (u_\epsilon + u_{c+\epsilon h}) k_2 - 2k_3
\]

the second integral may be rewritten as

\[
\int_\Omega (-\nabla \cdot (h_1 \nabla (u_{c+\epsilon h} - u_\epsilon)) + \lambda h_2 (u_{c+\epsilon h} - u_\epsilon)) L_{p,q}^{-1} w_\epsilon
\]

\[
= h_1 \nabla (u_{c+\epsilon h} - u_\epsilon) \cdot \nabla (L_{p,q}^{-1} w_\epsilon) + \lambda h_2 (u_{c+\epsilon h} - u_\epsilon) L_{p,q}^{-1} w_\epsilon.
\]

Now from (2.4) \( w_\epsilon \) is uniformly bounded in \( \epsilon \) in \( L^2(\Omega) \) and, as \( L_{p,q}^{-1} \) may be extended uniquely as a bounded linear operator from \( L^2(\Omega) \) to \( W^{1,2}(\Omega) \), \( L_{p,q}^{-1} w_\epsilon \) is bounded
independently of $\epsilon$ in $W^{1,2}(\Omega)$. From the boundedness of $\nabla$ on $W^{1,2}(\Omega)$ to $L^2(\Omega) \times L^2(\Omega)$ it follows that $|\nabla(L^{-1}w_\epsilon)|$ is bounded independently of $\epsilon$ in $L^2(\Omega)$. From (2.3) it now follows that the second integral in (2.9) tends to zero with $L$ value of $\epsilon$ and that $\nabla\cdot(-(\partial \nabla u_\epsilon) + \lambda h_2 u_\epsilon - h_3)$ lies in $W^{1,2}(\Omega)$; also note that the third integral vanishes as $\epsilon \to 0$ via (2.3) after an integration by parts. \hfill \Box

3. The uniqueness problem. We now make precise the statement that the solutions $u$ of (1.1) determine the coefficients in the equation. Before proceeding, it is worth noting that under the conditions (1.2) and (1.3), and assuming the (physically reasonable) condition that the Dirichlet data $\phi > 0$, the solutions $u_{p,q,f,\lambda}$ are positive everywhere in $\Omega$; this follows from the strong maximum principle for (1.1) (see, for example, [8, Theorem 3.5]).

3.1. The one-coefficient case. If two coefficients are presumed known, the remaining coefficient essentially is determined by a knowledge of $u$ and $F$ are known, as $\lambda > 0$ and $u_{p,q,f,\lambda}$ are positive, from (3.2). Finally, note that under the conditions (1.2) and (1.3), and assuming the (physically reasonable) condition that the Dirichlet data $\phi > 0$, the solutions $u_{p,q,f,\lambda}$ are positive everywhere in $\Omega$; this follows from the strong maximum principle for (1.1) (see, for example, [8, Theorem 3.5]).

The remaining case requires more effort. If we presume that $Q$ and $F$ are known, then the uniqueness question could be formulated as follows. Assume that the functions $W = u_{p,q,f,\lambda}$ and $w = u_{p,q,f,\lambda}$ satisfy

$$\begin{align*}
-\nabla \cdot (P(x)\nabla W) + \lambda Q(x)W &= F(x), \quad x \in \Omega, \\
-\nabla \cdot (p(x)\nabla w) + \lambda Q(x)w &= F(x), \quad x \in \Omega,
\end{align*}
$$

and that $W = w$ on $\Omega$. Under what conditions on $W$ is it true that $P = p^2$? The following theorem gives a sufficient condition for this uniqueness to hold.

**Theorem 3.1.** If

$$\begin{align*}
\inf_{x \in \Omega} \left\{ \max\{ |\nabla W(x)|, |\Delta W(x)| \} \right\} > 0,
\end{align*}
$$

then $P = p$ in $\Omega$. If $n = 2$ the condition (3.2) may be weakened to the requirement that at points $x = (\xi, \eta) \in \Omega$ for which $\nabla W(x) = 0$ not all of $W_{xx}(x)$, $W_{xy}(x)$, $W_{yy}(x)$ vanish.

**Remark.** The condition (3.2) is satisfied if for example $\nabla W \neq 0$ or $\Delta W \neq 0$ on $\Omega$; as is noted in [16], this condition may be interpreted as requiring that $\Delta W(x) \neq 0$ at all points $x \in \Omega$ at which $\nabla W(x) = 0$.

**Proof.** From (3.1) we have $\nabla \cdot ((P - p)(x)\nabla W) = 0$, i.e.,

$$\begin{align*}
\nabla((P - p)(x)\nabla W(x)) + (P - p)(x)\Delta W(x) = 0.
\end{align*}
$$

As $(P - p)|_{\partial \Omega} = 0$, it is enough to show that the first-order hyperbolic equation (3.3) has a unique solution, and this depends on the flow generated by the vector field $\nabla W$ on $\Omega$. Under the condition (3.2) it follows by the method outlined in [16, p. 218] that $P - p = 0$ in $\Omega$. This involves splitting $\Omega$ into two regions $\Omega_1$ and $\Omega_2$ for which

$$\begin{align*}
|\nabla W| \text{ or } \Delta W > 0 \text{ in } \Omega_1, \quad |\nabla W| \text{ or } -\Delta W > 0 \text{ in } \Omega_2
\end{align*}
$$

and using the technique given in an earlier part of the same paper. It should be noted that while this does result in an interface across which $P - p$ would normally be discontinuous, in the present case no such discontinuity appears because the limit for $P - p$ is zero from each side of the interface.

In the case $n = 2$, if $\nabla W(x) = 0$ but $\Delta W(x) \neq 0$ then $(P - p)(x) = 0$ from (3.3). Assume that $(P - p)(x) > 0$ for some $x \in \Omega$. (A similar argument applies to a point
Let \( E = \{ x \in \Omega : (P - p)(x) > 0 \} \). \( E \) is open and nonempty and \((P - p)(x) = 0\) for all \( x \in \partial E \). Further, by the observations above it follows that when \( x \in E \) and \( \nabla W(x) = 0 \) we know that \( \Delta W(x) = 0 \). If we let \( x = (\xi, \eta) \) represent one of these special \( x \) values, then the eigenvalues \( \alpha \) of the derivative

\[
D(\nabla W)(x) = \begin{pmatrix}
W_{\xi\xi}(x) & W_{\xi\eta}(x) \\
W_{\xi\eta}(x) & W_{\eta\eta}(x)
\end{pmatrix}
\]

satisfy

\[
\alpha^2 = W_{\xi\eta}^2(x) + W_{\xi\xi}^2(x)
\]

(recall that \( \Delta W(x) = W_{\xi\xi}(x) + W_{\eta\eta}(x) = 0 \)). As not all of the second-order derivatives of \( W \) are allowed to be zero, it follows that there must be exactly one positive and one negative eigenvalue. One can now use the Hartman–Grobman theorem on the local behavior of a \( C^1 \) vector field at a singular point (see, for example, [15, p. 119]) to assert that each of the singular points in \( E \) must be a saddle point. Moreover, it also follows that such singular points must be isolated and thus cannot accumulate in the interior of \( E \) (although they could accumulate on \( \partial E \)). At such a singular point \( x \) one has exactly two orbits entering the singular point and two orbits exiting. Notice also that, as this is a gradient flow, the values of \( W \) must strictly increase (or decrease) along an orbit and hence there can be no closed orbits in \( E \). It now follows that the orbit through any \( x \in E \) must reach the boundary. This is certainly true if the orbit does not contain any singular points (the \( \omega \)-limit set of such an orbit is nonempty and contained in the boundary by the Poincaré–Bendixson theorem (see, for example, [9, p. 248])). If singular points are encountered, the fact that each is a saddle point and that orbits cannot return to a singular point, resulting in the formation of a closed orbit, combine to ensure that orbits with singular points also reach the boundary.

We now have a contradiction because if an orbit from \( x \in E \) reaches the boundary, \((P - p)(x) \neq 0\), but (3.3) for \( P - p \) reduces, on the orbit (which is a characteristic), to a first-order ordinary differential equation with zero initial value, and thus zero as the unique solution.

**Example 1.** When \( \Omega = [-1, 1] \times [-1, 1] \), \( F = 1 \), and \( P(x, y) = Q(x, y) = e^{(x^2 - y^2)} \) for some positive constant \( c \), it is not hard to check that a solution \( W = u_{P,Q,F,\lambda} \) of (1.1) is given by

\[
W(x, y) = \frac{1}{\lambda} e^{c(y^2 - x^2)}, \quad (x, y) \in \Omega.
\]

We know that \( Q \) and \( F \) are uniquely determined by \( W \). As for \( P \), some calculations show that

\[
\nabla W(x, y) = \frac{2c}{\lambda} e^{c(y^2 - x^2)} \begin{pmatrix}
-x \\
y
\end{pmatrix}
\]

and

\[
\Delta W(x, y) = \frac{4c^2}{\lambda} (x^2 + y^2) e^{c(y^2 - x^2)};
\]

thus \((0, 0)\) is the only singular point for \( \nabla W \), and here we have \( \Delta W(0, 0) = 0 \) so that condition (3.2) of Theorem 3.1 fails. But \( W_{\xi\xi}(0, 0) = -2c/\lambda \neq 0 \), so that the weakened form of condition (3.2) does hold. It is not hard to see that in fact the flow \( \nabla W \) has a saddle point at \((0, 0)\).
3.2. The two-coefficient case. If one coefficient is presumed known, then the other two coefficients are determined from a knowledge of \(u_{p,q,f,\lambda}\) for two values of \(\lambda\), modulo some additional conditions.

In the case that \(P\) is known and one has two solutions, \(u_{p,q,f,\lambda_1}\) and \(u_{p,q,f,\lambda_2}\), where \(\lambda_1 \neq \lambda_2\) it is straightforward to see from (1.1) that \(Q\) and \(F\) are uniquely determined if and only if

\[
\lambda_1 u_{p,q,f,\lambda_1}(x) \neq \lambda_2 u_{p,q,f,\lambda_2}(x) \quad \text{for almost all } x \in \Omega.
\]

Next assume that the function \(Q\) is known and that we are given functions \(W = u_{p,q,f,\lambda_1}, W = u_{p,q,f,\lambda_2}\), and \(v = u_{p,q,f,\lambda_2}\) satisfying

\[
\begin{align*}
- \nabla \cdot (P(x) \nabla W) + \lambda_1 Q(x) W &= F, \quad x \in \Omega, \\
- \nabla \cdot (p(x) \nabla w) + \lambda_1 Q(x) w &= f, \quad x \in \Omega, \\
- \nabla \cdot (P(x) \nabla V) + \frac{\lambda_2 Q(x)}{2} V &= F, \quad x \in \Omega, \\
- \nabla \cdot (p(x) \nabla v) + \frac{\lambda_2 Q(x)}{2} v &= f, \quad x \in \Omega.
\end{align*}
\]

If \(W = w\) and \(V = v\) we seek conditions on \(W, V\) to ensure that \(P = p\) and \(F = f\).

One answer is given by the following theorem.

**Theorem 3.2.** If \(W - V\) satisfies the conditions satisfied by \(W\) in Theorem 3.1, then \(P = p\) and \(F = f\) on \(\Omega\).

*Proof.* From (3.6), as \(W = w\) and \(V = v\),

\[
\begin{align*}
- \nabla \cdot ((P - p)(x) \nabla W) &= F - f, \\
- \nabla \cdot ((P - p)(x) \nabla V) &= F - f;
\end{align*}
\]

hence

\[
\begin{equation}
- \nabla \cdot ((P - p)(x) \nabla (W - V)) = 0, \quad x \in \Omega.
\end{equation}
\]

This is (3.3) with \(W\) replaced with \(W - V\); as we have the same conditions on \(W - V\) that were placed on \(W\) in Theorem 3.1, it follows that \(P = p\) by the same argument. That \(F = f\) follows from (3.7). \(\Box\)

The last possibility is more difficult. Here we assume that \(F\) is known, and we seek uniqueness for \(P\) and \(Q\). Thus for \(\lambda_1 \neq \lambda_2\) we are given functions \(W = u_{p,q,f,\lambda_1}, W = u_{p,q,f,\lambda_2}\), and \(v = u_{p,q,f,\lambda_2}\), satisfying

\[
\begin{align*}
- \nabla \cdot (P(x) \nabla W) + \lambda_1 Q(x) W &= F, \quad x \in \Omega, \\
- \nabla \cdot (p(x) \nabla w) + \lambda_1 Q(x) w &= f, \quad x \in \Omega, \\
- \nabla \cdot (P(x) \nabla V) + \frac{\lambda_2 Q(x)}{2} V &= F, \quad x \in \Omega, \\
- \nabla \cdot (p(x) \nabla v) + \frac{\lambda_2 Q(x)}{2} v &= f, \quad x \in \Omega,
\end{align*}
\]

and \(W = w, V = v\) on \(\Omega\). The following theorem is a restatement of the uniqueness question in the present context.

**Theorem 3.3.** If the flow generated by the vector field

\[
M(x) = \lambda_1 W(x) \nabla V(x) - \lambda_2 V(x) \nabla W(x)
\]

on \(\Omega\) has the property that every point exits at the boundary of \(\Omega\) (i.e., lies on a flow line starting at the boundary), then \(P = p\) and \(Q = q\) in \(\Omega\).

*Proof.* From (3.9) we have that

\[
\begin{align*}
- \nabla \cdot ((P - p)(x) \nabla W) + \lambda_1 (Q - q)(x) W &= 0, \\
- \nabla \cdot ((P - p)(x) \nabla V) + \lambda_2 (Q - q)(x) V &= 0.
\end{align*}
\]
and hence that
\[ \lambda_1 W(x) \nabla \cdot ((P - p)(x) \nabla V(x)) - \lambda_2 V(x) \nabla \cdot ((P - p)(x) \nabla W(x)) = 0, \]
from which we obtain
\[ (3.12) \quad \nabla (P - p)(x) \cdot (\lambda_1 W(x) \nabla V(x) - \lambda_2 V(x) \nabla W(x)) + (P - p)(x)(\lambda_1 W(x) \Delta V(x) - \lambda_2 V(x) \Delta W(x)) = 0. \]

As usual \((P - p)|_{\partial \Omega} = 0\) and it is clear from the given property on the vector field \(M\) that (3.12) will have the unique solution \(P - p = 0\); that \(Q = q\) now follows from (3.11).

For general \(n\) the flow \(M\) can be extremely complicated and simple conditions on \(W, V\) that give rise to the kind of flow required by Theorem 3.3 are not readily available, even in the case \(n = 3\). For the case \(n = 2\), however, we have the following theorem.

**Theorem 3.4.** Assume that \(\nabla W \cdot \nabla V\) is of one sign on \(\Omega\), with the possible exception of a set of measure zero in \(\Omega\) where it may be zero, and assume that whenever \(M(x) = 0\) (\(M\) is defined by (3.10)), \(F(x) \neq 0\), and if the sign of \(\nabla W \cdot \nabla V\) is positive, \(\lambda_1 W(x) \neq \lambda_2 V(x)\), whereas if the sign is negative, it is assumed that for \(x = (\xi, \eta)\) not all of
\[ (W - V)_{\xi}(x), (W - V)_{\eta}(x), (W - V)_{\xi\eta}(x) \]
are zero. Then \(P = p\) and \(Q = q\) on \(\Omega\).

**Proof.** Assume by way of contradiction that the set \(E = \{ x \in \Omega : (P - p)(x) > 0 \}\) is not empty (a similar argument applies to the set of negative values for \(P - p\)); the set \(E\) is clearly open and \((P - p)(x) = 0\) for all \(x \in \partial E\). Set
\[ (3.13) \quad S(x) = \lambda_1 W(x) \Delta V(x) - \lambda_2 V(x) \Delta W(x). \]

For any \(x \in \Omega\) with \(M(x) = 0\) and \(S(x) \neq 0\) it follows from (3.12) that \((P - p)(x) = 0\) and hence such points do not lie in \(E\); thus for all singular points of the vector field \(M\) in \(E\) we may assume that \(S(x) = 0\). Note also that if \(M(x) = 0\), then
\[ (3.14) \quad \lambda_1 W(x) = \lambda_2 V(x) \text{ if and only if } S(x) = 0. \]

To see this recall that
\[ (3.15) \quad -\nabla P(x) \cdot \nabla W(x) - P(x) \Delta W(x) + \lambda_1 Q(x) W(x) = F(x), \]
\[ -\nabla P(x) \cdot \nabla V(x) - P(x) \Delta V(x) + \lambda_2 Q(x) V(x) = F(x). \]

If \(S(x) = 0\), one can multiply the first equation by \(\lambda_2 V(x)\) and the second by \(\lambda_1 W(x)\) and subtract to obtain \((\lambda_1 W(x) - \lambda_2 V(x))F(x) = 0\) from which one of the implications follows; the reverse implication is proven in a similar fashion.

There are now two cases, depending on the sign of \(\nabla W \cdot \nabla V\). If
\[ (3.16) \quad \nabla W \cdot \nabla V(x) \geq 0, \quad x \in \Omega, \]
then from the arguments and assumptions above it follows that there are no singular points of the vector field \(M\) in \(E\). There are also no closed orbits of \(M\) in \(E\). To see
this first note that
\[ \nabla \cdot \left( ((P - p) M) \right) \]
\[ = \lambda_1 W \nabla \cdot \left( ((P - p) \nabla V) \right) - \lambda_2 V \nabla \cdot \left( ((P - p) \nabla W) \right) \]
\[ + (\lambda_1 - \lambda_2) (P - p) \nabla W \cdot \nabla V \]
\[ = (\lambda_1 - \lambda_2) (P - p) \nabla W \cdot \nabla V \]
(3.17)
from (3.11). Let \( C \) be a closed orbit of \( M \) in \( E \). If \( C \) encloses any part of \( \partial E \), then this part of \( \partial E \) can be added to \( C \) to create a curve \( C' \) with a simply connected interior \( U \). Then by the divergence theorem and (3.17) we have that
\[ \int \int_U (\lambda_1 - \lambda_2) (P - p) \nabla W \cdot \nabla V = \int \int_U \nabla \cdot \left( ((P - p) M) \right) = \int_{C'} (P - p) M \cdot n, \]
where \( n \) denotes the outward normal to \( C' \). On the parts of \( C' \) corresponding to \( C \) \( M \cdot n = 0 \) and on the rest of \( C' \) \( P - p = 0 \), so the line integral is zero. But from the assumptions on \( \nabla W \cdot \nabla V \) it is clear that the double integral is not zero, and we have a contradiction (this is basically Dulac’s criterion; see [15, p. 246]). So \( E \) now has no closed orbits and no singular points. Consider the orbit \( \Gamma \) of a point \( x \in E \). By the Poincaré–Bendixon theorem ([9, p. 248]) as \( \Gamma \subset E \) and \( E \) is compact the \( \omega \)-limit set of \( \Gamma \) lies in the boundary of \( E \), and so every point in \( E \) exits under the flow \( M \). In particular, by the argument used earlier, \( (P - p)(x) = 0 \) and we have a contradiction.

Finally, if
\[ \nabla W \cdot \nabla V(x) \leq 0, \quad x \in \Omega, \]
(3.18)
let \( x \) be a singular point for \( M \) in \( E \). Then \( S(x) = 0 \) and hence by (3.14) we have that \( \lambda_1 W(x) = \lambda_2 V(x) \) and thus
\[ \nabla W(x) = \nabla V(x), \]
\[ \Delta(W - V)(x) = 0. \]
(3.19)
From (3.18) and (3.19)
\[ \nabla W(x) = \nabla V(x) = 0. \]
(3.20)
Now if \( x = (\xi, \eta) \) we have that
\[ D(M)(x) = \lambda_1 W(x) \left( \begin{array}{cc} V_{\xi \xi}(x) - W_{\xi \xi}(x) & V_{\xi \eta}(x) - W_{\xi \eta}(x) \\ V_{\eta \xi}(x) - W_{\eta \xi}(x) & V_{\eta \eta}(x) - W_{\eta \eta}(x) \end{array} \right) \]
on using (3.14) and (3.20). From (3.19) it then follows that the eigenvalues \( \alpha \) of \( D(M)(x) \) satisfy
\[ \alpha^2 = (V_{\xi \eta}(x) - W_{\xi \eta}(x))^2 + (V_{\xi \xi}(x) - W_{\xi \xi}(x))^2, \]
and as not all second-order derivatives of \( V - W \) are allowed to be zero at \( x \), there must be exactly one positive and one negative eigenvalue for \( D(M)(x) \). There are also no closed orbits of \( M \) in \( E \) by the same argument given above. One can now obtain a contradiction via the argument used in the last part of the proof of Theorem 3.1 above. \( \square \)
Example 2. Consider again Example 1 given above. Choose two values of \( \lambda \), say \( \lambda_1 < \lambda_2 \), and set

\[
W(x, y) = \frac{1}{\lambda_1} e^{c(y^2-x^2)} \quad \text{and} \quad V(x, y) = \frac{1}{\lambda_2} e^{c(y^2-x^2)}.
\]

Notice that \( \lambda_1 W(x, y) = \lambda_2 V(x, y) \) for all \( (x, y) \in \Omega \), so that (3.5) fails. Here \( Q \) and \( F \) (with \( P \) assumed known) are in fact not determined by the two solutions \( W \) and \( V \) because \( \nabla \cdot (P \nabla W) = \nabla \cdot (P \nabla V) = 0 \); this failure, while occurring in a somewhat contrived situation, points to the fact that uniqueness does come with some strings attached. A short calculation very much like that of Example 1, together with Theorem 3.2, shows that \( P \) and \( F \) (with \( Q \) now assumed known) are uniquely determined by \( W \) and \( V \). Finally, if \( F \) is assumed known, we could try to appeal to Theorem 3.4. Here we have

\[
M(x, y) = \lambda_1 W \nabla V - \lambda_2 V \nabla W
\]

\[
= 2ce^{2c(y^2-x^2)} \left( \frac{1}{\lambda_2} - \frac{1}{\lambda_1} \right) \begin{pmatrix} -x \\ y \end{pmatrix}
\]

(with a unique singular point at \( (0, 0) \)) and

\[
\nabla W \cdot \nabla V(x, y) = \frac{(2c)^2}{\lambda_1 \lambda_2} e^{2c(y^2-x^2)} (x^2 + y^2) \geq 0,
\]

but, as indicated above, we do not have \( \lambda_1 W(0, 0) \neq \lambda_2 V(0, 0) \) so the sufficiency condition is not satisfied. In fact \( P \) and \( Q \) are uniquely determined, by the argument used in the second part of this theorem, because at \( (0, 0) \), the only singular point for \( M \), we also have \( \nabla W(0, 0) = \nabla V(0, 0) = (0, 0) \).

3.3. The three-coefficient case. Assume that \( \lambda_1, \lambda_2, \text{and} \lambda_3 \) are given, together with functions \( R = u_{p,q,f,\lambda_1}, r = u_{p,q,f,\lambda_1}, S = u_{p,q,f,\lambda_2}, s = u_{p,q,f,\lambda_2}, T = u_{p,q,f,\lambda_3}, \text{and} t = u_{p,q,f,\lambda_3} \), satisfying

\[
\begin{align*}
-\nabla \cdot (p(x) \nabla R) + \lambda_1 Q(x) R &= F, & x \in \Omega, \\
-\nabla \cdot (p(x) \nabla r) + \lambda_1 q(x) r &= f, & x \in \Omega, \\
-\nabla \cdot (p(x) \nabla S) + \lambda_2 Q(x) S &= F, & x \in \Omega, \\
-\nabla \cdot (p(x) \nabla s) + \lambda_2 q(x) s &= f, & x \in \Omega, \\
-\nabla \cdot (p(x) \nabla T) + \lambda_3 Q(x) T &= F, & x \in \Omega, \\
-\nabla \cdot (p(x) \nabla t) + \lambda_3 q(x) t &= f, & x \in \Omega,
\end{align*}
\]

and \( R = r, S = s, \text{and} T = t \) on \( \Omega \). By analogy with Theorem 3.3 we have the following theorem.

Theorem 3.5. With the above notation, if the flow generated by the vector field

\[
V(x) = (\lambda_2 S(x) - \lambda_3 T(x)) \nabla R(x) + (\lambda_3 T(x) - \lambda_1 R(x)) \nabla S(x) + (\lambda_1 R(x) - \lambda_2 S(x)) \nabla T(x)
\]

on \( \Omega \) has the property that every point exits at the boundary of \( \Omega \) (i.e., lies on a flow line starting at the boundary) and at least one of

\[
\lambda_2 S(x) - \lambda_3 T(x), \quad \lambda_3 T(x) - \lambda_1 R(x), \quad \lambda_1 R(x) - \lambda_2 S(x)
\]

is not zero at every \( x \in \Omega \), then \( P = p, Q = q, \text{and} F = f \) in \( \Omega \).
Proof. From (3.23) we have that
\begin{align}
-\nabla \cdot ((P - p)(x) \nabla R) + \lambda_1 (Q - q)(x) R &= F - f, \\
-\nabla \cdot ((P - p)(x) \nabla S) + \lambda_2 (Q - q)(x) S &= F - f, \\
-\nabla \cdot ((P - p)(x) \nabla T) + \lambda_3 (Q - q)(x) T &= F - f
\end{align}
(3.26)
and hence that
\begin{align}
-\nabla \cdot ((P - p)(x) \nabla (R - S)) + (\lambda_1 R - \lambda_2 S)(Q - q)(x) &= 0, \\
-\nabla \cdot ((P - p)(x) \nabla (R - T)) + (\lambda_1 R - \lambda_3 T)(Q - q)(x) &= 0.
\end{align}
(3.27)
Eliminating the terms in \( Q - q \) then gives
\begin{align}
-\nabla (P - p)(x) \cdot [\left( \lambda_2 S - \lambda_3 T \right) \nabla R + (\lambda_3 T - \lambda_1 R) \nabla S + (\lambda_1 R - \lambda_2 S) \nabla T] \\
+ (P - p)(x)[(\lambda_2 S - \lambda_3 T) \Delta R + (\lambda_3 T - \lambda_1 R) \Delta S + (\lambda_1 R - \lambda_2 S) \Delta T] &= 0.
\end{align}
(3.28)
It is clear from the given property on the vector field \( V \) that (3.28) will have the unique solution \( P - p = 0 \) and that \( Q = q \) and \( F = f \) now follows from (3.25), (3.27), and (3.26). \( \Box \)
A sufficient condition for uniqueness of three coefficients is given by the following theorem.

**Theorem 3.6.** Let \( n = 2 \) and
\begin{align}
L &= (\lambda_2 - \lambda_1) \nabla R \cdot \nabla S + (\lambda_1 - \lambda_3) \nabla R \cdot \nabla T + (\lambda_3 - \lambda_2) \nabla T \cdot \nabla S, \\
N &= (\lambda_2 S - \lambda_3 T) \Delta R + (\lambda_3 T - \lambda_1 R) \Delta S + (\lambda_1 R - \lambda_2 S) \Delta T.
\end{align}
(3.29)
If \( L \) is of one sign in \( \Omega \) with the possible exception of a set of measure zero in \( \Omega \), where it may be zero, and \( V(x) \) and \( N(x) \) are never both zero, then \( P = p, Q = q, \) and \( F = f \) in \( \Omega \).

**Proof.** If \( V \) is defined by (3.24), observe that
\begin{align}
V(x) &= (\lambda_1 R(x) - \lambda_3 T(x)) \nabla (R - S)(x) + (\lambda_2 S(x) - \lambda_1 R(x)) \nabla (R - T)(x).
\end{align}
Thus
\begin{align}
\nabla \cdot ((P - p)V) \\
&= \nabla \cdot [(\lambda_1 R - \lambda_3 T)(P - p) \nabla (R - S)] + \nabla \cdot [(\lambda_2 S - \lambda_1 R)(P - p) \nabla (R - T)] \\
&= (\lambda_1 R - \lambda_3 T) \nabla \cdot [(P - p) \nabla (R - S)] \\
&\quad + (\lambda_2 S - \lambda_1 R) \nabla \cdot [(P - p) \nabla (R - T)] \\
&\quad + (P - p) \nabla (\lambda_1 R - \lambda_3 T) \cdot \nabla (R - S) \\
&\quad + (P - p) \nabla (\lambda_2 S - \lambda_1 R) \cdot \nabla (R - T) \\
&= (P - p) \nabla (\lambda_1 R - \lambda_3 T) \cdot \nabla (R - S) + \nabla (\lambda_2 S - \lambda_1 R) \cdot \nabla (R - T),
\end{align}
by (3.27), and hence
\begin{align}
\nabla \cdot ((P - p)V) &= (P - p)L
\end{align}
(3.30)
after some algebra.

If by way of contradiction we construct \( E \) as the subset of \( \Omega \) on which \( P - p \) is positive, it follows from the assumptions that \( V \) never vanishes on \( E \) and by the Dulac
argument used earlier, used in conjunction with (3.30), that there are no closed orbits of the vector field \( V \) in \( E \). This leads to a contradiction similar to that obtained in the first case of Theorem 3.4.

**Example 3.** If we recall that in Example 2 uniqueness failed for \( Q \) and \( F \) (given \( P \)), it should come as no surprise that the sufficiency in Theorem 3.6 also fails in this case, as \( L = 0 \) on \( \Omega \).

**Example 4.** A slightly more realistic example can be constructed as follows. Set \( P = Q = F = 1 \); then

\[
\begin{align*}
    u_{P,Q,F,\lambda}(x,y) &= e^{\sqrt{x}\lambda} - \frac{1}{\lambda},
\end{align*}
\]

satisfies (1.1) in \( \Omega = [0,1] \times [0,1] \). Let \( \lambda_1 = 1 \), \( \lambda_2 = 4 \), and \( \lambda_3 = 9 \) and set

\[
\begin{align*}
    R(x,y) &= e^x - 1, \\
    S(x,y) &= e^{2x} - \frac{1}{4}, \\
    T(x,y) &= e^{3x} - \frac{1}{9}.
\end{align*}
\]

Then one can show that

\[
V(x,y) = \begin{pmatrix}
    2e^{3x}(1 - 3e^x + 3e^{2x}) \\
    0
\end{pmatrix},
\]

which is never \((0,0)\) in \( \Omega \), and

\[
L(x,y) = 6e^{3x}(1 - 4e^x + 5e^{2x}) > 0, \quad (x,y) \in \Omega.
\]

Theorem 3.6 may then be applied to assert that \( P, Q, \) and \( F \) are uniquely determined by \( R, S, \) and \( T \).

**4. Some applications.** In the practical determination of the aquifer parameters \( p, S, \) and \( R(x,t) \) one now has several options, and these are currently undergoing numerical testing. As noted earlier, one can apply a Laplace transform to (1.5) to obtain an equation of the form (1.1) where \( f = f(x,\lambda) \). One could then use a functional of the form (1.7) to obtain \( p, S, \) and \( R \), at least when \( R = R(x) \); the latter assumption may be justified when for example the aquifer is being supplied steadily by a stream in a nonrainfall time period. In general, when the sufficiency criteria prove inconclusive, one can in practice resort to using computer graphics to plot the relevant vector fields, in effect using Theorems 3.3 and 3.5 (or their analogues) directly; it is usually obvious when the flow is suitable, especially in the two-dimensional case.

As noted earlier, there is no universally applicable method for estimating \( R(x,t) \) [2, p. 152] and the typical approach is to assume a spatially uniform recharge rate across the water table equal to some percentage of the annual precipitation. This is done even though it is well documented [18, 19, 20, 21] that typically there are significant spatial and temporal variations in \( R \).

One could attempt to estimate \( p, S, R(x,t) \) directly by making some physically reasonable assumptions about the form of \( R \). For example, if the data were gathered over a time interval \([0,T]\) and \( 0 = a_0 < a_1 < \cdots < a_n = T \) is a partition of \([0,T]\), then one could assume that, for a sufficiently fine partition, \( R \) was constant in time over each subinterval \((a_{i-1},a_i)\), i.e., that

\[
R(x,t) = \sum_{i=1}^{n} R_i(x)\chi_{(a_{i-1},a_i)}(t),
\]
and seek to determine the functions $p, q = S$ and $R_i(x), 1 \leq i \leq n$.

The approach also has the potential to resolve another difficulty in this area. The use of a scalar function $p$ amounts to the assumption that the porous medium is isotropic (roughly, the local flow behavior is the same in all directions). In a real aquifer, due to the presence of fractures and other interfaces, this assumption is often invalid. For an anisotropic porous medium one has to replace the scalar coefficient $p$ with an $n \times n$ matrix $p = (p_{ij})$. The problem then is to determine the $n^2 + 2$ coefficients $p_{ij}, q, f$, given $n^2 + 2$ values of $\lambda$ and corresponding solutions of (1.1). The theory of the functional $G$ with matrix principal coefficient in the differential equation is essentially the same so that appropriate sums of the functionals $G$ can be minimized as before. Uniqueness results, however, are not known for this case at this time.

REFERENCES

