

Apportionment 2

Divisor Methods

In this lecture we will consider the so-called *divisor methods* which change the standard divisor (up or down) in order to be able to *round* consistently to an apportionment. Except for the last below, all were proposed early in U.S. history. They may be known by other names elsewhere in the world.

- Jefferson's method (proposed by Thomas Jefferson)
- Adams' method (proposed by John Quincy Adams)
- Webster's method (proposed by Daniel Webster)
- Huntington-Hill method (proposed by chief statistician Joseph Hill at the Census Bureau and mathematician Edward Huntington)

Much debate surrounded the choice of apportionment method in U.S. history.

Jefferson's Method

- Let M denote the number of seats in the house.
- **Step 1.** Find a *modified divisor* D such that the states' *modified quotas*

$$\text{Modified Quota} = \frac{\text{State's Population}}{D}$$

have *integer parts* which total M.

- **Step 2.** Apportion to each state the *integer part* of its modified quota.
- **Note.** Integer part means **round down!**

History. Congress adopted Jefferson's method in 1791, after President Washington vetoed Hamilton's method (the first presidential veto in U.S. history). Jefferson's method was readopted subsequently after each census until Webster's method was adopted for the first time in 1842.

Parador – Jefferson’s method							
State	A	B	C	D	E	F	Total
Population (1000s)	1,646	6,936	154	2,091	685	988	12,500
House: $M = 250$	Standard Divisor: $D = 50$ (in 1000s)						
Standard Quota	32.92	138.72	3.08	41.82	13.70	19.76	250
Round Down							
Modified Divisor: $D =$							
Modified Quota							
Jefferson App’t							

Note that we **round down** for the Jefferson apportionment.

Violations of the Quota

Note that in Jefferson's method applied to Parador, the final Jefferson apportionment to State B was 140 seats, while the standard quota for B was only 138.72 seats.

- Rounding the standard quota up – B gets 139 seats.
- Rounding the standard quota down – B gets 138 seats.

Remember what the quota represents. Jefferson's method gives B 140 seats – seems like a lion's share!

Quota Rule. An apportionment method should apportion to a state, whose standard quota has a fractional part, either the integer immediately above, or the integer part of, that state's standard quota.

Jefferson's method violates the Quota Rule. If a state gets more than the integer immediately above its standard quota, we call it a *violation* of the *upper quota*.

Example – Finding a modified divisor for Jefferson

The small country of Southland has three states: AL, GA, and MS, with a total population of 12,600. The legislature of Southland has 100 seats. The seats are to be apportioned using Jefferson's method.

State	AL	GA	MS	Total
Population	3,718	6,762	2,120	12,600
House: $M = 100$		Standard Divisor: $D =$		
Standard Quota				
Round Down				

Will the standard divisor work for Jefferson's method?

We will try several Modified Divisors.

Should we start with a smaller or a larger Modified Divisor?

Southland – Jefferson’s Method

State	AL	GA	MS	Total
Population	3,718	6,762	2,120	12,600
Modified Divisor: $D =$				
Modified Quota				
Round Down				
Modified Divisor: $D =$				
Modified Quota				
Round Down				
Modified Divisor: $D =$				
Modified Quota				
Round Down				

Adams' Method

- Let M denote the number of seats in the house.
- **Step 1.** Find a *modified divisor* D such that the states' *modified quotas*

$$\text{Modified Quota} = \frac{\text{State's Population}}{D}$$

when rounded up to the next highest integer, total M.

- **Step 2.** Apportion to each state the *integer just above* its modified quota.
- **Note.** Integer just above means **round up!**

History. John Quincy Adams proposed this method in 1830, but it was never adopted by Congress. Like Lowndes' method, it was felt (by the big states?) that it favored small states too much.

Parador – Adams' method

State	A	B	C	D	E	F	Total
Population (1000s)	1,646	6,936	154	2,091	685	988	12,500
House: $M = 250$		Standard Divisor: $D = 50$ (in 1000s)					
Standard Quota	32.92	138.72	3.08	41.82	13.70	19.76	250
Round up							
Modified Divisor: $D =$							
Modified Quota							
Adams App't							

Quota Rule. An apportionment method should apportion to a state, whose standard quota has a fractional part, either the integer immediately above, or the integer part of, that state's standard quota.

Note that in Adams's method applied to Parador, the final Adams' apportionment for State B was 137 seats, while the standard quota for B was 138.72.

If State B expected either 138 or 139 seats based upon its standard quota, State B might feel a bit cheated by Adams' method.

Adams' method violates the Quota Rule. If a state gets less than the integer part of its standard quota, we call it a *violation* of the *lower quota*.

When Adams' method violates the quota, it can only violate the lower quota. Why?

When Jefferson's method violates the quota, it can only violate the upper quota. Why?

Example – Finding a modified divisor for Adams

We revisit Southland and its three states: AL, GA, and MS, with a total population of 12,600. The legislature of Southland has 100 seats, but now the seats are to be apportioned using Adams' method.

State	AL	GA	MS	Total
Population	3,718	6,762	2,120	12,600
House: $M = 100$		Standard Divisor: $D =$		
Standard Quota				
Round Up				

Will the standard divisor work for Adams' method?

We will try several Modified Divisors.

Should we start with a smaller or a larger Modified Divisor?

Southland – Adams’ Method				
State	AL	GA	MS	Total
Population	3,718	6,762	2,120	12,600
Modified Divisor: $D =$				
Modified Quota				
Round Up				
Modified Divisor: $D =$				
Modified Quota				
Round Up				
Modified Divisor: $D =$				
Modified Quota				
Round Up				

Webster's Method

- Let M denote the number of seats in the house.
- **Step 1.** Find a *modified divisor* D such that the states' *modified quotas*

$$\text{Modified Quota} = \frac{\text{State's Population}}{D}$$

When rounded in the *conventional way*, total M .

- **Step 2.** Apportion to each state its modified quota, rounded conventionally.
 - Up for ≥ 0.5 and down for < 0.5 .

History. Daniel Webster proposed this method in 1830 as a compromise between Jefferson's and Adams' methods. It was adopted by Congress in 1842, replaced by Hamilton's method in 1852, and re-adopted in 1901. It was replaced by the Huntington-Hill method in 1941, the method still in use today.

Parador – Webster’s method							
State	A	B	C	D	E	F	Total
Population (1000s)	1,646	6,936	154	2,091	685	988	12,500
House: $M = 250$		Standard Divisor: $D = 50$ (in 1000s)					
Standard Quota	32.92	138.72	3.08	41.82	13.70	19.76	250
Round Conv.							
Modified Divisor: $D =$							
Modified Quota							
Webster App’t							

Violations of the quota can occur in Webster’s method, either upper or lower (why?), though neither occurred in this example.

Example – Finding a modified divisor for Webster

We make one last visit to Southland and its three states: AL, GA, and MS. Meanwhile the population has changed a bit. The 100 seats in the legislature are to be apportioned using Webster's method.

State	AL	GA	MS	Total
Population	3,718	6,762	2,120	12,600
House: $M = 100$		Standard Divisor: $D =$		
Standard Quota				
Round Conv.				

Will the standard divisor work for Webster's method?

We will try several Modified Divisors.

Should we start with a smaller or a larger Modified Divisor?

Southland – Webster’s Method

State	AL	GA	MS	Total
Population	3,718	6,762	2,120	12,600
Modified Divisor: $D =$				
Modified Quota				
Round Conv.				
Modified Divisor: $D =$				
Modified Quota				
Round Conv.				
Modified Divisor: $D =$				
Modified Quota				
Round Conv.				

Parador – Comparison of Methods							
State	A	B	C	D	E	F	Total
Population (1000s)	1,646	6,936	154	2,091	685	988	12,500
House: $M = 250$	Standard Divisor: $D = 50$ (in 1000s)						
Standard Quota	32.92	138.72	3.08	41.82	13.70	19.76	250
Hamilton	33	139	3	42	13	20	250
Lowndes	33	138	4	41	14	20	250
Jefferson	33	140	3	42	13	19	250
Adams	33	137	4	42	14	20	250
Webster	33	138	3	42	14	20	250

Each method yields a different apportionment in Parador.

Southland – Comparison of Methods

State	AL	GA	MS	Total
Population	3,718	6,762	2,120	12,600
Legislature: M = 100		Standard Divisor: D = 126		
Standard Quota	29.508	53.667	16.825	100
Hamilton	29	54	17	100
Lowndes	30	53	17	100
Jefferson	29	54	17	100
Adams	30	53	17	100
Webster	29	54	17	100

Three of the five methods above give the same apportionment in Southland. Do you see any pattern?

Example – Northland – Divisor Methods

The small country of Northland has three states: NY, PA, and MA, with a total population of 43,600. The legislature of Northland has 100 seats. Compare divisor methods for apportionment.

State	NY	PA	MA	Total
Population	21,010	14,990	7,600	43,600
Legislature: $M = 100$		Standard Divisor: $D =$		
Standard Quota				
Round Down				
Round Up				
Round Conv.				

Will any of the divisor methods work using the Standard Divisor?

Northland – Jefferson’s Method				
State	NY	PA	MA	Total
Population	21,010	14,990	7,600	43,600
Modified Divisor: $D =$				
Modified Quota				
Round				
Modified Divisor: $D =$				
Modified Quota				
Round				
Modified Divisor: $D =$				
Modified Quota				
Round				

Northland – Adams’ Method				
State	NY	PA	MA	Total
Population	21,010	14,990	7,600	43,600
Modified Divisor: D =				
Modified Quota				
Round				
Modified Divisor: D =				
Modified Quota				
Round				
Modified Divisor: D =				
Modified Quota				
Round				

Northland – Webster’s Method				
State	NY	PA	MA	Total
Population	21,010	14,990	7,600	43,600
Modified Divisor: D =				
Modified Quota				
Round				
Modified Divisor: D =				
Modified Quota				
Round				
Modified Divisor: D =				
Modified Quota				
Round				

Northland – Webster’s Method				
State	NY	PA	MA	Total
Population	21,010	14,990	7,600	43,600
Modified Divisor: D =				
Modified Quota				
Round				
Modified Divisor: D =				
Modified Quota				
Round				
Modified Divisor: D =				
Modified Quota				
Round				

Summary of Apportionment Methods

Types of Methods

- Quota Methods
 - Hamilton, Lowndes
- Divisor Methods
 - Jefferson, Adams, Webster, Huntington-Hill, Dean

Observations

- Satisfy Quota Rule: Hamilton, Lowndes
- May violate Quota Rule
 - May violate Upper Quota: Jefferson, Webster, Huntington-Hill, Dean
 - May violate Lower Quota: Adams, Webster, Huntington-Hill, Dean
- Favor large states: Hamilton, Jefferson
- Favor small states: Lowndes, Adams
- Neutral to state size: Webster, Huntington-Hill