PH 201-4A spring 2007

Dynamics of Uniform Circular Motion

Lectures 11-13

Chapter 5
(Cutnell & Johnon, Physics 7th edition)
Uniform circular motion is motion with constant speed along a circular path.

All the velocity vectors have the same magnitude (speed) but they differ in direction.

Sometimes it is more convenient to describe uniform circular motion by specifying the period of the motion, rather than the speed.

The period $T$ is the time required to travel once around the circle, that is to make one complete revolution

$$v = \frac{2\pi r}{T}$$

Sometimes we use frequency of rotation $f$. $T = \frac{1}{f}$

Uniform circular motion is accelerated motion.
Centripetal acceleration

\[
\frac{\text{magn. of } \Delta \mathbf{v}}{\text{magn. of } \Delta \mathbf{r}} = \frac{v}{r}
\]

\[
\text{magn. of } \Delta \mathbf{v} = \frac{v}{r} \times \text{dist. traveled in time } \Delta t
\]

\[
\mathbf{a} = \frac{\text{magn. of } \Delta \mathbf{v}}{\Delta t} = \frac{v}{r} \times \frac{\text{dist. traveled in time } \Delta t}{\Delta t}
\]

\[
\mathbf{a} = \frac{v}{r} \times v
\]

\[
\mathbf{a} = \frac{v^2}{r}
\]

\(\mathbf{a}\) points along the radius, toward the center of the circle.
\[ a = \frac{v^2}{r} \]

Instantaneous acceleration vectors for a particle in uniform circular motion.
An ultracentrifuge spins a small test tube in a circle of radius 10 cm at 1000 revolutions per second. What is the centripetal acceleration of the test tube?

\[
\text{Given: uniform circular motion}
\]

\[ t = 10 \text{ cm} = 0.1 \text{ m} \]

\[ f = 1000 \text{ rev/s} \]

Find \( \alpha = ? \)

1. \[ v = \frac{\text{distance travel}}{\text{time taken}} = \frac{2\pi t}{T} = \frac{0.63 \text{ m}}{10^{-3} \text{ s}} = 630 \text{ m/s} \]

2. One revolution corresponds to the circumference of the circular path.

\[ 2\pi r = 2\pi \times 0.1 = 0.63 \text{ m} \]

3. Time taken for one revolution \( T = \frac{1}{f} = \frac{1}{1000} = 10^{-3} \text{ s} \)

4. \[ a = \frac{v^2}{r} = \frac{(630 \times 10^3 \text{ m/s})^2}{0.1 \text{ m}} = 4.0 \times 10^6 \text{ m/s}^2 \]
The Earth moves around the Sun in a circular path of radius $1.50 \times 10^{11}$ m at uniform speed. What is the magnitude of the centripetal acceleration of the Earth toward the Sun?

Given: uniform motion

$T = 365.25 \times \frac{24 \times 60 \times 60}{14}$ sec

$= 3.156 \times 10^7$ sec

Find $a = ?$

$v = \frac{2\pi r}{T} = \frac{2\pi \times 1.50 \times 10^{11}}{3.156 \times 10^7} = 2.986 \times 10^4$ m/s

$a = \frac{v^2}{r} = \frac{(2.986 \times 10^4 \text{ m/s})^2}{1.50 \times 10^{11} \text{ m}} = 5.94 \times 10^{-3}$ m/s$^2$
Concept at a glance

External Forces
1. Gravitational Force
2. Normal Force
3. Friction Force
4. Tension Force

Newton’s Second Law
\[ \sum F = ma \]

Centripetal Acceleration
\[ a_c = \frac{v^2}{r} \]

Centripetal force \( F_c \)

2\textsuperscript{nd} Newton’s Law indicates that whenever an object accelerates, there must be a net force to create the acceleration.

The net force causing the centripetal acceleration is called the centripetal force \( F_c \) and points in the same direction as acceleration, toward the center of the circle.
\[ a = \frac{v^2}{r} \]
\[ F_c = ma = \frac{mv^2}{r} \]

**Magnitude:** Centripetal force is the net force required to keep an object of mass “m” moving at a speed \( v \), on a circular path of radius \( r \) and has a magnitude of \( F_c = \frac{mv^2}{r} \).

**Direction:** It always points toward the center of the circle.
We must exert a pull toward the center of the circle to produce a centripetal acceleration and to keep the stone in uniform circular motion.
A “swing” ride at a carnival consists of chairs that are swung in a circle by 12.0 m cables attached to a vertical rotating pole, as the drawing shows. Suppose the total mass of a chair and its occupant is 220kg.

a) Determine the tension in the cable attached to the chair.

b) Find the speed of the chair.
Centripetal Force

What is the minimum coefficient of static friction necessary to allow a penny to rotate along with a 33 1/3 rpm record (diameter = 0.300 m) when the penny is placed at the outer edge of the record?

Friction provides the centripetal force holding the penny in the circular path.

\[
\frac{v^2}{r} = 2\pi f = 2\pi \left( \frac{33.3}{60} \right)
\]

\[
0.523 \text{ m/s} = \frac{v^2}{r}
\]

\[
\mu_s = \frac{v^2}{2g} = \frac{(0.523 \text{ m/s})^2}{2 \times 9.80 \text{ m/s}^2} = 0.186
\]
Banked curves

When a car travels without skidding around an unbanked curve, the static frictional force between the tires and the road provides the centripetal force. The reliance on friction can be eliminated completely for a given speed, however, if the curve is banked at an angle relative to the horizontal.

Friction free banked curve

\[ X: F_c = F_N \sin \theta = \frac{mv^2}{r} \quad (1) \]
\[ Y: F_N \cos \theta - mg = 0; \quad F_N \cos \theta = mg \quad (2) \]

\[(1) : (2) \rightarrow F_N \sin \theta / F_N \cos \theta = (mv^2/r) / mg \]
\[ \tan \theta = \frac{v^2}{rg} \]

For a given speed \( v \), the centripetal force needed for a turn of radius \( r \) can be obtained from the normal force by banking the turn at an angle \( \theta \), independent of the mass of the vehicle.
Banked Curves

A racetrack has the shape of an inverted cone, as the drawing shows. On this surface the cars race in circles that are parallel to the ground. For a speed of 2.70 m/s, at what value of the distance \( d \) should a driver locate his car if he wishes to stay on a circular path without depending on friction?

The radius of the circle in which the car travels is
\[
\mathbf{r} = d \cos \theta = \frac{2.70 \text{ m/s}}{\sin \theta}.
\]

In the horizontal direction,

\[
N \sin \theta = \frac{m v^2}{r} \quad \Rightarrow \quad \theta = \sin^{-1} \left( \frac{m v^2}{N r} \right).
\]

In the vertical direction,

\[
N \cos \theta - mg = 0 \quad \Rightarrow \quad N = \frac{mg}{\cos \theta} = \frac{v^2}{\sin \theta}.
\]

\[
\theta = \sin^{-1} \left( \frac{m v^2}{N r} \right) = \sin^{-1} \left( \frac{2.70 \text{ m/s}}{\frac{v^2}{\sin \theta}} \right) = \sin^{-1} \left( \frac{2.70 \text{ m/s}}{\frac{v^2}{\sin \theta}} \right) = \sin^{-1} \left( \frac{2.70 \text{ m/s}}{\frac{v^2}{\sin \theta}} \right) = 105 \text{ m}.
\]
Vertical circular motion

A motorcycle driver drives his cycle around a vertical circular track with a constant speed $v$. Find the apparent weight of the cycle plus rider at points 1, 2, 3, 4, 5. Total mass (rider + cycle) = $m$

1. $N_1 - mg = \frac{mv^2}{r}; \quad \{W_{A1} = |N_1| = m\left(\frac{v^2}{r} + g\right)\}$
2. $N_2 = \frac{mv^2}{r}; \quad \{W_{A2} = |N_2| = \frac{mv^2}{r}\}$
3. $N_3 + mg = \frac{mv^2}{r}; \quad \{W_{A3} = |N_3| = \frac{mv^2}{r} - g\}$
4. $N_4 = \frac{mv^2}{r}; \quad \{W_{A4} = |N_4| = \frac{mv^2}{r}\}$
5. $N_5 + mg \cos 45^\circ = \frac{mv^2}{r}; \quad \{W_{A5} = |N_5| = m\left(\frac{v^2}{r} - g \cos 45^\circ\right)\}$
What is the force of a 700 kg sports car on the road surface as it goes 10 m/s over the crest of a hill having a radius of curvature 35 m measured in the vertical plane?

\[ mg - N = \frac{mv^2}{r} \]

\[ N = mg - \frac{mv^2}{r} = m(g - \frac{v^2}{r}) = \]

\[ = (700 \text{ kg}) \left[ 9.81 \text{ m/s}^2 - \frac{(10 \text{ m/s})^2}{35 \text{ m}} \right] = 4867 \text{ N} \]

According to the 2nd Newton's Law force of the 700 kg sports car on the road surface (weight) is equal to tangential reaction of the surface \( N \) and opposite in direction.

\[ \Rightarrow \ W_{AP} = 4867 \text{ N} \]
Circular orbit of a planet around the Sun

\[ F = \frac{GM_s m}{r^2} \quad F = ma = \frac{mv^2}{r} \]

\[ \Rightarrow v^2 = \frac{GM_s}{r} \quad v = \sqrt{\frac{GM_s}{r}} \]

There is only one speed that a satellite can have if the satellite is to remain in an orbit with a fixed radius.

\[ v = \frac{2\pi r}{T} \]

\[ T \text{ – period (time for 1 rev.)} \]

\[ 4\pi^2 r^2 / T^2 = \frac{GM_s}{r} \]

\[ T^2 = \frac{4\pi^2 r^3}{GM_s} \]
A communication satellite in a circular orbit around the Earth

\[ T = 1 \text{ day} \]
\[ r = ? \]

Synchronous geostationary orbit

\[ T^2 = \frac{4\pi^2 r^3}{GM_E} \]
\[ \Rightarrow \]
\[ r = \left( \frac{GM_E T^2}{4\pi^2} \right)^{\frac{1}{3}} = \left[ \frac{6.67 \times 10^{-11} \text{ Nm}^2/\text{kg}^2 \times 5.98 \times 10^24 \times (24 \times 60 \times 60)}{4\pi^2} \right]^{\frac{1}{3}} \approx 4.23 \times 10^7 \text{ m} \]

Since the radius of the earth is \( \sim 6.38 \times 10^7 \text{ m} \) the height above the earth’s surface is \( H = 4.23 \times 10^7 - 0.64 \times 10^7 = 3.59 \times 10^7 \text{ m} \) !!!
Global Positioning System (GPS)

Used to determine the position of an object to within 15 m or less

- Each GPS satellite carries a highly accurate atomic clock whose time is transmitted to the ground continually by means of radio waves
- A car carries a computerized GPS receiver that can detect the waves and is synchronized to the satellite clock.
- The receiver determines the distance between the car and the satellite from the knowledge of the travel time of the waves and the speed of light.
Imagine that somewhere in interstellar space a small pebble is in a circular orbit around a spherical asteroid of mass 1000kg. If the radius of the circular orbit is 1km, what is the period of the motion.

Given:
\[ t = 10^3 \text{m} \]
\[ M_A = 10^3 \times 2 \]
Find \( T = ? \)

\[
F = \frac{G M_A m}{r^2} = \frac{m v^2}{r} \quad \Rightarrow \quad v^2 = \frac{G M_A}{r} \quad \Rightarrow \quad v = \sqrt{\frac{G M_A}{r}}
\]

\[
v = \frac{2\pi r}{T} \quad \Rightarrow \quad T = \frac{2\pi r}{v} = \frac{2\pi r \sqrt{r}}{v} = \frac{2\pi r \sqrt{r}}{\sqrt{G M_A}} = \sqrt{\frac{4\pi^2 r^3}{G M_A}} = \sqrt{\frac{39.438 \times (10^3)^2}{(6.67 \times 10^{-11})(10^3)}} = 7.7 \times 10^5 = 7.7 \times 10^8 \times \frac{1 \text{min}}{60 \text{s}} \times \frac{1 \text{hr}}{60 \text{min}} \times \frac{1 \text{day}}{24 \text{hr}} \times \frac{1 \text{year}}{365 \text{day}} = 24 \text{years}
\]
Apparent Weightlessness and Free Fall
Artificial Gravity

The surface of the rotating space station pushes on an object with which is in contact and thereby provides the centripetal force that keeps the object moving along a circular path.

At what speed must the surface of the space station $r = 1700$ m move so that the astronaut at point $P$ experiences a push on his feet that equals his earth weight?

$$F_c = mg = \frac{mv^2}{r}$$

$$v = \sqrt{\frac{gr}{}} = \sqrt{1700 \times 9.80 \times \frac{m}{s^2}} = 130 \frac{m}{s}$$
Artificial Gravity Problem 35: To create artificial gravity, the space station shown in the drawing is rotating at a rate of 1.00 rpm. The radii of the cylindrically shaped chambers have the ratio $r_a/r_b = 4.00$. Each chamber A simulates an acceleration due to gravity of 10.0 m/s$^2$. Find values for:

a) $r_a$  
b) $r_b$  
c) acceleration due to gravity that is simulated in Chamber B

\[
|W_{AP}| = |N| = \frac{mv_{AS}^2}{r_A}
\]
(a) \[ A_A = \frac{V_{B_A}^2}{2A} \]
\[ V_{AS} = \frac{2\pi Z_A}{f} \]
\[ A_A = \frac{4\pi^2 Z_A f^2}{Z_A} \]
\[ Z_A = \frac{A_A}{4\pi^2 f^2} = \frac{(10.0 \text{ m/s}^2)}{4\pi^2 \left(\frac{1.60 \text{ min}}{60}\right)} = \boxed{912 \text{ m}} \]

(b) \[ Z_B = \frac{Z_A}{4.80} = \boxed{228 \text{ m}} \]

(c) \[ A \text{ point on the rim of chamber } B \]
\[ A \text{ has a centripetal acceleration} \]
\[ A_B = \frac{V_{B_A}^2}{2B} \]
\[ V_{B_A} = 2\pi Z_A f \]
\[ A_B = \frac{4\pi^2 Z_B f}{Z_B} = \frac{4\pi^2 (228 \text{ m})}{60 \text{ s}} = \boxed{2.50 \text{ m/s}^2} \]
Kepler’s First Law
The orbits of the planets are ellipses with Sun at one focus.

The sum of the distances from one focus and the other focus is the same for all points on the ellipse.
Kepler’s Second Law

The radial line segment from the Sun to the planet sweeps out equal areas in equal times.
The speed of a planet at aphelion and perihelion

\[ PQ = u_1 e = \frac{1}{5} u_1 \]
\[ \text{area } SPQ = \frac{1}{2} r_1 u_1 \]
\[ \text{area } SQ'P' = \frac{1}{2} r_2 u_2 \]

\[ \frac{1}{2} r_2 u_1 = \frac{1}{2} r_2 u_2 \]

\[ \frac{u_2}{u_1} = \frac{r_2}{r_1} \]
Kepler’s Third Law

The square of the period is proportional to the cube of the semi-major axis of the planetary orbit.
Calculate the orbital period of Sputnik 1 from its apogee distance of $7.33 \times 10^3$ km and perigee distance of $6.60 \times 10^3$ km. The square of the period is proportional to the cube of the semimajor axis of the planetary orbit.

\[
T^2 = 4\pi^2 \frac{a^3}{GM}
\]

\[
a = \frac{r_p + r_o}{2} = \frac{6.60 \times 10^6 + 7.33 \times 10^6}{2}
\]

\[
T = 2\pi \sqrt{\frac{a^3}{GM}} = 2\pi \sqrt{\frac{(6.97 \times 10^5)^3}{(6.67 \times 10^{-11})(5.98 \times 10^{24})}} = 5790 s = 96.5 \text{ min}
\]
For a trip to Mars, we want to place a spacecraft into an elliptical orbit such that its perihelion coincides with mean Earth-Sun distance, but its aphelion coincides with the mean Mars-Sun distance. What is the period of such an elliptical orbit? How long does the spacecraft take to coast from the perihelion (at the Earth) to the aphelion (at Mars)?

\[
T = 2\pi \sqrt{\frac{\frac{3}{2}}{G M_s}} = 2\pi \sqrt{\frac{(r_p^3 + r_a^3)}{6 M_s}} = 2\pi \sqrt{\frac{(1.89 \times 10^{11} m)^3}{(6.67 \times 10^{-11} N\cdot m^2/kg^2)(1.99 \times 10^{30} kg)}} = 4.48 \times 10^7 \text{ s} = 1.42 \text{ year} \quad \text{and} \quad t = \frac{1.42}{2} = 0.7 \text{ year}
\]

Given:
\[
\begin{align*}
r_p &= 150 \times 10^9 \text{ m} \\
r_a &= 228 \times 10^9 \text{ m} \\
M_s &= 1.99 \times 10^{30} \text{ kg}
\end{align*}
\]