PH 221-3A Fall 2007

Motion in Two and Three Dimensions

Lectures 4, 5

Chapter 4
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
In this chapter we will continue to study the motion of objects without the restriction we put in chapter 2 to move along a straight line. Instead we will consider motion in a plane (two dimensional motion) and motion in space (three dimensional motion). The following vectors will be defined for two- and three- dimensional motion:

- Displacement
- Average and instantaneous velocity
- Average and instantaneous acceleration

We will consider in detail projectile motion and uniform circular motion as examples of motion in two dimensions.

Finally we will consider relative motion, i.e. the transformation of velocities between two reference systems which move with respect to each other with constant velocity.
Position Vector

The position vector $\vec{r}$ of a particle is defined as a vector whose tail is at a reference point (usually the origin O) and its tip is at the particle at point P.

Example: The position vector in the figure is:

\[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]

\[ \vec{r} = \left( -3\hat{i} + 2\hat{j} + 5\hat{k} \right) m \]
Displacement Vector

For a particle that changes position vector from $\mathbf{r}_1$ to $\mathbf{r}_2$, we define the displacement vector $\Delta \mathbf{r}$ as follows:

$$\Delta \mathbf{r} = \mathbf{r}_2 - \mathbf{r}_1$$

The position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ are written in terms of components as:

$$\mathbf{r}_1 = x_1 \hat{i} + y_1 \hat{j} + z_1 \hat{k}$$

$$\mathbf{r}_2 = x_2 \hat{i} + y_2 \hat{j} + z_2 \hat{k}$$

The displacement $\Delta \mathbf{r}$ can then be written as:

$$\Delta \mathbf{r} = (x_2 - x_1) \hat{i} + (y_2 - y_1) \hat{j} + (z_2 - z_1) \hat{k} = \Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k}$$

$$\Delta x = x_2 - x_1$$

$$\Delta y = y_2 - y_1$$

$$\Delta z = z_2 - z_1$$
Problem 2. A watermelon seed has the following coordinates: $x = -5.0m$, $y = 8.0m$, $z = 0m$. Find its position vector (a) in unit-vector notation and as (b) a magnitude and (c) an angle relative to the positive direction of the x axis. (d) Sketch the vector on a right handed coordinate system. If the seed is moved to the xyz coordinates (3.00m, 0m, 0m) what is its displacement (e) in unit-vector notation as as (f) a magnitude and (g) an angle relative to the positive x direction?

(a) The position vector, according to is $\mathbf{r} = x\hat{i} + y\hat{j} + z\hat{k}$ is $\mathbf{r} = (-5.0 \text{ m})\hat{i} + (8.0 \text{ m})\hat{j}$.

(b) The magnitude is $|\mathbf{r}| = \sqrt{x^2 + y^2 + z^2} = \sqrt{(-5.0 \text{ m})^2 + (8.0 \text{ m})^2 + (0 \text{ m})^2} = 9.4 \text{ m}$.

(c) Many calculators have polar $\leftrightarrow$ rectangular conversion capabilities which make this computation more efficient than what is shown below. Noting that the vector lies in the $xy$ plane and using $\tan \theta = \frac{y}{x}$, we obtain:

$$\theta = \tan^{-1}\left(\frac{8.0 \text{ m}}{-5.0 \text{ m}}\right) = -58^\circ \text{ or } 122^\circ$$

where the latter possibility ($122^\circ$ measured counterclockwise from the $+x$ direction) is chosen since the signs of the components imply the vector is in the second quadrant.

(d) The sketch is shown on the right. The vector is $122^\circ$ counterclockwise from the $+x$ direction.

(e) The displacement is $\Delta\mathbf{r} = \mathbf{r}' - \mathbf{r}$ where $\mathbf{r}$ is given in part (a) and $\mathbf{r}' = (3.0 \text{ m})\hat{i}$. Therefore, $\Delta\mathbf{r} = (8.0 \text{ m})\hat{i} - (8.0 \text{ m})\hat{j}$.

(f) The magnitude of the displacement is $|\Delta\mathbf{r}| = \sqrt{(8.0 \text{ m})^2 + (-8.0 \text{ m})^2} = 11 \text{ m}$.

(g) The angle for the displacement, using $\tan \theta = \frac{y}{x}$, is

$$\tan^{-1}\left(\frac{8.0 \text{ m}}{-8.0 \text{ m}}\right) = -45^\circ \text{ or } 135^\circ$$

where we choose the former possibility ($-45^\circ$, or $45^\circ$ measured clockwise from $+x$) since the signs of the components imply the vector is in the fourth quadrant. A sketch of $\Delta\mathbf{r}$ is shown on the right.
Average and Instantaneous Velocity

Following the same approach as in chapter 2 we define the average velocity as:

\[
\text{average velocity} = \frac{\text{displacement}}{\text{time interval}}
\]

\[
\vec{v}_{\text{avg}} = \frac{\Delta \vec{r}}{\Delta t} = \frac{\Delta x \hat{i} + \Delta y \hat{j} + \Delta z \hat{k}}{\Delta t} = \frac{\Delta x \hat{i}}{\Delta t} + \frac{\Delta y \hat{j}}{\Delta t} + \frac{\Delta z \hat{k}}{\Delta t}
\]

We define as the instantaneous velocity (or more simply the velocity) as the limit:

\[
\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} = \frac{d \vec{r}}{dt}
\]
If we allow the time interval $\Delta t$ to shrink to zero, the following things happen:

1. Vector $\vec{r}_2$ moves towards vector $\vec{r}_2$ and $\Delta \vec{r} \to 0$

2. The direction of the ratio $\frac{\Delta \vec{r}}{\Delta t}$ (and thus $\vec{v}_{avg}$) approaches the direction of the tangent to the path at position 1

3. $\vec{v}_{avg} \to \vec{v}$

$$\vec{v} = \frac{d}{dt} \left( x\hat{i} + y\hat{j} + z\hat{k} \right) = \frac{dx}{dt}\hat{i} + \frac{dy}{dt}\hat{j} + \frac{dz}{dt}\hat{k} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k}$$

The three velocity components are given by the equations:

\[
\begin{align*}
    v_x &= \frac{dx}{dt} \\
    v_y &= \frac{dy}{dt} \\
    v_z &= \frac{dz}{dt}
\end{align*}
\]
Problem 6. An electron position is given by \( r = 3.00t \hat{i} - 4.00t^2 \hat{j} + 2.00 \hat{k} \), with \( t \) in seconds and \( \vec{r} \) in meters. (a) In unit-vector notation, what is the electron's velocity \( \vec{v}(t) \)? At \( t = 2.00 \) s, what is \( \vec{v} \) (b) in unit-vector notation and as (c) a magnitude and (d) an angle relative to the positive direction of the \( x \) axis?

To emphasize the fact that the velocity is a function of time, we adopt the notation \( v(t) \) for \( dx/dt \).

(a) Eq. \[ \vec{v} = \frac{d\vec{r}}{dt} \]
leads to
\[ v(t) = \frac{d}{dt} (3.00t \hat{i} - 4.00t^2 \hat{j} + 2.00 \hat{k}) = (3.00 \text{ m/s}) \hat{i} - (8.00 \text{ m/s}) \hat{j} \]

(b) Evaluating this result at \( t = 2.00 \) s produces \( \vec{v} = (3.00 \hat{i} - 16.0 \hat{j}) \) m/s.

(c) The speed at \( t = 2.00 \) s is \( v = |\vec{v}| = \sqrt{(3.00 \text{ m/s})^2 + (-16.0 \text{ m/s})^2} = 16.3 \text{ m/s} \).

(d) The angle of \( \vec{v} \) at that moment is
\[ \tan^{-1} \left( \frac{-16.0 \text{ m/s}}{3.00 \text{ m/s}} \right) = -79.4^\circ \text{ or } 101^\circ \]
where we choose the first possibility (79.4° measured \textit{clockwise} from the +x direction, or 281° counterclockwise from +x) since the signs of the components imply the vector is in the fourth quadrant.
Average and Instantaneous Acceleration

The average acceleration is defined as:

\[
\text{average acceleration} = \frac{\text{change in velocity}}{\text{time interval}}
\]

\[
\vec{a}_{\text{avg}} = \frac{\vec{v}_2 - \vec{v}_1}{\Delta t} = \frac{\Delta \vec{v}}{\Delta t}
\]

We define as the instantaneous acceleration as the limit:

\[
\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} = \frac{d \vec{v}}{dt} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}
\]

\[
\Delta t \to 0
\]

**Note:** Unlike velocity, the acceleration vector does not have any specific relationship with the path.

The three acceleration components are given by the equations:

\[
a_x = \frac{dv_x}{dt} \quad a_y = \frac{dv_y}{dt} \quad a_z = \frac{dv_z}{dt} \quad \vec{a} = \frac{d\vec{v}}{dt}
\]
Problem 13. The position $\mathbf{r}$ of a particle moving in an xy plane is given by $\mathbf{r} = (2.00t^3 - 5.00t) \hat{i} + (6.00 - 7.00t^4) \hat{j}$, with $\mathbf{r}$ in meters and $t$ in seconds. In unit-vector notation, calculate (a) $\mathbf{r}$, (b) $\mathbf{v}$, (c) $\mathbf{a}$ for $t = 2.00$ s. (d) What is the angle between the positive direction of the $x$ axis and a line tangent to the particle at $t=2.00$ s?

In parts (b) and (c), we use $\mathbf{v} = \frac{d\mathbf{r}}{dt}$ and $a_x = \frac{dv_x}{dt}$. For part (d), we find the direction of the velocity computed in part (b), since that represents the asked-for tangent line.

(a) Plugging into the given expression, we obtain

$$\mathbf{r} \bigg|_{t=2.00} = [2.00(8) - 5.00(2)] \hat{i} + [6.00 - 7.00(16)] \hat{j} = (6.00 \hat{i} - 106 \hat{j}) \text{ m}$$

(b) Taking the derivative of the given expression produces

$$\mathbf{v}(t) = (6.00t^2 - 5.00) \hat{i} - 28.0t^3 \hat{j}$$

where we have written $v(t)$ to emphasize its dependence on time. This becomes, at $t = 2.00$ s, $\mathbf{v} = (19.0 \hat{i} - 224 \hat{j}) \text{ m/s}$.

(c) Differentiating the $\mathbf{v}(t)$ found above, with respect to $t$ produces $12.0t \hat{i} - 84.0t^2 \hat{j}$, which yields $\mathbf{a} = (24.0 \hat{i} - 336 \hat{j}) \text{ m/s}^2$ at $t = 2.00$ s.

(d) The angle of $\mathbf{v}$, measured from $+x$, is either

$$\tan^{-1} \left( \frac{-224 \text{ m/s}}{19.0 \text{ m/s}} \right) = -85.2^\circ \text{ or } 94.8^\circ$$

where we settle on the first choice ($-85.2^\circ$, which is equivalent to $275^\circ$ measured counterclockwise from the $+x$ axis) since the signs of its components imply that it is in the fourth quadrant.
Projectile Motion

The motion of an object in a vertical plane under the influence of gravitational force is known as “projectile motion”

The projectile is launched with an initial velocity $\vec{v}_o = v_{ox}\hat{i} + v_{oy}\hat{j}$

The horizontal and vertical velocity components are:

$\nu_{ox} = \nu_o \cos \theta_o \quad \nu_{oy} = \nu_o \sin \theta_o$

Projectile motion will be analyzed in a horizontal and a vertical motion along the x- and y-axes, respectively. These two motions are independent of each other. Motion along the x-axis has zero acceleration. Motion along the y-axis has uniform acceleration $a_y = -g$
Horizontal Motion: \( a_x = 0 \)  
The velocity along the x-axis does not change
\[
v_x = v_0 \cos \theta_0 \quad \text{(eqs. 1)}
\]
\[
x - x_o = v_{ox} t = (v_0 \cos \theta_0) t \quad \text{(eqs. 2)}
\]

Vertical Motion: \( a_y = -g \)  
Along the y-axis the projectile is in free fall
\[
v_y = v_0 \sin \theta_0 - gt \quad \text{(eqs. 3)}
\]
\[
y - y_o = v_{oy} t - \frac{gt^2}{2} = (v_0 \sin \theta_0) t - \frac{gt^2}{2} \quad \text{(eqs. 4)}
\]
If we eliminate \( t \) between equations 3 and 4 we get:
\[
v_y^2 = (v_0 \sin \theta_0)^2 - 2g(y - y_o)
\]

Here \( x_o \) and \( y_o \) are the coordinates of the launching point. For many problems the launching point is taken at the origin. In this case \( x_o = 0 \) and \( y_o = 0 \)

Note: In this analysis of projectile motion we neglect the effects of air resistance
The equation of the path:

\[ x = (v_o \cos \theta_o)t \]  
\[ y = (v_o \sin \theta_o)t - \frac{gt^2}{2} \]

If we eliminate \( t \) between equations 2 and 4 we get:

\[ y = (\tan \theta_o)x - \frac{g}{2(v_o \cos \theta_o)^2}x^2 \]

This equation describes the path of the motion.

The path equations has the form: \( y = ax + bx^2 \)  
This is the equation of a parabola.

Note: The equation of the path seems too complicated to be useful. Appearances can deceive: Complicated as it is, this equation can be used as a short cut in many projectile motion problems.
\[ v_x = v_0 \cos \theta_0 \quad \text{(eqs.1)} \]
\[ x = \left( v_0 \cos \theta_0 \right) t \quad \text{(eqs.2)} \]
\[ v_y = v_0 \sin \theta_0 - gt \quad \text{(eqs.3)} \]
\[ y = \left( v_0 \sin \theta_0 \right) t - \frac{gt^2}{2} \quad \text{(eqs.4)} \]

**Horizontal Range:** The distance OA is defined as the horizontal range \( R \)

At point A we have: \( y = 0 \)  
From equation 4 we have:

\[
\left( v_0 \sin \theta_0 \right) t - \frac{gt^2}{2} = 0 \quad \rightarrow \quad t \left( v_0 \sin \theta_0 - \frac{gt}{2} \right) = 0 \quad \text{This equation has two solutions:}
\]

Solution 1. \( t = 0 \)  
This solution correspond to point O and is of no interest

Solution 2. \( v_0 \sin \theta_0 - \frac{gt}{2} = 0 \)  
This solution correspond to point A

From solution 2 we get: \( t = \frac{2v_0 \sin \theta_0}{g} \)  
If we substitute \( t \) in eqs.2 we get:

\[ R = \frac{2v_0^2}{g} \sin \theta_0 \cos \theta_0 = \frac{v_0^2}{g} \sin 2\theta_0 \]

\( R \) has its maximum value when \( \theta_0 = 45^\circ \)

\[ R_{max} = \frac{v_0^2}{g} \]
Maximum height $H$

$$H = \frac{v_0^2 \sin^2 \theta_0}{2g}$$

The $y$-component of the projectile velocity is: $v_y = v_0 \sin \theta_0 - gt$

At point A: $v_y = 0 \quad \rightarrow \quad v_0 \sin \theta_0 - gt \quad \rightarrow \quad t = \frac{v_0 \sin \theta_0}{g}$

$$H = y(t) = \left(v_0 \sin \theta_0\right) t - \frac{gt^2}{2} = \left(v_0 \sin \theta_0\right) \frac{v_0 \sin \theta_0}{g} - \frac{g}{2} \left(\frac{v_0 \sin \theta_0}{g}\right)^2 \rightarrow$$

$$H = \frac{v_0^2 \sin^2 \theta_0}{2g}$$
Maximum height $H$ (encore)

We can calculate the maximum height using the third equation of kinematics for motion along the $y$-axis:

$$v_y^2 = v_{yo}^2 + 2a(y - y_o)$$

In our problem: $y_o = 0$, $y = H$, $v_{yo} = v_o \sin \theta_o$, $v_y = 0$, and $a = -g$ →

$$-v_{yo}^2 = -2gH \rightarrow H = \frac{v_{yo}^2}{2g} = \frac{v_o^2 \sin^2 \theta_o}{2g}$$
Problem: A car drives straight off the edge of a cliff that is 54m high. The police at the scene of the accident note that the point of impact is 130m from the base of a cliff. How fast was the car traveling when it went over the cliff?

During this time the car travels a horizontal distance of 130m

\[ t = \sqrt{\frac{2\Delta y}{g}} = \sqrt{\frac{2(-54m)}{9.80 \text{ m/s}^2}} = 3.8 \text{ s} \]

\[ \frac{x}{t} = \frac{130 \text{ m}}{3.8 \text{ s}} = 34.2 \text{ m/s} \]
Problem: A diver springs upward from a board that is three meters above the water. At the instant she contacts the water her speed is 8.90 m/s and her body makes an angle of 75.0 degrees with respect to the horizontal surface of the water. Determine her initial velocity, both magnitude and direction.

Given:

$V = 8.90 \text{ m/s}$

**X Direction**

- $a_x = 0$
- $v_x = \text{const}$
- $x = v_0 x t$

**Y Direction**

<table>
<thead>
<tr>
<th>$y$</th>
<th>$g$</th>
<th>$v_{0y}$</th>
<th>$v_y$</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>$9.81$</td>
<td>?</td>
<td>$8.90 \times \sin 75.0^\circ$</td>
<td></td>
</tr>
</tbody>
</table>

Consider the $X$ direction:

$v_x = V \cos 75.0^\circ = (8.90 \text{ m/s}) \cos 75.0^\circ = 2.80 \text{ m/s}$

2) Consider the $Y$ direction:

$v_y^2 = v_{0y}^2 + 2gy$

$v_{0y}^2 = v_y^2 + 2gy = (8.90 \text{ m/s})^2 \sin^2 75.0^\circ + 2(9.81 \text{ m/s}^2)(-3.00 \text{ m})$

$\Rightarrow v_y = 3.89 \text{ m/s}$

3) $v_y = \sqrt{(2.30 \text{ m/s})^2 + (3.89 \text{ m/s})^2} = 4.52 \text{ m/s}$

4) $\theta = \tan^{-1} \left(\frac{v_y}{v_x}\right) = 59.4^\circ$
The motion of a ballistic missile can be regarded as the motion of a projectile because along the greatest part of its trajectory the missile is in free fall. Suppose that a missile is to strike a target 1000 km away. What minimum speed must the missile have at the beginning of its trajectory? What maximum height does it reach when launched with this minimum speed? How long does it take to reach the target? For these calculations assume that \( g = 9.8 \text{ m/s}^2 \) everywhere along the trajectory.

\[
\begin{align*}
given:
  x_{\text{max}} &= 1000 \text{ km} = 10^6 \text{ m} \\
g &= -9.8 \text{ m/s}^2 \\
\theta &= 45^\circ \\
\text{Find:}
  \quad \nu_0 = ? \\
  \quad y_{\text{max}} = ? \\
  \quad t_f = ? \\
\end{align*}
\]

\( \nu_0 = \frac{\sqrt{2g x_{\text{max}}}}{\sin \theta} \), \( \nu_0 = \nu_{0, \text{min}} \), which \( \sin \theta \) is max 
\( \Rightarrow \sin \theta = 1 \) 
\( \theta = 45^\circ \)

\( \nu_0 = \sqrt{2 \times 9.8 \times 10^6 \sin^2 45^\circ} = 3.13 \times 10^3 \text{ m/s} \)

\( t_f = \frac{\nu_y t_h}{g} \Rightarrow t_f = \frac{\nu_y}{g} \)

\( y_{\text{max}} = \nu_y t_h - \frac{1}{2} g t_h^2 = \nu_y^2 - \frac{1}{2} \frac{\nu_y^2}{g} - \frac{2 \nu_y^2}{2g} = \frac{\nu_y^2}{2} - \frac{2 \nu_y^2}{2 \times 9.8} = \frac{2 \times 13 \times 10^3 \sin 45^\circ}{2 \times 9.8} = 2.5 \times 10^3 \text{ m} \)

\( t_f = 2 t_h = \frac{2 \nu_y}{g} = \frac{2 \times 313 \times 10^3 \times \sin 45^\circ}{9.8} = 4525 \text{ s} \approx 7.5 \text{ min} \)
**Problem:** A projectile is fired at an initial velocity of 35.0 m/s at an angle of 30.0 degrees above the horizontal from the roof of a building 30.0 m high, as shown. Find

a) The maximum height of the projectile
b) The time to rise to the top of the trajectory
c) The total time of the projectile in the air
d) The velocity of the projectile at the ground
e) The range of the projectile

Given:
- \( v_0 = 35.0 \text{ m/s} \)
- \( \theta = 30.0^\circ \)
- \( y_0 = -30.0 \text{ m} \)

Find:
- \( y_{\text{max}} \)
- \( t_f \)
- \( v \)
- \( x_{\text{max}} \)

\[
\begin{align*}
\theta &= 30.0^\circ \\
\dot{\theta} &= 0 \\
\ddot{\theta} &= 0 \\
\dot{x} &= v_x = v_0 \cos \theta = (35.0 \text{ m/s}) \cos 30.0^\circ = 30.2 \text{ m/s} \\
\dot{y} &= v_y = v_0 \sin \theta = (35.0 \text{ m/s}) \sin 30.0^\circ = 17.5 \text{ m/s} \\
\dot{y}^2 &= -2gy \Rightarrow y_{\text{max}} = \frac{-v_y^2}{2g} = \frac{17.5^2}{2(9.80)} = 15.6 \text{ m} \\
t_f &= \frac{v_y}{-g} = \frac{17.5}{9.80} = 1.8 \text{ s} \\
y &= v_0 t - \frac{1}{2} gt^2 \\
-30.0 &= 17.5 t_f - \frac{1}{2} 9.80 t_f^2 \\
4.90 t_f^2 - 17.5 t_f - 30.0 &= 0 \\
(4.90) t_f^2 - (17.5) t_f - 30.0 &= 0 \\
\Delta = 17.5^2 - 4(4.90)(-30.0) = 17.5 \pm 2.99 \\
t_f = \frac{17.5 \pm \sqrt{17.5^2 - 4(4.90)(-30.0)}}{2(4.90)} = 4.94 \text{ s} \\
\dot{x}_{\text{max}} = v_x \cdot t_f = (30.2 \text{ m/s}) (4.94 \text{ s}) = 147 \text{ m} \\
v_f = v_y - gt_f = 17.5 - 9.80 \cdot 4.94 = -29.9 \% \\
v = \sqrt{v_x^2 + v_y^2} = \sqrt{30.2^2 + 17.5^2} = 36.3 \text{ m/s}
\end{align*}
\]
Problem 37. A ball is shot from the ground into the air. At a height of 9.1 m, its velocity is 
\( \vec{v} = (7.6 \hat{i} + 6.1 \hat{j}) \text{ m/s, with } \hat{i} \text{ horizontal and } \hat{j} \text{ upward.} \) (a) To what maximum height does the ball rise? (b) What total horizontal distance does the ball travel? What are the (c) magnitude and (d) angle (below the horizontal) of the ball's velocity just before it hits the ground?

We designate the given velocity \( \vec{v} = (7.6 \text{ m/s}) \hat{i} + (6.1 \text{ m/s}) \hat{j} \) as \( \vec{v}_i \) as opposed to the velocity when it reaches the max height \( \vec{v}_2 \) or the velocity when it returns to the ground \( \vec{v}_3 \) and take \( \vec{v}_o \) as the launch velocity, as usual. The origin is at its launch point on the ground.

(a) Different approaches are available, but since it will be useful (for the rest of the problem) to first find the initial \( y \) velocity, that is how we will proceed.

\[
\begin{align*}
v_{i,y}^2 &= v_{0,y}^2 - 2 \cdot g \cdot \Delta y \\
&= (6.1 \text{ m/s})^2 - 2(9.8 \text{ m/s}^2)(9.1 \text{ m})
\end{align*}
\]

which yields \( v_{0,y} = 14.7 \text{ m/s} \). Knowing that \( v_{2,y} \) must equal 0, we use \( \Delta y = h \) for the maximum height:

\[
\begin{align*}
v_{2,y}^2 &= v_{0,y}^2 - 2 \cdot g \cdot h \\
&= (14.7 \text{ m/s})^2 - 2(9.8 \text{ m/s}^2)h
\end{align*}
\]

which yields \( h = 11 \text{ m} \).

(b) Recalling the derivation of Eq. 4-26, but using \( v_{0,y} \) for \( v_0 \sin \theta_0 \) and \( v_{0,x} \) for \( v_0 \cos \theta_0 \), we have

\[
0 = v_{0,y}t - \frac{1}{2}gt^2, \quad R = v_{0,x}t
\]

which leads to \( R = 2v_{0,x}v_{0,y} / g \). Noting that \( v_{0,x} = 7.6 \text{ m/s} \), we plug in values and obtain

\[
R = 2(7.6 \text{ m/s})(14.7 \text{ m/s})/(9.8 \text{ m/s}^2) = 23 \text{ m}.
\]

(c) Since \( v_{3,x} = v_{1,x} = 7.6 \text{ m/s} \) and \( v_{3,y} = -v_{0,y} = -14.7 \text{ m/s} \), we have

\[
v_3 = \sqrt{v_{3,y}^2 + v_{3,x}^2} = \sqrt{(7.6 \text{ m/s})^2 + (-14.7 \text{ m/s})^2} = 17 \text{ m/s}.
\]

(d) The angle (measured from horizontal) for \( \vec{v}_3 \) is one of these possibilities:

\[
\tan^{-1} \left( \frac{-14.7 \text{ m}}{7.6 \text{ m}} \right) = -63^\circ \text{ or } 117^\circ
\]

where we settle on the first choice (\(-63^\circ\), which is equivalent to \(297^\circ\)) since the signs of its components imply that it is in the fourth quadrant.
Uniform circular Motion

A particle is in uniform circular motion if it moves on a circular path of radius $r$ with constant speed $v$. Even though the speed is constant, the velocity is not. The reason is that the direction of the velocity vector changes from point to point along the path. The fact that the velocity changes means that the acceleration is not zero. The acceleration in uniform circular motion has the following characteristics:

1. Its vector points towards the center $C$ of the circular path, thus the name “centripetal”
2. Its magnitude $a$ is given by the equation: $a = \frac{v^2}{r}$

The time $T$ it takes to complete a full revolution is known as the “period”. It is given by the equation:

$$T = \frac{2\pi r}{v}$$
\[ \vec{v} = v_x \hat{i} + v_y \hat{j} = (-v \sin \theta) \hat{i} + (v \cos \theta) \hat{j} \quad \sin \theta = \frac{y_p}{r} \quad \cos \theta = \frac{x_p}{r} \]

Here \( x_p \) and \( y_p \) are the coordinates of the rotating particle.

\[ \vec{v} = \left(-v \frac{y_p}{r}\right) \hat{i} + \left(v \frac{x_p}{r}\right) \hat{j} \quad \text{Acceleration} \quad \vec{a} = \frac{d\vec{v}}{dt} = \left(-v \frac{dy_p}{r} \frac{dt}{dt}\right) \hat{i} + \left(v \frac{dx_p}{r} \frac{dt}{dt}\right) \hat{j} \]

We note that: \( \frac{dy_p}{dt} = v_y = v \cos \theta \) and \( \frac{dx_p}{dt} = v_x = -v \sin \theta \)

\[ \vec{a} = \left(-\frac{v^2}{r} \cos \theta\right) \hat{i} + \left(-\frac{v^2}{r} \sin \theta\right) \hat{j} \quad a = \sqrt{a_x^2 + a_y^2} = \frac{v^2}{r} \sqrt{(\cos \theta)^2 + (\sin \theta)^2} = \frac{v^2}{r} \]

\[ \tan \phi = \frac{a_y}{a_x} = \frac{-\left(v^2 / r\right) \sin \theta}{-\left(v^2 / r\right) \cos \theta} = \tan \theta \rightarrow \phi = \theta \rightarrow \vec{a} \text{ points towards } C \]

\[ v_x = -v \sin \theta \quad v_y = v \cos \theta \]

\[ (\cos \theta)^2 + (\sin \theta)^2 = 1 \]
Problem 60. An earth satellite moves in a circular orbit 640 km above Earth's surface with a period of 98.0 min. What are the (a) speed and (b) magnitude of the centripital acceleration of the satellite?

We apply $T = \frac{2\pi r}{v}$ to solve for speed $v$ and $a = \frac{v^2}{r}$ to find acceleration $a$.

(a) Since the radius of Earth is $6.37 \times 10^6$ m, the radius of the satellite orbit is

$$r = (6.37 \times 10^6 + 640 \times 10^3) \text{ m} = 7.01 \times 10^6 \text{ m}.$$ 

Therefore, the speed of the satellite is

$$v = \frac{2\pi r}{T} = \frac{2\pi (7.01 \times 10^6 \text{ m})}{(98.0 \text{ min})(60 \text{ s/min})} = 7.49 \times 10^3 \text{ m/s}.$$ 

(b) The magnitude of the acceleration is

$$a = \frac{v^2}{r} = \frac{(7.49 \times 10^3 \text{ m/s})^2}{7.01 \times 10^6 \text{ m}} = 8.00 \text{ m/s}^2.$$
Relative Motion in One Dimension:

The velocity of a particle P determined by two different observers A and B varies from observer to observer. Below we derive what is known as the “transformation equation” of velocities. This equation gives us the exact relationship between the velocities each observer perceives. Here we assume that observer B moves with a known constant velocity $v_{BA}$ with respect to observer A. Observer A and B determine the coordinates of particle P to be $x_{PA}$ and $x_{PB}$, respectively.

$$x_{PA} = x_{PB} + x_{BA}$$  
Here $x_{BA}$ is the coordinate of B with respect to A.

We take derivatives of the above equation:

$$\frac{d}{dt}(x_{PA}) = \frac{d}{dt}(x_{PB}) + \frac{d}{dt}(x_{BA}) \rightarrow$$

$$v_{PA} = v_{PB} + v_{BA}$$

If we take derivatives of the last equation and take into account that $\frac{dv_{BA}}{dt} = 0 \rightarrow a_{PA} = a_{PB}$

Note: Even though observers A and B measure different velocities for P, they measure the same acceleration.
Relative Motion in Two Dimensions:
Here we assume that observer B moves with a known constant velocity \( v_{BA} \) with respect to observer A in the xy-plane.

Observers A and B determine the position vector of particle P to be \( \vec{r}_{PA} \) and \( \vec{r}_{PB} \), respectively.

\[
\vec{r}_{PA} = \vec{r}_{PB} + \vec{r}_{BA}
\]

We take the time derivative of both sides of the equation

\[
\frac{d}{dt} \vec{r}_{PA} = \frac{d}{dt} \vec{r}_{PB} + \frac{d}{dt} \vec{r}_{BA} \rightarrow \vec{v}_{PA} = \vec{v}_{PB} + \vec{v}_{BA}
\]

If we take the time derivative of both sides of the last equation we have:

\[
\frac{d}{dt} \vec{v}_{PA} = \frac{d}{dt} \vec{v}_{PB} + \frac{d}{dt} \vec{v}_{BA} \quad \text{If we take into account that} \quad \frac{d\vec{v}_{BA}}{dt} = 0 \rightarrow \vec{a}_{PA} = \vec{a}_{PB}
\]

Note: As in the one dimensional case, even though observers A and B measure different velocities for P, they measure the same acceleration.
Problem 74. A light plane attain an airspeed of 500 km/h. The pilot sets out for a destination 800 km due north but discovers that the plane must be headed 20.0° east of due north to fly there directly. The plane arrives in 2.00 h. What were the (a) magnitude and (b) direction of the wind velocity?

The destination is \( \vec{D} = 800 \text{ km } \hat{j} \) where we orient axes so that +\( y \) points north and +\( x \) points east. This takes two hours, so the (constant) velocity of the plane (relative to the ground) is \( \vec{v}_{pg} = (400 \text{ km/h}) \hat{j} \). This must be the vector sum of the plane’s velocity with respect to the air which has \( (x, y) \) components \((500 \cos 70°, 500 \sin 70°)\) and the velocity of the air (wind) relative to the ground \( \vec{v}_{ag} \). Thus,

\[
(400 \text{ km/h}) \hat{j} = (500 \text{ km/h}) \cos 70° \hat{i} + (500 \text{ km/h}) \sin 70° \hat{j} + \vec{v}_{ag}
\]

which yields

\[
\vec{v}_{ag} = (-171 \text{ km/h}) \hat{i} - (70.0 \text{ km/h}) \hat{j}.
\]

(a) The magnitude of \( \vec{v}_{ag} \) is

\[
|\vec{v}_{ag}| = \sqrt{(-171 \text{ km/h})^2 + (-70.0 \text{ km/h})^2} = 185 \text{ km/h}.
\]

(b) The direction of \( \vec{v}_{ag} \) is

\[
\theta = \tan^{-1}\left(\frac{-70.0 \text{ km/h}}{-171 \text{ km/h}}\right) = 22.3° \text{ (south of west)}. \]