PH 202-7B Summer 2007

Electrostatic Potential and Energy

Lectures 2-3

Chapter 19
(Cutnell & Johnson, Physics 7th edition)
Electrostatic Potential and Energy

Recall:
- A general criterion for a conservative force - the work the force performs on any round trip must be zero.
- If a force is conservative, then the work done by this force on a particle during a displacement can be expressed as a difference between two potential energies, one for the starting point and one for the end point of the displacement.

\[ W_{12} = (U_2 - U_1) = m g (y_1 - y_2) \]

\[ U_2 = U_1 - W_{12} \]

The electric force, that a static distribution of charges exerts on a point charge is a conservative force.

For the electrical potential energy

\[ U_2 = U_1 - W_{12} \] or \[ U_2 - U_1 = -W_{12} \]

Work done by the electric force results in a negative change in potential energy.
Uniform electric field in the space between two charged parallel plates

- Introduce a point charge $q$ in the field $E$
- $F = qE$
- $W = F\Delta y = qE(y_2 - y_1) = -qEy_1 + qEy_2 = U_1 - U_2$
- $U = -qEy$ potential energy
- The conserved energy $= \frac{1}{2}mv^2 - qEy = [\text{const}]$

$E = \frac{\sigma}{\varepsilon_0} = \frac{Q}{A\varepsilon_0}$
Electrostatic Potential $V$ at a given point is the electric potential energy of a small test charge $q$ situated at that point divided by the charge itself

$$E = \frac{F}{q}$$

$[\text{Electric field}] = \frac{[\text{electric force}]}{[\text{charge}]}$

$$[\text{electrostatic potential}] = \frac{[\text{electric potential energy}]}{[\text{charge}]}$$

$$V = \frac{U}{q}$$

$V$ is the potential energy per unit charge.

For uniform electric field with $U=-E \cdot y$:

$$V = -E \cdot y$$

The unit of Electrostatic potential is the volt $(V)$.

1 volt $= 1$ joule/coulomb $= 1 \frac{J}{C}$

The unit of electric field

$$1 \frac{V}{m} = 1 \frac{N \cdot m}{C \cdot m} = 1 \frac{N}{C \cdot m} = 1 \frac{V}{m}$$

$$V_2 - V_1 = \frac{E \cdot P_{E2}}{q} - \frac{E \cdot P_{E1}}{q} = -\frac{W_{12}}{q}$$

$$\Delta V = -\frac{W_{12}}{q}$$

Neither the potential $V$ nor the potential energy $EPE$ can be determined in an absolute sense, because only the differences $\Delta V$ and $\Delta EPE$ are measurable in terms of the work $W_{AB}$. 
The electric potential difference created by a point charge

The positive test charge \( +q_0 \) experiences a repulsive force \( F \) due to the positive point charge \( +q \). As a result, work is done by this force when the test charge moves from A to B. Consequently, the electric potential is higher (uphill) at A and lower (downhill) at B.

\[
W_{AB} = \frac{kqq_0}{r_A} - \frac{kqq_0}{r_B}
\]

\[
V_B - V_A = -W_{AB}/q = kq/r_B - kq/r_A
\]

For \( r_B = \infty \), \( kq/r_B = 0 \) \( \Rightarrow \) \( V_B = 0 \)

\( \Rightarrow V_A = kq/r_A \) or \( V = kq/r \)

The symbol \( V \) does not refer to the potential in any absolute sense. Rather, \( V = kq/r \) stands for the amount by which the potential at a distance \( r \) from a point charge differs from the potential at an infinite distance away. In other words, \( V \) refers to a potential difference with the arbitrary assumption that the potential at infinity is zero.
Example. The potential of a point charge for a zero reference potential at infinity

A point charge of \( q = 4.0 \times 10^{-8} \text{C} \) creates a potential at a spot 1.2 m away.

The potential is (a) \( V = \frac{kq}{r} = \frac{(8.99 \times 10^9)(4.0 \times 10^{-8})}{1.2} = +300 \text{V} \)
when the charge is positive

and (b) \(-300 \text{V}\) when the charge is negative
The acceleration of positive and negative charges

A positive charge accelerates from a region of higher electric potential toward a region of lower electric potential.

A negative charge accelerates from a region of lower electric potential toward a region of higher electric potential.
The electric potential energy of a point charge $q$ for non-uniform electric field

$$W = U_1 - U_2$$

Energy = $K + U = \frac{1}{2} mv^2 + U = \text{constant}$

$$U = qV$$

Energy = $\frac{1}{2} mv^2 + qV = \text{constant}$

$$U = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r^2}$$

Potential energy of two interacting point charges

For two charges of equal signs, the electric potential energy is positive $U$ and inversely proportional to the distance.

For two charges of opposite signs, $U < 0$ and this negative potential energy increases with distance from a large negative value to zero.
Electrostatic Potential of a point charge

The electrostatic potential produced by the point charge is

\[ V = \frac{1}{4\pi\varepsilon_0} \frac{q'}{r} \]

\( q' = +\text{c} \)

\( q' = -\text{c} \)
One electron volt is the magnitude of the amount by which the potential energy of an electron changes when the electron moves through the potential difference of 1 volt.

\[
\text{Electron-volt (eV)} \quad 1\text{eV} = (1.6 \times 10^{-19})(1\text{V}) = 1.6 \times 10^{-19}
\]

In chemical reactions among atoms or molecules, the energy released or absorbed by each atom or molecule is typically 1 or 2 eV.

The mechanical energy of a point charge moving in an electric field.

\[
\text{Energy} = K + U = \frac{1}{2} mv^2 + \frac{1}{4\pi\varepsilon_0} \frac{q^2}{r} = E\text{const}
\]

the law of conservation of energy for the motion of a point charge \( q \) in an electric field \( E \).

The total energy remains constant during the motion.
Problem: The electric potential difference between the positive and negative poles of an automotive battery is 12 volts. In order to charge the battery fully, the charging device must force \(2.0 \times 10^5\) C from the negative terminal of the battery to the positive terminal. How much work must the charging device do during this process?

\[
W = U_1 - U_2
\]

\[
U_2 = U_1 + W
\]

\[
U_1 = 0 \implies W = U_2
\]

\[
U_2 = QV
\]

\[
W = QV = 2 \times 10^5 \text{C} \times 12 \text{volts} = 2.4 \times 10^6 \text{J}
\]
A headlight connected to a 12-V battery
Problem: A proton sits at the origin of coordinates. How much work must you do against the electric force of the proton to push an electron from the point \( x = 10 \times 10^{-10} \text{ m}, \ y = 0 \) in the \( x-y \) plane to the point \( x = 2.5 \times 10^{-10} \text{ m}, \ y = 2.5 \times 10^{-10} \text{ m} \)?

\[
U_1 = \frac{-1}{4\pi \varepsilon_0} \frac{e^2}{r^2} = \frac{-9 \times 10^9 \times (1.6 \times 10^{-19})^2}{1 \times 10^{-10}} = -2.3 \times 10^{-18} \text{ J}
\]

\[
U_2 = \frac{-1}{4\pi \varepsilon_0} \frac{e^2}{r^2} = \frac{-9 \times 10^9 \times (1.6 \times 10^{-19})^2}{\sqrt{(2.5 \times 10^{-10})^2 + (2.5 \times 10^{-10})^2}} = -6.5 \times 10^{-19} \text{ J}
\]

Work done by the electric force

\[
W = U_1 - U_2 = -2.3 \times 10^{-18} \text{ J} + 6.5 \times 10^{-19} \text{ J} = -1.6 \times 10^{-18} \text{ J} = -\frac{-1.6 \times 10^{-18} \text{ J}}{1.6 \times 10^{-13} \text{ eV}} = -10 \text{ eV}
\]

Work to be done against the electric force = +10 eV
Problem
A proton is accelerated from rest through a potential of $2.5 \times 10^5 \text{V}$. What is its final speed?

\[ E_0 = \frac{m v_0^2}{2} + e V_0 \]

\[ E_i = \frac{m v_i^2}{2} + e V_i \]

\[ E_0 = E_i \]

\[ e V_0 - e V_i = \frac{m v_i^2}{2} \]

\[ \frac{m v_i^2}{2} + e V_i = \frac{m v_0^2}{2} + e V_0 \]

\[ \frac{m v_i^2}{2} = e (V_0 - V_i) \]

\[ v_i = \sqrt{\frac{2 e (V_0 - V_i)}{m}} = \sqrt{\frac{2 \times 1.6 \times 10^{-19} \times (2.5 \times 10^5 \text{V})}{1.67 \times 10^{-27} \text{kg}}} \]

\[ = 6.9 \times 10^6 \text{ m/s} \]
Problem: A positive point charge with mass \( m \), is released at a distance \( d \) from a fixed positive point charge \( Q \). How fast is the charge moving when the distance has grown to three times the initial value?

\[ V = \sqrt{\frac{Qx}{3\pi \varepsilon_0 md}} \]

\[ E_1 = \frac{mV^2}{2} + \frac{1}{4\pi \varepsilon_0} \frac{Qx}{d} \]

\[ E_2 = \frac{mV^2}{2} + \frac{1}{4\pi \varepsilon_0} \frac{Qx}{3d} \]

\[ E_1 = E_2 = \text{const} \]

\[ \frac{1}{4\pi \varepsilon_0} \frac{Qx}{d} = \frac{mV^2}{2} + \frac{1}{4\pi \varepsilon_0} \frac{Qx}{3d} \]

\[ \frac{mV^2}{2} = \frac{2Qx}{4\pi \varepsilon_0 3d} \]

\[ V = \sqrt{\frac{Qx}{3\pi \varepsilon_0 md}} \]
Equipotential surface for a very large sheet with a uniform distribution of charge

Definition: Equipotential surface – a mathematical surface on which the electrostatic potential has a fixed constant value
Equipotential surfaces for a positive point charge. The equipotentials are concentric spheres. Points at the same distance from the central charge are at the same potential.

\[ \vec{U} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q \cdot q'}{r} \]

\[ V = \frac{U}{q'} = \frac{1}{4\pi\varepsilon_0} \cdot \frac{q}{r} \]
Properties of the equipotential surfaces and some useful conclusions

1) The electric field is everywhere to the equipotentials. If there were $\nabla V$ to equip surface than the field will do work $W = U_1 - U_2$. Since $dV = \nabla V \cdot d\mathbf{x}$ we have a conflict with requirement $dV = 0$.

2) If the electric field is everywhere to a given surface - this surface must be an equipotential surface.

   $E \perp$ surface $\Rightarrow W = 0 \Rightarrow du = 0 \Rightarrow AV = 0$

3) All points within a conducting body are at the same electrostatic potential. $E$ in the conducting body is zero. Equilibrium is zero $\Rightarrow V = 0$.

Ground is a conductor. Convention: potential of the surface of the earth is zero, $V = 0$. Any conductor connected to the earth is said to be grounded and has $V = 0$.

Conclusion of property #3. Faraday cage: in a closed, empty cavity within a homogeneous conductor, the electric field is zero. By contradiction: field exists - it begins and ends on the surface of the cavity. Transport a positive charge on a path that follows the line $W = U_1 - U_2$; $W > 0$ because $\mathbf{E}$. $\Rightarrow U_1 < U_2$; $\Rightarrow V_1 > V_2$.

but it is wrong $V_1 = V_2$. 

The electric potential energy of a system of point charges is the work needed to assemble the point charges in their current state from a state in which every charge is at an infinite distance from every other charge and so has zero potential energy. For a pair of point charges $q_1$ and $q_2$ separated by a distance $r_{12}$, it requires zero work to bring $q_1$ to any position in the absence of $q_2$. It then takes an amount of work

$$U = \frac{1}{4\pi\varepsilon_0} \frac{q_1 q_2}{r_{12}}$$

to bring $q_2$ from infinity to a position $r_{12}$ from $q_1$.

To bring in a third charge $q_3$ from infinity to a position $r_{13}$ from $q_1$ and $r_{23}$ from $q_2$ requires an additional amount of work

$$\frac{1}{4\pi\varepsilon_0} \frac{q_1 q_3}{r_{13}} + \frac{1}{4\pi\varepsilon_0} \frac{q_2 q_3}{r_{23}}$$

The total amount of work needed to assemble all three charges is, then,

$$U = \frac{1}{4\pi\varepsilon_0} \left[ \frac{q_1 q_2}{r_{12}} + \frac{q_1 q_3}{r_{13}} + \frac{q_2 q_3}{r_{23}} \right]$$

The three charges, and their separations $r_{12}$, $r_{13}$, and $r_{23}$, are shown in Fig.
This expression for \( U \) can be rewritten as

\[
U = \frac{1}{2} V_{\text{other}(1)} q_1 + \frac{1}{2} V_{\text{other}(2)} q_2 + \frac{1}{2} V_{\text{other}(3)} q_3
\]

\[
V_{\text{other}(1)} = \frac{1}{4 \pi \varepsilon_0} \frac{q_2}{r_{12}} + \frac{1}{4 \pi \varepsilon_0} \frac{q_3}{r_{13}}
\]

which is just the potential at charge \( q_1 \) produced by charges \( q_2 \) and \( q_3 \). Similarly, \( V_{\text{other}(2)} \) is the potential at charge \( q_2 \) produced by charges \( q_1 \) and \( q_3 \), and \( V_{\text{other}(3)} \) is the potential at \( q_3 \) caused by \( q_1 \) and \( q_2 \). Since this expression for \( U \) can be applied to any number of charges, the electric potential energy for \( n \) point charges can be written as

\[
U = \frac{1}{2} V_{\text{other}(1)} q_1 + \frac{1}{2} V_{\text{other}(2)} q_2 + \ldots + \frac{1}{2} V_{\text{other}(n)} q_n
\]

To see how useful this expression can be, let us apply it to evaluate the potential energy of a single charged conductor with total charge \( Q \) whose surface is at a potential \( V \). We can imagine the total charge \( Q \) to be made up of a large number of very small charges \( q \), so that \( V_{\text{other}} = V \) for each of the many small charges \( q \) contributing to the total charge \( Q \). Then the total electric energy is just given by

\[
U = \frac{1}{2} V q_1 + \frac{1}{2} V q_2 + \frac{1}{2} V q_3 + \ldots
\]

\[
= \frac{1}{2} V (q_1 + q_2 + q_3 + \ldots) = \frac{1}{2} V Q
\]

That is, the electric energy of a charged conductor is one-half the product of the charge times the electric potential. If we have a large number of charged conductors with charges \( Q_1, Q_2, Q_3, \ldots \) and respective potentials \( V_1, V_2, V_3, \ldots \), the total electric energy is given directly by

\[
U = \frac{1}{2} Q_1 V_1 + \frac{1}{2} Q_2 V_2 + \frac{1}{2} Q_3 V_3 + \ldots
\]

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Energy of a system of charged conductors

Conducting bodies have constant potentials $V_1 = \text{const.}$, $V_2 = \text{const.}$...

$V = \text{const.}$

$U_1 = \frac{1}{2} \Delta \varphi \int \varphi_1 \varphi_2 \text{d}x$,

$= \frac{1}{2} \int \Delta \varphi \text{d}x = \frac{1}{2} \int \varphi_1 \varphi_2$,

$= \frac{1}{2} \int (\varphi_1 \varphi_2 + \varphi_2 \varphi_2 + \cdots)$

$U_{net} = \frac{1}{2} \int \varphi_1 \varphi_1 + \frac{1}{2} \int \varphi_2 \varphi_2 + \frac{1}{2} \int \varphi_3 \varphi_3 + \cdots$
Problem: Suppose that at one instant the electrons and the nucleus of a helium atom occupy the positions shown in Fig. 2. At this instant, the electrons are at a distance of $0.2 \times 10^{-10}$ m from the nucleus. What is the electric potential energy of this arrangement? Treat the electrons and the nucleus as point charges.

\[
U = \frac{1}{4\pi \varepsilon_0} \left( \frac{q_1 q_2}{r_{12}} + \frac{q_2 q_3}{r_{23}} + \frac{q_1 q_3}{r_{13}} \right) = \\
= \frac{1}{4\pi \varepsilon_0} \left( \frac{-2e^2}{0.2 \times 10^{-10}} + \frac{-2e^2}{0.2 \times 10^{-10}} + \frac{e^2}{0.4 \times 10^{-10}} \right) = \\
= (9 \times 10^9) \left( -4 \frac{(1.6 \times 10^{-19})^2}{0.2 \times 10^{-10}} + \frac{1.6 \times 10^{-19}}{0.4 \times 10^{-10}} \right) = \\
= -4.0 \times 10^{-17} \text{J} = \frac{-4 \times 10^{-17} \text{J}}{1.6 \times 10^{-19} \text{C}} = -250 \text{eV}
\]
Problem. Four equal particles of positive charges $q$ and equal masses $m$ are initially held at the four corners of a square of side $L$. If these particles are released simultaneously, what will be their speeds when they have separated by a very large distance?

\[ V_{\text{initial}} = -9 \frac{q^2}{\epsilon_0 L} \]

\[ V_{\text{final}} = \frac{q}{4 \pi \epsilon_0 (q_2/L + q_3/\sqrt{2L} + q_4/L)} \]

1) \( U = \frac{1}{2} m \left( \frac{q^2}{\epsilon_0 L} \left( \frac{2}{2} + \frac{2}{\sqrt{2} L} + \frac{1}{L} \right) \right) \times 4 = \frac{q^2}{24 \pi \epsilon_0} \left( \frac{2}{2} + \frac{1}{\sqrt{2} L} \right) \)

2) KE of all 4 particles = $U$

KE = $q \left( \frac{1}{2} m v^2 \right)$ where $v$ = speed of each

\[ 2 m v^2 = \frac{q^2}{24 \pi \epsilon_0 L} \left( 2 + \frac{1}{\sqrt{2} L} \right) \]

\[ v^2 = \frac{q^2}{4 \pi \epsilon_0 m \sqrt{2 + \frac{1}{\sqrt{2} L}}} \]
Energy density of an electric field. We can easily calculate the electric energy $U$ of a pair of large, parallel metal plates separated by a distance $d$ and carrying respective charges of $+Q$ and $-Q$.

\[
U = \frac{1}{2}Q_1 V_1 + \frac{1}{2}Q_2 V_2
= \frac{1}{2}Q(V_1 - V_2) = \frac{1}{2}Q (-E_y, +E_y) = \frac{1}{2} Q Ed = \frac{Q^2 d}{2 \varepsilon_0 A}
\]

The electric field in the region between plates is given by

\[
E = \frac{Q}{\varepsilon A}
\]

Thus

\[
U = \frac{1}{2} \varepsilon_0 E^2 \times \text{[volume]}
\]

which gives an energy per unit volume of the field

\[
u = \frac{1}{2} \varepsilon_0 E^2
\]

It can be shown that this is a very general result, true for all electric fields, whether uniform or nonuniform.

The energy is concentrated in the regions of space where the electric field is strong.
Problem. A pair of parallel plates, each measuring 30 cm x 30 cm, are separated by a gap of 1.0 mm. How much work do you do against the electric forces to charge these plates with +10 x 10^-6 C and -10 x 10^-6 C, respectively?

\[ E = \frac{\Delta V}{d} \]

\[ V = V_1 - V_2 \quad U_1 = 0 \]

\[ U_2 = \frac{1}{2} \varepsilon_0 V_1 + \frac{1}{2} \varepsilon_2 V_2 = \frac{1}{2} \varepsilon_0 (V_1 - V_2) = \]

\[ = \frac{1}{2} \varepsilon_0 \left[ -E_y^1 - (-E_y^2) \right] = \frac{1}{2} \varepsilon_0 E d = \]

\[ = \frac{1}{2} \varepsilon_0 \left( \frac{E}{\varepsilon_0 A} \right) d = \frac{1}{2} \frac{E^2 d}{\varepsilon_0 A} \]

\[ W = -\frac{1}{2} \frac{E^2 d}{\varepsilon_0 A} = -\frac{1}{2} \left( \frac{10 \times 10^{-6}}{885 \times 10^{-12}} \right)^2 \frac{0.01}{0.03} = -6.3 \times 10^{-4} \text{J} \]

\[ W_{\text{against el}} = +6.3 \times 10^{-4} \text{J} \]
Problem Near the surface of the nucleus of a lead atom, the electric field has a strength $3.4 \times 10^{21} \text{V/m}$. What is the energy density in this field?

Energy density $I = \frac{1}{2} \varepsilon_0 E^2$

$= \frac{1}{2}(8.85 \times 10^{-12}/(3.4 \times 10^{21}))^2 = 5.1 \times 10^{-32} \text{ J/m}^3$

(units $\varepsilon_0 = \left[ \frac{C^2}{N \cdot m^2} \right] = \left[ \frac{C \cdot C}{N \cdot m \cdot m} \right] = \left[ \frac{C}{g \cdot m} \right] = \left[ \frac{C}{V \cdot m} \right]$)

units $E = \left[ \frac{V}{m} \right]$

(units $\varepsilon_0 \times$ units $E^2 = \left[ \frac{C}{V \cdot m} \right] \times \left[ \frac{V^2}{m^2} \right] = \left[ \frac{C \cdot V}{m^3} \right] = \frac{J}{m^3}$)
Capacitance

Any arrangement of conductors that is used to store electric charge is called a capacitor.

The capacitor will also store electric potential energy whenever it stores electric charge.

Example of a capacitor:

An isolated metallic sphere of radius $R$ with $Q$ stores electric charge $Q$.

$$V = \frac{1}{4\pi \varepsilon_0} \frac{Q}{R}$$

So $Q$ is directly proportional to $V$.

This proportionality holds in general for any conductor of arbitrary shape.

The constant of proportionality, $C$, is called the capacitance of the conductor.

A large value of $C$ implies that a large amount of charge can be stored at low voltage.
The unit of capacitance is the farad (F):
1 farad = 1 F = 1 coulomb/volt (C = \( \frac{Q}{V} \))

In practice we prefer to use the microfarad (1 \( \mu \)F = \( 10^{-6} \) F) and the picofarad (1 pF = \( 10^{-12} \) F)

\[ 1 F = 1 \frac{C}{V} = 1 \frac{C^2}{CV} = 1 \frac{C^2}{V^2} = 1 \frac{C^2}{N \cdot m} \]

So the constant \( \varepsilon_0 \) can be written:

\[ \varepsilon_0 = 8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2} = \boxed{8.85 \times 10^{-12} \frac{F}{m}} \]

The capacitance of a spherical conductor:

\[ C = \frac{Q}{V} = \frac{1}{4\pi \varepsilon_0 R} = 4\pi \varepsilon_0 R \]

The capacitance of a sphere conductor increases with its radius.

The most common variety of capacitor consists of two metallic conductors insulated from each other and carrying opposite amount of charge \( \pm Q \). 

\( C \) is defined in terms of the difference of potential \( \Delta V \) between conductors.

\[ C = \frac{Q}{\Delta V} \]
Capacitance of a parallel plate capacitor

\[ C = \frac{\frac{\Delta Q}{\Delta V}}{E} = \frac{Q}{\epsilon_0 A} = \frac{\epsilon_0 A}{d} \]

In order to store a large amount of charge at a low potential, \( A \gg d \)

Capacitance of a concentric spheres capacitor

\[ \Delta V = \frac{Q}{4\pi \epsilon_0} \left[ \frac{1}{R_1} - \frac{1}{R_2} \right] \]

So \( C = \frac{Q}{\Delta V} = 4\pi \epsilon_0 \left[ \frac{R_1}{R_2} \right] \) when \( R_2 \) approaches infinity

\( C \to 4\pi \epsilon_0 R_1 \), value for an isolated sphere
Potential energy in capacitor

Consider a two conductor capacitor with charges $\pm Q$ on its plates $Q_1 = +Q$; $Q_2 = -Q$

$$U = \frac{1}{2} Q_1 V_1 + \frac{1}{2} Q_2 V_2 = \frac{1}{2} Q (V_1 - V_2)$$

$$U = \frac{1}{2} Q \Delta V$$

$$Q = C \Delta V \Rightarrow$$

$$U = \frac{1}{2} \varepsilon (\Delta V)^2$$

**Variable capacitors**

- movable plate
- fixed plate

are used in the tuning circuits of radios

 Capacitor microphone

Flexible diaphragm

Rigid disk

The periodic fluctuations of the air pressure alternately push and pull the diaphragm toward and away from the rigid plate. $C$ is vary. => The change in $C$ results in a change in the amount of electric charge on the plates, and electric current. So we transform a sound signal into an electrical signal.
Problem: Consider an isolated metallic sphere of radius $R$ and another isolated metallic sphere of radius $3R$. If both spheres are at the same potential, what is the ratio of their charges?

If both spheres carry the same charge, what is the ratio of their potentials?

\[ \frac{Q_2}{Q_1} \]

\[ R_2 = 3R, \]

a) Find $\frac{Q_2}{Q_1}$ if $V_1 = V_2 = V$

b) Find $\frac{V_2}{V_1}$ if $Q_1 = Q_2 = Q$

\[ \frac{1}{4\pi\varepsilon_0} \frac{Q_1}{R_1} = V_1 = V_2 = \frac{1}{4\pi\varepsilon_0} \frac{Q_2}{3R}, \]

\[ \Rightarrow \frac{Q_2}{Q_1} = \frac{3R_1}{R_1} = 3 \]

\[ V_2 = \frac{Q_2}{C_2} = \frac{Q}{C_2} = \frac{Q}{4\pi\varepsilon_0 (3R)}, \quad \frac{V_2}{V_1} = \frac{1}{3} \]
Problem

Your head is approximately a conducting sphere of radius 10 cm. What is the capacitance of your head? What will be the charge on your head if, by means of an electrostatic machine, you raise your head (and your body) to a potential of 100,000 V?

\[ C = 4\pi \varepsilon_0 R = 4\pi \times (8.85 \times 10^{-12} \text{ F/m}) \times (0.1 \text{ m}) = 1.1 \times 10^{-11} \text{ F} = 11 \text{ pF} \]

\[ Q = CV = \left(1.1 \times 10^{-11} \text{ F} \right) \times (100,000 \text{ V}) = 1.1 \times 10^{-6} \text{ C} \]
Problem: Two parallel conducting plates of area 0.5 m² placed in a vacuum have a potential difference of 2.0 x 10⁵ V when charges of +4.0 x 10⁻⁸ C are placed on them respectively.

(a) What is the capacitance of the pair of plates?
(b) What is the distance between them?
(c) What is the electric field between them?
(d) What is the potential energy?

\[ Q = \pm 4.0 \times 10^{-8} \text{ C} \]
\[ \Delta V = 2.0 \times 10^5 \text{ V} \]
\[ A = 0.5 \text{ m}^2 \]

(a) \[ C = \frac{Q}{\Delta V} = \frac{4.0 \times 10^{-8} \text{ C}}{2.0 \times 10^5 \text{ V}} = 2.0 \times 10^{-13} \text{ F} \]

(b) \[ C = \frac{\varepsilon_0 A}{d} \quad \Rightarrow \quad d = \frac{\varepsilon_0 A}{C} = \frac{(8.85 \times 10^{-12}\text{ F/m}) (0.5 \text{ m}^2)}{(2.0 \times 10^{-13}\text{ F})} \]
\[ = 2.2 \times 10^{-4} \text{ m} \]

(c) \[ E = \frac{\Delta V}{d} = \frac{2.0 \times 10^5 \text{ V}}{2.2 \times 10^{-4} \text{ m}} = 9 \times 10^8 \text{ V/m} \]

(d) \[ U = \frac{1}{2} C (\Delta V)^2 = \frac{1}{2} \times (2.0 \times 10^{-13} \text{ F}) (2.0 \times 10^5 \text{ V})^2 = 400 \text{ J} \]
Problem: What is the electric field in a 3.0-μF capacitor with parallel plates of area 15 m² charged to 4.4 volts?

\[ E = \frac{V}{d} \quad C = \frac{\varepsilon_0 A}{d} \quad d = \frac{\varepsilon_0 A}{C} \]

\[ \Rightarrow E = \frac{V}{d} = \frac{V \cdot \varepsilon_0}{\varepsilon_0 A} = \frac{(4.4V)(3 \times 10^{-6} F)}{(8.85 \times 10^{-12} F/m)(15 m^2)} = \]

\[ = 9.9 \times 10^4 \frac{V}{m} \]
Problem: A 4.00 μF capacitor has been charged by a 9.00-Volt battery. How many electrons must be moved from the negative plate to the positive plate of the capacitor to reverse the electric field inside of the capacitor?

\[ Q = CV = (4.00 \times 10^{-6} \text{ F}) (9.00 \text{ V}) = 36 \times 10^{-6} \text{ C} \]

\[ \Delta Q = 2Q = 72 \times 10^{-6} \text{ C} \]

\[ n = \frac{\Delta Q}{e} = \frac{72 \times 10^{-6} \text{ C}}{1.6 \times 10^{-19} \text{ C}} = 4.5 \times 10^{14} \]
Problem: In many computer keyboards, the switches under the keys consist of small parallel-plate capacitors. The key is attached to the upper plate, which is movable when you push the key down, you push the upper plate toward the lower plate, and yet alter the plate separation $d$ and the capacitance. The capacitor is connected to an external circuit that maintains a constant potential difference $V$ across the plates. The change of capacitance therefore triggers a transfer of charge between the capacitor and the computer circuit. Suppose that the initial plate separation is 5.0 mm and $C = 6 \times 10^{-3} \text{F}$. $d' = 0.2 \text{mm}$. The constant potential difference $V = 8 \text{V}$.

(a) What is the change in capacitance when you depress the key?
(b) What is the amount of charge transferred between the capacitor and the computer circuit?

\[ C = \frac{\varepsilon_0 A}{d} \quad \text{and} \quad C' = \frac{\varepsilon_0 A}{d'} \]

\[ C' = \frac{\varepsilon_0 A}{d' \varepsilon_0 A} = \frac{d}{d'} = \frac{5 \times 10^{-3} \text{m}}{0.2 \times 10^{-3} \text{m}} = 25 \]

\[ \Delta C = C' - C = 25C - C = 24C = 24(6 \times 10^{-3} \text{F}) = 144 \times 10^{-3} \text{C} \]

\[ \Delta Q = \Delta C \cdot V = (144 \times 10^{-3} \text{F})(8 \text{V}) = 1.152 \times 10^{-1} \text{C} \]
Dielectrics

The space between the plates of the capacitor is usually filled with an electric insulator or dielectric, which drastically reduces the strength of the electric field from what it would be in vacuum.

Why? — In an insulator all the charges are bound, the \( E \) cannot wander about as in a conductor.

- In response to the electric force the charges will move very slightly without leaving their atoms.
- \( \vec{Q} \) will move in the direction of \( \vec{E} \)

The opposite displacement create electric dipoles within the dielectric, with

\[ P = \varepsilon_0 \varepsilon \vec{E} \]

Such dielectrics are called linear.

The details of the mechanism of displacement and separation of charges depend on the dielectric.
1. In glass, nylon, solids the creation of dipole moments involves a distortion of the molecules or atoms.

<table>
<thead>
<tr>
<th>E=0</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
<tr>
<td>0</td>
</tr>
</tbody>
</table>

$\uparrow$ $\uparrow$ $\uparrow$

The electric field produces a distortion of the molecules.

$E$ stretches the molecule, producing charge separation.

2. In other dielectrics - distilled water or carbon dioxide - the charge separation results from a realignment of existing dipoles.

$E=0$

$\uparrow\uparrow\uparrow$

$E\neq0$

E produces an alignment of already distorted molecules.

Consequently there is an excess of positive charge on the upper surface of the slab and negative on the opposite.

The dielectric is polarized.
The total electric field, consisting of the sum of the fields of the free charges on the conducting plates $E_{\text{free}}$ plus the field of the bound charges on the dielectric surfaces is smaller than $E_{\text{free}}$ alone.

$$E_{\text{free}} - E_d = E$$

In a linear dielectric, the amount by which the dielectric reduces $E_{\text{free}}$ is characterized by the dielectric constant $K > 1$.

$$E = \frac{1}{K} E_{\text{free}}$$

With the dielectric, the potential difference between the plates is $\Delta V = E_d = (E_{\text{free}})^2 = \Delta V_0$

$$\Rightarrow \Delta V_0 \text{ is reduced by the factor } \frac{1}{K}$$

The capacitance with the dielectric:

$$C = \frac{\Delta \mathcal{W}}{\Delta V_0} = \frac{E_{\text{free}}}{\Delta V_0} = C_0 \cdot K$$

$C_0$ - capacitance without the dielectric.
The dielectric increases the capacitance by a factor $k$.

### Dielectric Constants of Some Materials

<table>
<thead>
<tr>
<th>Material</th>
<th>$k$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Vacuum</td>
<td>1</td>
</tr>
<tr>
<td>Air</td>
<td>1.00054</td>
</tr>
<tr>
<td>Rubber</td>
<td>2.8</td>
</tr>
<tr>
<td>Transformer Oil</td>
<td>3</td>
</tr>
<tr>
<td>Plexiglass</td>
<td>3.4</td>
</tr>
<tr>
<td>Epoxy Resin</td>
<td>3.6</td>
</tr>
<tr>
<td>Paper</td>
<td>4</td>
</tr>
<tr>
<td>Glass</td>
<td>6</td>
</tr>
<tr>
<td>Water distilled</td>
<td>80</td>
</tr>
</tbody>
</table>

In a dielectric material, the correct formula for the energy density is $u = \frac{1}{2} k \varepsilon_0 E^2$. 
Problem

You wish to construct a capacitor out of sheet of polyethylene of thickness $5 \times 10^{-2}$ mm and $k = 2.3$ sandwiched between two aluminum sheets. If the capacitance is to be $3.0 \mu F$, what must be the area of the sheets?

\[ C = \frac{k \varepsilon_0 A}{d} \]

\[ \Rightarrow A = \frac{C d}{k \varepsilon_0} = \frac{(3 \times 10^{-6} \, \mu F)(5 \times 10^{-5} \, m)}{2.3 \times (8.85 \times 10^{-12} \, F/m)} = 7.4 \, m^2 \]

Problem

What is the capacitance of a sphere of radius $R$ immersed in a large volume of gas of dielectric constant $k$?

In free space $Q = CV; \ V = \frac{Q}{4\pi \varepsilon_0 \ R}$

\[ C = \frac{Q}{4\pi \varepsilon_0 \ R} = \frac{Q}{4\pi \varepsilon_0 \ R} = \frac{Q}{4\pi \varepsilon_0 \ R} = 4\pi \varepsilon_0 \ R \]

In gas with dielectric constant $k$

\[ C' = kC = 4\pi k\varepsilon_0 \ R \]
Problem: In order to measure the dielectric constant of a dielectric material, a slab of this material 2.0 cm thick is slowly inserted between a pair of parallel conducting plates separated by a distance of 2.0 cm. Before insertion of the dielectric, the potential difference across these capacitor plates is $3.0 \times 10^5$ V. During insertion, the charge on the plates remains constant. After insertion, the potential difference is $1.8 \times 10^5$ V. What is the value of the dielectric constant?

Initial capacitance = $C_0$

After insertion $C = kC_0$, with dielectric filling the capacitor

$$Q = C_0 V_0 = CV$$

$$C = \frac{C_0 V_0}{V} = kC_0$$ ; \hspace{1cm} k = \frac{V}{V} = \frac{3 \times 10^5}{1.8 \times 10^5} = 1.7$$
Problem  A parallel-plate capacitor is filled with carbon dioxide at 1 atm pressure. Under these conditions the capacitance is 0.5 μF. We charge the capacitor by means of a 48-V battery and then disconnect the battery so that the electric charge remains constant thereafter. What will be the change in the potential difference if we now pump the carbon dioxide out of the capacitor, leaving it empty?

\[
C = \frac{k\varepsilon_0 A}{d}; \quad Q = CV
\]

Therefore, for constant charge

\[
C_1 V_1 = C_2 V_2 \implies \frac{V_2}{V_1} = \frac{C_1}{C_2}
\]

Here, \( C_1 \sim k_{\text{CO}_2} = 1.0098 \); \( C_2 \sim k_{\text{vacuum}} = 1.000 \)

Therefore,

\[
V_2 = \frac{C_1}{C_2} V_1 = \left(\frac{1.0098}{1.000}\right) \times 48 \text{ V} = 48.05 \text{ V}
\]

\[
\Delta V = V_2 - V_1 = 48.05 - 48.00 = 0.05 \text{ V}
\]