Diffraction
Lectures 26-27

Chapter 36
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
In Chapter 35, we saw how light beams passing through different slits can interfere with each other and how a beam after passing through a single slit flares—diffracts—in Young's experiment. Diffraction through a single slit or past either a narrow obstacle or an edge produces rich interference patterns. The physics of diffraction plays an important role in many scientific and engineering fields.

In this chapter we explain diffraction using the wave nature of light and discuss several applications of diffraction in science and technology.
Diffraction and the Wave Theory of Light

Diffraction pattern from a single narrow slit.

These patterns cannot be explained using geometrical optics (Ch. 34)!

Fresnel Bright Spot.
When the path length difference between rays \( r_1 \) and \( r_2 \) is \( \lambda/2 \), the two rays will be out of phase when they reach \( P_1 \) on the screen, resulting in destructive interference at \( P_1 \). The path length difference is the distance from the starting point of \( r_2 \) at the center of the slit to point \( b \).

For \( D \gg a \), the path length difference between rays \( r_1 \) and \( r_2 \) is \( (a/2) \sin \theta \).
Repeat previous analysis for pairs of rays, each separated by a vertical distance of $a/2$ at the slit.

Setting path length difference to $\lambda/2$ for each pair of rays, we obtain the first dark fringes at:

$$\frac{a}{2} \sin \theta = \frac{\lambda}{2} \rightarrow a \sin \theta = \lambda \quad \text{(first minimum)}$$

For second minimum, divide slit into 4 zones of equal widths $a/4$ (separation between pairs of rays). Destructive interference occurs when the path length difference for each pair is $\lambda/2$.

$$\frac{a}{4} \sin \theta = \frac{\lambda}{2} \rightarrow a \sin \theta = 2\lambda \quad \text{(second minimum)}$$

Dividing the slit into increasingly larger even numbers of zones, we can find higher order minima:

$$a \sin \theta = m\lambda, \quad \text{for } m = 1, 2, 3\ldots \quad \text{(minima-dark fringes)}$$

Fig. 36-5
Intensity in Single-Slit Diffraction, Qualitatively

To obtain the locations of the minima, the slit was equally divided into $N$ zones, each with width $\Delta x$. Each zone acts as a source of Huygens wavelets. Now these zones can be superimposed at the screen to obtain the intensity as a function of $\theta$, the angle to the central axis.

To find the net electric field $E_\theta$ (intensity $\propto E_\theta^2$) at point $P$ on the screen, we need the phase relationships among the wavelets arriving from different zones:

$$
\Delta \phi = \left(\frac{2\pi}{\lambda}\right)(\Delta x \sin \theta)
$$

$$
\left(\frac{\text{phase difference}}{\text{path length difference}}\right) = \left(\frac{2\pi}{\lambda}\right)
$$

Fig. 36-6

(N=18)

$\theta = 0$

1st side max.

1st min.

$\theta$ small
Here we will show that the intensity at the screen due to a single slit is:

\[ I(\theta) = I_m \left( \frac{\sin \alpha}{\alpha} \right)^2 \]  

(36-5)

where \( \alpha = \frac{1}{2} \phi = \frac{\pi a}{\lambda} \sin \theta \)  

(36-6)

In Eq. 36-5, minima occur when:

\( \alpha = m\pi, \quad \text{for } m = 1, 2, 3 \ldots \)

If we put this into Eq. 36-6 we find:

\[ m\pi = \frac{\pi a}{\lambda} \sin \theta, \quad \text{for } m = 1, 2, 3 \ldots \]

or \( a \sin \theta = m\lambda, \quad \text{for } m = 1, 2, 3 \ldots \)

(minima-dark fringes)
Proof of Eqs. 36-5 and 36-6

If we divide the slit into infinitesimally small zones of width $\Delta x$, the arc of the phasors approaches the arc of a circle. The length of the arc is $E_m$. $\phi$ is the difference in phase between the infinitesimal vectors at the left and right ends of the arc. $\phi$ is also the angle between the 2 radii marked $R$.

The dashed line bisecting $f$ forms two triangles, where: 

$$\sin \frac{1}{2} \phi = \frac{E_{\theta}}{2R}.$$  

In radian measure:  

$$\phi = \frac{E_m}{R}.$$  

Solving the previous 2 equations for $E_{\theta}$ one obtains: 

$$E_{\theta} = \frac{E_m}{\frac{1}{2} \phi} \sin \frac{1}{2} \phi.$$  

The intensity at the screen is therefore: 

$$\frac{I(\theta)}{I_m} = \frac{E^2_{\theta}}{E^2_m} \rightarrow I(\theta) = I_m \left( \frac{\sin \alpha}{\alpha} \right)^2$$  

$\phi$ is related to the path length difference across the entire slit: 

$$\phi = \left( \frac{2\pi}{\lambda} \right) (a \sin \theta).$$  

Fig. 36-8
Diffraction by a Circular Aperture

Distant point source, e.g., star

Image is not a point, as expected from geometrical optics! Diffraction is responsible for this image pattern.

\[
\sin \theta = 1.22 \frac{\lambda}{d} \quad (1\text{st min.- circ. aperture})
\]

\[
\sin \theta = \frac{\lambda}{a} \quad (1\text{st min.- single slit})
\]
Resolvability

Rayleigh’s Criterion: Two point sources are barely resolvable if their angular separation $\theta_R$ results in the central maximum of the diffraction pattern of one source’s image centered on the first minimum of the diffraction pattern of the other source’s image.

$$\theta_R = \sin^{-1} \left( 1.22 \frac{\lambda}{d} \right)$$  \hspace{1cm} \theta_R \text{ small} \approx 1.22 \frac{\lambda}{d} \quad \text{(Rayleigh's criterion)}$$

Fig. 36-10
Some spy satellites carry cameras with lenses 30 cm in diameter and with a focal length of 2.4 m.

(a) What is the angular resolution of the camera according to Rayleigh criterion? Assume the wavelength of light is 550 nm.

(b) If such a satellite looks down on the Earth from a height of 150 km, what is the distance between two points on the ground that the camera can barely resolve?

(c) The lens projects images of the two points on a film at the focal plane of the lens. What is the distance between the two images projected on the film?

(a) \( \Delta \theta = 1.22 \frac{\lambda}{d} = 1.22 \times \frac{5.5 \times 10^{-7}}{0.30 \text{ m}} = 2.24 \times 10^{-6} \text{ rad} \)

(b) \( \Delta \theta = \frac{d}{z} \) (\( d \): distance between points, \( z \): distance to points)

\[ \Rightarrow d = z \Delta \theta; \quad d = 150,000 \text{ m} \times 2.24 \times 10^{-6} = 0.34 \text{ m} \]

(c) Distance \( = l = z \Delta \theta = 2.4 \text{ m} \times 2.24 \times 10^{-6} \)

\[ = 5.4 \times 10^{-6} \text{ m} = 0.0054 \text{ mm} \]
Diffraction by a Double Slit

In the double-slit experiment described in Ch. 35 we assumed that the slit width \( a \ll \lambda \). What if this is not the case?

Two vanishingly narrow slits \( a \ll \lambda \)

\[ I(\theta) = I_m \left( \cos^2 \beta \right) \left( \frac{\sin \alpha}{\alpha} \right)^2 \]  
(double slit)

\[ \beta = \frac{\pi d}{\lambda} \sin \theta \]

\[ \alpha = \frac{\pi a}{\lambda} \sin \theta \]
Diffraction Gratings

A device with $N$ slits (rulings) can be used to manipulate light, such as separate different wavelengths of light that are contained in a single beam. How does a diffraction grating affect monochromatic light?

$$d \sin \theta = m\lambda \quad \text{for } m = 0, 1, 2 \ldots \quad \text{(maxima-lines)}$$
Sodium light with wavelengths $\lambda = 589.59 \text{ nm}$ and $\lambda = 589.99 \text{ nm}$ is incident on a grating with 5500 lines/cm. A screen is placed 3.0 m beyond the grating. What is the distance between the two spectral lines in the first order spectrum on the screen? In the second-order spectrum?

\[ d \text{ between slits} = \frac{1}{5500} \text{ cm} = 1.82 \times 10^{-4} \text{ cm} = 1.82 \times 10^{-2} \text{ m} \]

Since $d \sin \theta = n \lambda$ - condition for max intensity

\[ \Rightarrow \theta = \arcsin \left( \frac{n \lambda}{d} \right) \]

For first order spectrum $n = 1$,

\[ \theta_1 = \arcsin \left( \frac{1 \times 589.99 \times 10^{-7}}{1.82 \times 10^{-2}} \right) = 18.982^\circ \]

\[ \theta'_1 = \arcsin \left( \frac{1 \times 589.59 \times 10^{-7}}{1.82 \times 10^{-2}} \right) = 18.922^\circ \]

Therefore, $\Delta \theta = 0.060^\circ$.

For second order spectrum $y = 1 \times \Delta \theta = 1.05 \times 10^{-3}$

\[ \theta_2 = \arcsin \left( \frac{2 \lambda}{d} \right) = \arcsin \left( \frac{2 \times 589.99 \times 10^{-7}}{1.82 \times 10^{-2}} \right) = 40.583^\circ \]

\[ \theta'_2 = \arcsin \left( \frac{2 \lambda}{d} \right) = \arcsin \left( \frac{2 \times 589.59 \times 10^{-7}}{1.82 \times 10^{-2}} \right) = 40.432^\circ \]

$\Rightarrow \Delta \theta = 0.063^\circ$.

$\Delta y = 2 \Delta \theta = 3.0 \text{ m} \times \theta_2 = 12.6 \times 10^{-3}$.
Width of Lines

The ability of the diffraction grating to resolve (separate) different wavelengths depends on the width of the lines (maxima).

Fig. 36-19

Fig. 36-20
Width of Lines, cont’d

In this course, a sound wave is roughly defined as any longitudinal wave (particles moving along the direction of wave propagation).

\[ Nd \sin \Delta \theta_{hw} = \lambda, \quad \sin \Delta \theta_{hw} \approx \Delta \theta_{hw} \]

\[ \Delta \theta_{hw} = \frac{\lambda}{Nd} \quad \text{(half width of central line)} \]

\[ \Delta \theta_{hw} = \frac{\lambda}{Nd \cos \theta} \quad \text{(half width of line at \( \theta \))} \]
Grating Spectroscope

Separates different wavelengths (colors) of light into distinct diffraction lines

Fig. 36-23

Fig. 36-22
Optically Variable Graphics

Gratings embedded in the device send out hundreds or even thousands of diffraction orders to produce virtual images that vary with the viewing angle. This is complicated to design and extremely difficult to counterfeit, so it makes an excellent security graphic.

Fig. 36-25
Gratings: Dispersion and Resolving Power

Dispersion: the angular spreading of different wavelengths by a grating

\[ D = \frac{\Delta \theta}{\Delta \lambda} \quad \text{(dispersion defined)} \]

\[ D = \frac{m}{d \cos \theta} \quad \text{(dispersion of a grating) (36-30)} \]

Resolving Power

\[ R = \frac{\lambda_{\text{avg}}}{\Delta \lambda} \quad \text{(resolving power defined)} \]

\[ R = Nm \quad \text{(resolving power of a grating) (36-32)} \]
Proof of Eq. 36-30

Angular position of maxima

\[ d \sin \theta = m \lambda \]

Differential of first equation (what change in angle does a change in wavelength produce?)

\[ d (\cos \theta) \, d\theta = m \, d\lambda \]

For small angles

\[ d\theta \rightarrow \Delta \theta \quad \text{and} \quad d\lambda \rightarrow \Delta \lambda \]

\[ d (\cos \theta) \Delta \theta = m \Delta \lambda \]

\[ \frac{\Delta \theta}{\Delta \lambda} = \frac{m}{d (\cos \theta)} \]
Proof of Eq. 36-32

\[ R = Nm \] (resolving power of a grating) (36-32)

Rayleigh's criterion for half width to resolve two lines

\[ d \sin \theta = m \lambda \]
\[ d(\cos \theta)d\theta = m d \lambda \]
\[ d(\cos \theta)d\theta = m \Delta \lambda \]

\[ \Delta \theta_{hw} = \frac{\lambda}{Nd \cos \theta} \]

Substituting for \( \Delta \theta \) in calculation on previous slide

\[ \Delta \theta_{hw} \rightarrow \Delta \theta \]
\[ \rightarrow \frac{\lambda}{N} = m \Delta \lambda \]

\[ R = \frac{\lambda}{\Delta \lambda} = Nm \]
Dispersion and Resolving Power Compared

**Table 36-1**

<table>
<thead>
<tr>
<th>Grating</th>
<th>$N$</th>
<th>$d$ (nm)</th>
<th>$\theta$</th>
<th>$D$ ($^\circ$/μm)</th>
<th>$R$</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>10 000</td>
<td>2540</td>
<td>13.4$^\circ$</td>
<td>23.2</td>
<td>10 000</td>
</tr>
<tr>
<td>B</td>
<td>20 000</td>
<td>2540</td>
<td>13.4$^\circ$</td>
<td>23.2</td>
<td>20 000</td>
</tr>
<tr>
<td>C</td>
<td>10 000</td>
<td>1360</td>
<td>25.5$^\circ$</td>
<td>46.3</td>
<td>10 000</td>
</tr>
</tbody>
</table>

Data are for $\lambda = 589$ nm and $m = 1$

*Fig. 36-26*
X-Ray Diffraction

X-rays are electromagnetic radiation with wavelength ~1 Å = 10^{-10} m (visible light ~5.5x10^{-7} m).

X-ray generation

\[ \sin^{-1} \left( \frac{m\lambda}{d} \right) = \sin^{-1} \left( \frac{1 \text{ (0.1 nm)}}{3000 \text{ nm}} \right) = 0.0019^\circ \]
X-Ray Diffraction, cont’d

Diffraction of x-rays by crystal: spacing $d$ of adjacent crystal planes on the order of 0.1 nm

→ three-dimensional diffraction grating with diffraction maxima along angles where reflections from different planes interfere constructively

$2d \sin \theta = m\lambda$ for $m = 0, 1, 2…$ (Bragg's law)

Fig. 36-28
X-Ray Diffraction, cont’d

Interplanar spacing \( d \) is related to the unit cell dimension \( a_0 \):

\[
5d = \sqrt{\frac{5}{4} a_0^2} \quad \text{or} \quad d = \frac{a_0}{20} = 0.2236a_0
\]

Not only can crystals be used to separate different x-ray wavelengths, but x-rays in turn can be used to study crystals, for example, to determine the type of crystal ordering and \( a_0 \).