Laser Physics I

PH481/581-2D (Mirov)

Optical Frequency Amplifiers

Lectures 3-5 chapter 2

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C. Davis, "Lasers and Electro-optics"
Optical Frequency Amplifiers

The intensity of a light wave, propagating through a medium can be changed due to stimulated emission and absorption processes. If a number of stimulated emissions is larger than absorptions, then we have built a light amplifier.

- Laser amplifier has useful gain over a particular frequency bandwidth.
- The operating frequency range will be determined by the lineshape of the transition.
- Line broadening affects in fundamental way not only the frequency bandwidth of the amplifier, but also its gain.

To turn a laser amplifier into an oscillator we need to supply an appropriate amount of positive feedback. The level of oscillation will stabilize because the amplifier saturates.

Two categories of laser amplifiers that saturate in different ways:

- The **homogeneously** broadened amplifier consists of a number of amplifying particles that are equivalent.
- The **in homogeneously** broadened amplifier consists of particles with a distribution of amplification characteristics.
Homogeneous Line Broadening

All energy states of atoms, molecules or ions are broadened over a finite range of energies.

At the fundamental level, this broadening of the energy is caused by the uncertainty involved in the energy measurement process. This gives rise to an intrinsic and unavoidable amount of line broadening called natural broadening.

Natural Broadening

The uncertainty in measured energy, \( \Delta E \), arises from the time, \( \Delta t \), involved in making such measurement:

\[
\Delta E \cdot \Delta t \sim \hbar
\]

Heisenberg's Principle

\[
\frac{\hbar}{2} = \frac{\hbar}{2m} \approx \frac{\hbar}{2}
\]

\( \hbar = 6.63 \times 10^{-34} \)

An excited particle can only be observed for a time \( \Delta t \times \tau = \tau \)

\[
\Delta E \sim \frac{\hbar}{\tau} = A \tau
\]

\[
\Delta \nu = \frac{\Delta E}{\hbar} \sim \frac{\frac{\hbar}{\tau}}{\frac{\hbar}{2}} = \frac{2\pi}{2\pi} = A
\]

uncertainty in emitted frequency
Consider the exponential intensity decay of a group of excited atoms. 

\[ e(t) = e^{-\frac{t}{\tau}} \]

The decay of each individual excited atom is modelled as an exponentially decaying (damped) cosinusoidal oscillation.

When the decay of the excited atom is viewed as a photon emission process, the atom initially placed in the excited state at time \( t = 0 \) emits a photon at time \( t \). The distribution of these times \( t \) among many such atoms varies as \( e^{-\frac{t}{\tau}} \).

The knowledge of when the photon is likely to be emitted with respect to \( t = 0 \) restricts our ability to be sure of its frequency.

The electric field of a decaying excited particle

\[ e(t) = E_0 e^{-\frac{t}{\tau}} \cos \omega t \]

is the instantaneous intensity \( i(t) \) emitted by an individual excited atom is

\[ i(t) \propto \frac{1}{e(t)} = E_0^2 e^{-\frac{2t}{\tau}} \cos^2 \omega t \]
If we observe many such atoms the total observed intensity is

\[ I(t) = \sum \sum_{\text{particles}} i(t) = \sum E_0^2 e^{-\frac{2\pi}{\tau_c}} \cos^2(\omega t + \xi_i) = \sum E_0^2 e^{-\frac{2\pi}{\tau_c}} \left[ 1 + \cos 2(\omega t + \xi_i) \right] \]

\[ \xi_i \text{ - time constant} \]

\[ \cos(\xi + \beta) = \cos \xi \cos \beta - \sin \xi \sin \beta \]
\[ \cos 2\xi = \cos^2 \xi - \sin^2 \xi = 1 - 2\sin^2 \xi = 2\cos^2 \xi - 1 \]

\[ \Rightarrow \cos^2 \xi = \frac{\cos 2\xi + 1}{2} \]

\( \xi_i \) - is the phase of the wave emitted by atom i. Individual atoms are emitting with random phases \( \Rightarrow \) in the summation the cosine term gets smeared

\[ \Rightarrow I(t) \propto e^{-\frac{2\pi}{\tau_c}} \]

\( \tau_c \)-time constant

Also we know \( I(t) \propto e^{-\frac{\xi}{\tau}} \); \( \tau \)-lifetime of the state.

\[ \Rightarrow \tau_c = 2\tau \]

\[ \Rightarrow E(t) = E_0 e^{-\frac{\xi}{\tau_c}} \cos \omega t \]
The electric field of a decaying excited particle

\[ e(t) = E_0 e^{-\frac{t}{2\varepsilon}} \cos \omega_0 t \]

To find the frequency distribution of this signal we take its Fourier transform

\[ E(w) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e(t) e^{-iwt} dt, \]

where

\[ e(t) = \frac{E_0}{2} \left( e^{i(w_0 + \frac{\omega}{2\varepsilon})t} + e^{-i(w_0 - \frac{\omega}{2\varepsilon})t} \right) \text{ for } t > 0 \]

\[ e(t) = 0 \text{ for } t < 0 \]

The start of the period of observation at \( t = 0 \), taken at an instant when all the particles are pushed into the excited state, allows the lower limit of integration to be changed to 0.

\[ E(w) = \frac{1}{2\pi} \int e(t) e^{-iwt} dt = \frac{E_0}{4\pi} \left[ \frac{i}{(w_0 - w + \frac{\omega}{2\varepsilon})} - \frac{i}{(w_0 - w - \frac{\omega}{2\varepsilon})} \right] \]

\[ e(t) = E_0 e^{-\frac{t}{2\varepsilon}} \cos \omega_0 t \]

\[ \cos \omega_0 t = \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} \]

Euler formula

\[ e(t) = E_0 \left[ e^{i(\omega_0 t - \frac{\omega}{2\varepsilon})} + e^{-i(\omega_0 t + \frac{\omega}{2\varepsilon})} \right] = \frac{E_0}{2} \left[ e^{it(\omega_0 - \frac{\omega}{2\varepsilon})} + e^{-it(\omega_0 - \frac{\omega}{2\varepsilon})} \right] \]

Since \( i^2 = -1 \)

\[ E_0 \left[ e^{it(\omega_0 + \frac{\omega}{2\varepsilon})} + e^{-it(\omega_0 + \frac{\omega}{2\varepsilon})} \right] = \frac{E_0}{2} \left[ e^{it(\omega_0 + \frac{\omega}{2\varepsilon})} + e^{-it(\omega_0 + \frac{\omega}{2\varepsilon})} \right] \]
The intensity of emitted radiation is

\[ I(\omega) \propto |E(\omega)|^2 = E(\omega) \cdot E^*(\omega) = \frac{1}{(\omega - \omega_0)^2 + (\frac{1}{2\pi \Gamma})^2} \]

In terms of ordinary frequency

\[ I(\nu) \propto \frac{1}{(\nu - \nu_0)^2 + (\frac{1}{4\pi \Gamma})^2} \]

The lineshape function for natural broadening

\[ \Delta \nu = \text{the full width at half maximum height (FWHM)} \]

this occurs when

\[ \frac{1}{(4\pi \Gamma)^2} = (\nu_{\pm} - \nu_0)^2 \]

\[ \Rightarrow \Delta \nu = \nu_{\pm} - \nu_{\pm} = (\nu_{\pm} - \nu_0)^2 \cdot (\nu_{\pm} - \nu_0) = 2 \cdot \frac{1}{4\pi \Gamma} = \frac{1}{2\pi \Gamma} = \frac{A}{2\pi} \]

\[ \Rightarrow I(\nu) \propto \frac{1}{(\nu - \nu_0)^2 + (\frac{\Delta \nu}{2})^2} \]

The lineshape function for natural broadening

\[ g(M_n) = \frac{\frac{3}{4} \Delta \nu}{1 + \left[2(\nu - \nu_0)/\Delta \nu\right]^2} = \frac{2 \Delta \nu}{\pi \left[4(\nu - \nu_0)^2 + (\Delta \nu)^2\right]} \]

The rate of change of this function is

\[ 8 \Delta \nu \cdot \frac{1}{I(\nu)} \left(\nu - \nu_0\right)^2 + (\Delta \nu)^2 \]
Natural broadening is the same for each particle. It is a homogeneous broadening mechanism.

Other mechanisms of homogeneous broadening

1. Collision of phonons with the particles of the lattice perturb the phase of any excited, emitting particles - soft collision.
   Constant vibrational motion of the crystalline lattice particles can carry energy in discrete amounts. The packets of acoustic energy are called phonons.

2. By pressure broadening: interaction of the emitting particle with its neighbors causes perturbation of its emitting frequency and broadening of the transition.
   2a) Collisions with neutral particles.
   2b) Collisions with charged particles.
      Stark broadening: external electric field perturbs the energy level of atom, ion, molecule.

2c) Van der Waals and resonance interaction
   Excited particle exchange energy with like neighbors.
Inhomogeneous Broadening

• When the environment of particles in an emitting sample are non-identical, inhomogeneous broadening can occur.
• The shifts and perturbations of emission frequencies differ from particle to particle.
   In a real crystal the presence of imperfections and impurities in the crystal structure alters the physical environment of atoms from one lattice site to another. The random distribution of lattice point environments leads to a distribution of particles whose center frequencies are shifted in a random way throughout the crystal.

Doppler Broadening

• In a gas the random distribution of particle velocities leads to a distribution in the emission center frequencies of different emitting particles seen by stationary observer.
• if \( v_x \) - component of atom's velocity towards the observer than the observed frequency of the transition \( \nu = \nu_0 + \frac{v_x}{c} \nu_0 \); \( \nu_0 \) - stationary frequency
• The Maxwell-Boltzmann distribution of atomic velocities for particles of mass \( m \) at \( T \).

\[
f(v_x, v_y, v_z) = \left( \frac{m}{2\pi kT} \right)^{3/2} \exp \left[ -\frac{m}{2kT} (v_x^2 + v_y^2 + v_z^2) \right]
\]
If $N$ is the total # of atoms per unit volume than the # of atoms per unit volume that have velocities simultaneously in the range:

\[ \begin{align*}
V_x & \rightarrow V_x + dV_x \\
V_y & \rightarrow V_y + dV_y \\
V_z & \rightarrow V_z + dV_z
\end{align*} \]

is

\[ N \int f(V_x, V_y, V_z) dV_x dV_y dV_z \]

The \( \left( \frac{M}{2\pi kT} \right)^{3/2} \) factor is a normalization constant that ensures

\[ \int_{-\infty}^{\infty} f(V_x, V_y, V_z) dV_x dV_y dV_z = 1 \]

The normalized one-dimensional distribution of velocities for the particles in a gas:

\[ f(V_x) = \sqrt{\frac{M}{2\pi kT}} e^{-\frac{MV_x^2}{2kT}} \]

represents the probability that the velocity of a particle towards an observer is in a range $V_x \rightarrow V_x + dV_x$.

- It is the same as the probability that the frequency be in the range:

\[ \nu + \frac{dV_x}{c} \nu \rightarrow \nu + \left( \frac{V_x + dV_x}{c} \right) \nu = \nu + \frac{V_x}{c} \nu + \frac{dV_x}{c} \nu = \nu + \frac{dV_x}{c} \nu \]
the probability that the frequency lies in the range \( \nu \to \nu + d\nu \) is the same as the probability of finding the velocity in the range

\[
\frac{(\nu - \nu_0) c}{\nu_0} \rightarrow \frac{(\nu - \nu_0) c}{\nu_0} + \frac{c}{\nu_0} d\nu
\]

\( \nu_x \rightarrow \nu_x + d\nu_x \)

\( \nu = \nu_0 + \frac{\nu_x}{c} \nu_0 \) — Doppler frequency

\( \rightarrow \nu_x = \frac{(\nu - \nu_0) c}{\nu_0} \)

\( \nu_x + d\nu_x = \frac{(\nu - \nu_0) c}{\nu_0} + \frac{c}{\nu_0} d\nu \)

\( \rightarrow \) the distribution of the emitted frequencies is

\[
g(\nu) = \frac{c}{\nu_0} \sqrt{\frac{M}{2\pi k T}} \exp \left[ -\frac{M}{2 k T} \frac{c^2}{\nu_0^2} (\nu - \nu_0)^2 \right]
\]

normalized Doppler broadening lineshape function

FWHM

\[
\Delta \nu_0 = 2 \nu_0 \sqrt{\frac{2 k T c^2}{M c^2}}
\]

\[
g(\nu) = \frac{2}{\Delta \nu_0} \sqrt{\frac{c^2}{2\pi}} e^{-\left[ 2 (\nu - \nu_0)/\Delta \nu_0 \right]^2} c^2 \nu_0^2
\]
Comparison of normalized Lorentzian and Gaussian line shapes

Example:

The 632.8 nm transition of neon is the most important transition for laser oscillation in He-Ne laser. Atomic mass of neon is 20.8 g/mole.

\[ M = \frac{A}{N_A} = \frac{20.8}{6.02 \times 10^{23} \text{ particles/mole}} = 3.3 \times 10^{-23} \text{ g/mole} \]

\[ \rho = \frac{C}{\lambda_0} = \frac{3 \times 10^8 \text{ m/s}}{632.8 \times 10^{-9} \text{ m}} = 4.74 \times 10^{14} \text{ Hz} \]

\[ \Delta \nu_0 = 2 \rho \sqrt{\frac{2kT \rho \chi^2}{Mc^2}} = 2 \times 4.7 \times 10^{14} \sqrt{\frac{2 \times (3.8 \times 10^{-23} \text{ kg}) \times 400 \text{ K} \times 6.6 \times 10^{-26} \text{ m}^2}{(3.3 \times 10^{-26} \text{ kg}) \times (3 \times 10^8 \text{ m/s})^2}} \]

\[ = 1.52 \times 10^9 \text{ Hz} = 1.52 \text{ GHz} \]
Homogeneous Line Broadening

1) \[ \Delta E \Delta t \sim \hbar \]
   \[ \Delta t \Delta \omega \Rightarrow \Delta E \sim \frac{\hbar}{\omega} = \Delta \hbar \]
   \[ \Delta \nu = \frac{\Delta E}{\hbar} \Rightarrow \Delta \nu = \frac{\Delta \hbar}{\hbar} = \frac{A}{2\pi} \times \frac{1}{h} = \frac{A}{2\pi} \]

2) Lorentzian lineshape function for natural broadening

\[ I(\nu) = I_0 \frac{1}{1 + \left(\frac{\nu - \nu_0}{\Delta \nu}\right)^2} \]

Mechanisms of homogeneous broadening
- natural broadening
- electron-phonon interactions
- collisional broadening
- Stark broadening
- resonance interactions with unexcited particles.
Inhomogeneous Broadening

\[ g \left( \frac{v}{v_0} \right) = \frac{2}{\Delta v_0} \sqrt{\frac{\ln 2}{\pi}} e^{-\left[ \frac{2 (v - v_0)^2}{\Delta v_0^2} \right]} \]

\[ \Delta v_0 = 2v_0 \sqrt{\frac{2kT \ln 2}{MC^2}} \text{ for gas medium} \]
Homogeneous broadening always occurs at the same time as inhomogeneous broadening, to a greater or lesser degree.

Homogeneous broadening of a group of particles in a gas that have the same velocity $v$ and center frequency $v_0$.

Overall profile (Voight profile)

Overall lineshape results from the superposition of Lorentzian lineshapes spread across the Gaussian distribution of Doppler shifted center frequencies.

- If $\Delta v_L \ll \Delta v$ - overall lineshape is a pure Gaussian in a homogeneously broadened system.

- If $\Delta v_L \ll \Delta v$ - homogeneously broadened system.
Optical Frequency Amplification with a Homogeneously Broadened Transition

Consider the general case, when the monochromatic radiation field and the center frequency of the radiation are not the same.

- The closer $v$ is to $v'$, the greater the number of transitions that can be stimulated.
- The stimulated transitions occur at frequency $v$.
- The number of stimulated transitions is
  \[ N_s = N_0 B A \frac{f(v')g(v,v)}{(2\pi A)^2} \]

  \[ g(v,v) = \frac{2}{\pi \Delta v} \left( \frac{v-v}{\Delta v} \right)^2 \]
  for $v = v'$. This function $g(v,v)$ is the homogeneous line shape function of individual particles.

- An excited atom can interact with a monochromatic radiation that overlaps its homogeneous line shape profile.
As the wave passes through the medium it grows in intensity if the number of stimulated emissions exceeds the number of absorptions.

The change in intensity of the wave in travelling a small distance $dx$ through the medium is

$$dI_y = \frac{\text{# stim. emiss.} \times \text{# of abs.}}{\text{volume}} \times h\nu \times dx = \left[ N_2 h \frac{N_2 B_2 g(y',y) I_y}{\gamma} - N_1 h \frac{N_1 B_1 g(y',y) I_y}{\gamma} \right] x h\nu \times dx$$

Using Einstein relations:

$$dI_y = \frac{I_y}{c} \left( N_2 - \frac{g_2 N_1}{g_1} \right) \frac{c^3 A_2}{8\pi^2} h\nu g(y',y) \frac{I_y}{c}$$

Solution:

$$I_y = I_y(0) e^{-\gamma x}$$

where $I_y(0)$ - initial at $x=0$

$$\gamma = \left( N_2 - \frac{g_2 N_1}{g_1} \right) \frac{c^3 A_2}{8\pi^2} g(y',y)$$

Gain coefficient.

If $N_2 > \frac{g_2}{g_1} N_1$, then $\gamma > 0$ we have an optical frequency amplifier.
- if \( N_2 < \frac{g_2}{g_1} N_1 \), then \( f(x) \leq 0 \) and we have net absorption of the incident radiation.

- For a system in thermal equilibrium:
  \[ \frac{N_2}{N_1} = \frac{g_2}{g_1} \exp^{-\frac{h\nu}{kT}} \quad \text{for } T > 0 \quad \exp^{-\frac{h\nu}{kT}} < 1 \]
we have no positive gain.

- For negative temperature we have population inversion \( N_2 > \frac{g_2}{g_1} N_1 \), we have positive gain.
  - It is not a true state of thermal equilibrium.
  - It can be maintained by feeding energy into the system.

- in the discussion we have neglected the occurrence of spontaneous emission.
  - Total amount of spont. emissions into a small solid angle is very small.
    \[ N_2 \propto \frac{8\nu^2}{c} \]
  - There is no a constant phase relationship with the incident wave.
The Stimulated Emission rate in a Homogeneously Broadened Transition

The stimulated emission rate $W_{21}(\nu)$ is the number of stimulated emissions per particle per second per unit volume caused by a monochromatic input wave at frequency $\nu$

$$W_{21}(\nu) = B_{21} g(\nu', \nu) \rho(\nu)$$

$$W_{21}(\nu) = \frac{A_{21} \cdot c^2 I_0}{8 \pi \hbar \nu^3} \cdot g(\nu', \nu)$$

The frequency variation of $W_{21}(\nu)$ follows the lineshape function $g(\nu', \nu)$

The total # of stimulated emissions is $N_s = N_2 W_{21}(\nu)$
Optical Frequency Amplification with Inhomogeneous Broadening

We can divide the atoms up into classes, each class consisting of atoms with a certain range of center emission frequencies and the same homogeneous lineshape.

- The class with center freq. $\nu''$ in the freq. range $d\nu''$ has $N\nu''(\nu',\nu'')\,d\nu''$ atoms in it.

- $g_\nu(\nu',\nu'')$ normalized inhomogeneous distribution of center frequencies.

- The inhomogeneous lineshape function centered at $\nu'$.

- This class of atoms contributes to the change in intensity of a monochromatic wave at frequency $\nu$ as

$$
\Delta\langle dI\nu \rangle (\text{from the group of particles in the band } d\nu'') =
\left[ N_2 B_2 \int g_\nu(\nu',\nu'')\,d\nu'' \right] \frac{I\nu}{c} - \int N_1 B_1 g_\nu(\nu',\nu'')\,d\nu'' \frac{I\nu}{c} \int hr\,d\tau
$$

where $g_\nu(\nu',\nu'')$ is the homogeneous lineshape function of an atom at center frequency $\nu''$.
The increase in intensity from all the classes of atoms is found by integrating over these classes over the range of frequencies.

\[ dI_v = \frac{I_v}{c} \left( N_2 B_2 - N_1 B_1 \right) \left[ \int g_0 (v', v) g_2 (v', v) dv' \right] dv. \]

**Solution**

\[ I_v = I_v(0) e^{-\sigma (v) t}. \]

\[ J (v) = \left( N_2 - \frac{N_2}{3} N_1 \right) \frac{A_z^2}{8\pi v^2} g (v', v) \]

where \( g (v', v) \) is the overall lineshape function, defined as

\[ g (v', v) = \int g_0 (v', v') g_2 (v', v') dv'. \]

Convolution of the homogeneous and inhomogeneous lineshape functions.

If we measure frequency relative to the center frequency of the overall lineshape \( v' = 0 \)

\[ J (0, v) = \int g_0 (0, v') g_2 (v', v) dv' = \int g_0 (0, v') g_2 (0, v - v') dv'. \]
\[ g(r) = \int g_2(r') g_2(r-r') \, dr' \]

It is a standard convolution integral of two functions \( g_2(r) \) and \( g_2(r) \).

- If \( g_2(r', r'') \) is Gaussian lineshape, \( g_2(r'', r') \) is Lorentzian then

\[ g(r', r) = \frac{2}{4\pi^2} \sqrt{\frac{\alpha/2}{\pi}} \int e^{-\frac{r^2}{4\alpha^2}} \, dr \]

\[ g = \Delta x \sqrt{\arctan 2 / \Delta x} \]

\[ x = 2(r-r') \sqrt{\arctan 2 / \Delta x} \]

The integral in Voigt profile cannot be evaluated analytically but must be evaluated numerically.

Normalized Voigt profile.
Optical Frequency Oscillation - Saturation

- If we can force a medium into a state of population inversion for a pair of its energy levels, the transitions between these levels forms an optical frequency amplifier.
- To turn an amplifier into an oscillator, apply positive feedback by inserting the medium between a pair of appropriate mirrors $M_1, AM, M_2$.
- Oscillation occurs when gain $>$ losses.
- The level at which the oscillation stabilizes is set by the way in which the amplifier saturates.

Homogeneous System

- Consider an amplifying transition at center frequency $\nu_0$ between two energy levels of an atom.
- Maintain this pair of levels in population inversion by feeding in energy.
- In equilibrium if $\delta(\nu)=0$ (absence of an external field), the rates $R_2$ and $R_1$, at which atoms are fed into this level, must be balanced by spontaneous emission $\delta$ and nonradiative loss processes.
If \( x_{2j} \) is the rate per particle per unit volume by which collisions depopulate level 2 and cause a particle to end up in a lower state \( j \)

\[
\frac{1}{\tau_2} = \sum_j (R_{2j} + x_{2j})
\]

In equilibrium

\[
\frac{dN_2^0}{dt} = R_2 - \frac{N_2^0}{\tau_2} = 0
\]

for level 2

where \( N_2^0/\tau_2 \) - total loss rate per unit volume from spontaneous emission + other deactivation processes.

\[
\Rightarrow N_2^0 = R_2 \tau_2
\]

\( N_2^0 \) indicates that the population is calculated in the absence of a radiative field.

\[
\frac{dN_1^0}{dt} = R_1 + N_2^0 A_{21} - \frac{N_1^0}{\tau_1}
\]

for level 1

\[
\Rightarrow N_1^0 = (R_1 + N_2^0 A_{21}) \tau_1 = (R_1 + R_2 \tau_2 A_{21}) \tau_1
\]

\[
(N_2^0 - \frac{\tau_1}{\tau_2} N_1^0) = \Delta N^0 = R_2 \tau_2 - \frac{\tau_2}{\tau_1} (R_1 + R_2 \tau_2 A_{21})
\]

the population of inversion.
When $g_1 = g_2$

\[ \Delta N^0 = R_2 \zeta_2 - \zeta_1 (R_1 + R_2 \zeta_2 A_{22}) \]

1. Now feed in an external electric field radiation, $P(x) = \frac{I(x)}{c}$

The rate at which this signal causes stimulated emission

\[ W_{21}(y) = \int B_{21} g(y_0, y) P(x) \, dx \]

$g(y_0, y)$ - homogeneous lineshape function.

for a white radiation

\[ W_{21}(y) = B_{21} \int_{-\infty}^{\infty} g(y, y) P(x) \, dx = B_{21} P(x) \]

for a monochrom. plane wave

\[ W_{21}(y) = B_{21} \frac{2}{c} \int_{-\infty}^{\infty} g(y, y) P(x) \, dx = B_{21} g(y, y) \frac{2}{c} \]

\[
\begin{array}{c}
\begin{array}{c}
N_2 \\
\hline
R_2 \\
\hline
W_0 \\
\hline
N_1 \\
\hline
R_1 \\
\hline
N_{21} A_{21}
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\frac{dN_2}{dt} = R_2 - \frac{N_2}{\tau_2} - N_2 B_{21} g(y_0, y) P(x) + N_1 B_{12} g(y_0, y) P(x) = 0
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{c}
\frac{dN_1}{dt} = R_1 + N_2 A_{21} - \frac{N_1}{\tau_1} + N_2 B_{21} g(y_0, y) P(x) - N_1 B_{12} g(y_0, y) P(x) = 0
\end{array}
\end{array}
\]

If $B_{21} g(y_0, y) P(x) = W_{21}(y)$ and $g_1 = g_2$

\[ W_{12}(y) = W_{21}(y) = W \]
\[
R_2 - \frac{N_2}{\varepsilon_2} - N_2 \, \text{W} + N_1 \, \text{W} = 0 \quad \Rightarrow \quad N_2 = \frac{N_1 \, \text{W} + R_2}{\varepsilon_2 + \text{W}}
\]

\[
R_1 + N_2 \, \text{A}_{21} - \frac{N_1}{\varepsilon_1} + N_2 \, \text{W} - N_1 \, \text{W} = 0 \quad \Rightarrow \quad N_2 = \frac{N_1 + N_1 \, \text{W} - R_1}{\varepsilon_{21} + \text{W}}
\]

\[
\frac{N_1 \, \text{W} + R_2}{\varepsilon_2 + \text{W}} = \frac{N_1}{\varepsilon_1} + N_2 \, \text{W} - R_1
\]

\[
N_1 = \frac{R_1 \, \text{W}_2 + R_2 \, \text{A}_{21} + \text{W} \, (R_1 + R_2)}{\frac{1}{\varepsilon_i} + \text{W} \left(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2} - \text{A}_{21}\right)}
\]

\[
N_2 - N_1 = \frac{N_1 \, \text{W} + R_2}{\varepsilon_2 + \text{W}} - N_1 = \frac{N_1 \, \text{W} + R_2 - \frac{N_1}{\varepsilon_2} - R_1 \, \text{W} - N_1 \, \text{W}}{\varepsilon_2 + \text{W}}
\]

\[
N_2 - N_1 = \frac{R_2 \, \text{W}_2 - R_1 \, \text{W}_1 - R_2 \, \text{W}_1 \, \text{W}_2 \, \text{A}_{21}}{1 + \text{W} \, \varepsilon_2 \left(1 + \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2} - \text{A}_{21} \, \text{W}\right)} = \frac{\Delta N^0}{1 + \frac{\Phi}{\text{W}_1 / \text{A}_{21}}}
\]

where \(\Phi = \text{A}_{21} \, \text{W}_2 \left[1 + \left(1 - \text{A}_{21} \, \text{W}_2\right) \frac{\text{W}_1}{\text{A}_{21}}\right]\)

we know that \(\text{W} = \frac{c^2 \, \text{A}_{21}}{874 \, \text{W}_3^3} \text{I}_y \, g(y_0, v)\)

if we define \(\text{I}_y^3 (v) = \frac{874 \, \text{W}_3^3}{c^2 \, \Phi} \text{I}_y (y_0, v)\)

\[
N_2 - N_1 = \frac{\Delta N^0}{1 + \frac{\Phi}{\text{I}_y^3 (v)}}
\]
gain of a laser amplifier

\[ \gamma(v) = (N_2 - N_1) \frac{c^2 A_2 I}{8\pi \hbar^2 v^2} g(v_0, v) \]

The gain as a function of intensity in a homogeneously broadened system

\[ \gamma(v) = \frac{\Delta N^0}{[1 + \frac{I}{I_s(v)}]} \frac{c^2 A_2 I}{8\pi \hbar^2 v^2} g(v_0, v) \]

\[ I_s = \frac{c^2 A_2 I}{8\pi \hbar^2 g(v_0, v)} \]

- Gain saturates as the strength of the amplified signal increases.
- Good amplifier should have a large value of \( I_s \).
- \( I_s \uparrow \phi \downarrow \phi = A_2 \frac{I_s}{I_s(v)} \left[ 1 + (1 - A_2 I_s(v)) \frac{I_s}{I_s(v)} \right] \)

\[ \phi = A_2 \frac{1}{\tau_2} \Rightarrow \phi \approx 1 \]

\[ \gamma_0 = (N_2 - \frac{2\pi}{\hbar^2} N_1) \left( \frac{c^2 A_2 I}{8\pi \hbar^2 v^2} g(v_0, v) \right) = (N_2 - \frac{2\pi}{\hbar^2} N_1) \gamma_0 \]

\[ I_s = \frac{N^0 h v^3}{c^2 \phi g(v_0, v)} = \frac{8\pi \hbar^2 v^2 \phi g(x, h v)}{c^2 A_2(g(x, h v)) \times \tau_2} = \frac{h v}{\sigma(x, v)} \frac{v^2}{\tau_2} \]

\[ \phi = A_2 \frac{1}{\tau_2} \left[ 1 + (1 - A_2 \frac{I_s}{I_s(v)}) \frac{I_s}{I_s(v)} \right] \]

\[ \sqrt{I_s} = \frac{h v}{\sigma(v) \tau_2} \]