ROBUST FITTING OF ELLIPSES TO NON-COMPLETE AND CONTAMINATED DATA

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1 Introduction

The circle and ellipse fitting problems are getting quite important in contemporary experimental physics due to the wide use of the RICH (Ring Imaging CHerenkov) detectors as the main instrument for the particle identification. Cherenkov radiation photons produced by charged particles transversing the RICH's sensitive volume (radiator) are then focused onto a detector and converted to electronic avalanches to be read out via multi-wires proportional chamber or other two-dimensional array of many thousand photosensitive sells (pads). During an event a number of Cherenkov photons produced by a detected particle falls on the pad plane and dependently on the particle incident angle forms rings or ellipses. Besides sometime a part of photons formed these ellipses are not detected beeing on the edge of a sensitive area or due to internal reflection from the detector surface (see for example [1, 2].

Thus in this paper we consider the problem of reconstructing the ellipse parameters from measurements of some part of an ellipse in presence of contaminating points. On the first glance the statistical problem of robust ellipse fitting looks as a simple generalization of the circular fitting, where an essential experience is accumulated in physical and other applications [3, 4, 9, 6, 5]. However some serious obstacles should be poited out that make the ellipse fitting problem much harder than circle fitting one from both, analytical and statistical points of view: (1) Unlike circles ellipses are not invariant to rotations, so all five basic ellipse parameters (its principal half-axes a and b, center co-ordinates x_c, y_c and the angle α of its main axis direction) must be searched. It makes the problem quite non-linear. (2) Considering an arbitrary ellipse in arbitrary axes one should use as its equation a general equation of the second order curve, but without the special ellipse constraint one could obtain some other conic section, for instance, hyperbola. (3) In order to obtain the best, most exact fit one should minimize the functional constructed from geometrical distances of given points to the searched ellips. That should inevitably increase the non-linearity of the problem. (4) As it was pointed above, in our par-

ticular application measured data can be incomplete and are contaminated by noise and points of other, close ellipses, therefore a special robust technique application (see, for instance [9]) is inavoidable.

2 Algebraic fitting

We have to fit an ellipse to the given set of measurements $(x_i, y_i), i = \overline{1,n}$. The ellipse equation in its canonical form is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

Shift the center to (x_c, y_c) and rotation on the angle α give the general equation of the second order curve

$$Q(x,y) = Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

The special ellipse constraint $(4AC - B^2) > 0$ is needed. Well known algebraic ellipse fit, i.e. minimizing of

$$L(A,B,C,D,E,F) = \sum_{i=1}^{n} Q(x_i, y_i)^2 \tag{2}$$

has quite a rich history (see surveys in [6, 5]). Its obvious disadvantage is that the value of $|Q(x_i, y_i)|$ does not represent the distance from the point (x_i, y_i) to the ellipse. That leads to many unpleasant diffuculties, in particular, in cases of non-complete and contaminated measurements [4].

3 Kepler's ellipse definition

It is clear now that the same problem should arise in the case of the algebraic ellipse fit by minimizing (2).We prefer Kepler's ellipse definition as a locus formed by points for each the sum of its distances from two fixed point (focuses (x_{f1}, y_{f1}) and (x_{f2}, y_{f2})) is a constant:

$$d_1 + d_2 = 2a, \tag{3}$$

where $d_1 = \sqrt{(x - x_{f1})^2 + (y - y_{f1})^2}, d_2 = \sqrt{(x - x_{f2})^2 + (y - y_{f2})^2}$.

Denoting $\rho_i = d_1 + d_2 - 2a$
we have the problem to minimize

$$L(x_{f1}, y_{f1}, x_{f2}, y_{f2}, a) = \sum_{i=1}^{n} \rho_i^2 \Rightarrow min_{x_{f1}, y_{f1}, x_{f2}, y_{f2}, a} \tag{4}$$

This approach is non-linear and involves us into an iterative minimization procedure, but it has some advantages.

- It would give us namely an ellipse without applying any constraints.

– Parameters are invariant to any shifts and rotations, besides it is easy to derive from them five basic ellipse parameters a, b, x_c, y_c, α needed for physicists.

– All these parameters are of the same order of magnitude what is important for choosing steps in parameter space during iterations.

As for non-linearity, we are going even to aggravate it due to two more requirements:

- to overcome data contamination by robust fitting;

- to obtain the most accurate parameter estimations by accomplishing the best geometric fit.

Our first attempt to carry out the weighted robust fit in the following iterative algorithm minimizing the weighted analog of (4):

$$L(x_{f1}, y_{f1}, x_{f2}, y_{f2}, a) = \sum_{i=1}^{n} w(\rho_i)\rho_i^2, \tag{5}$$

with the optimal functional weights [7]

$$w(\rho) = \frac{1}{1 + (\rho/const)^2}. \tag{6}$$

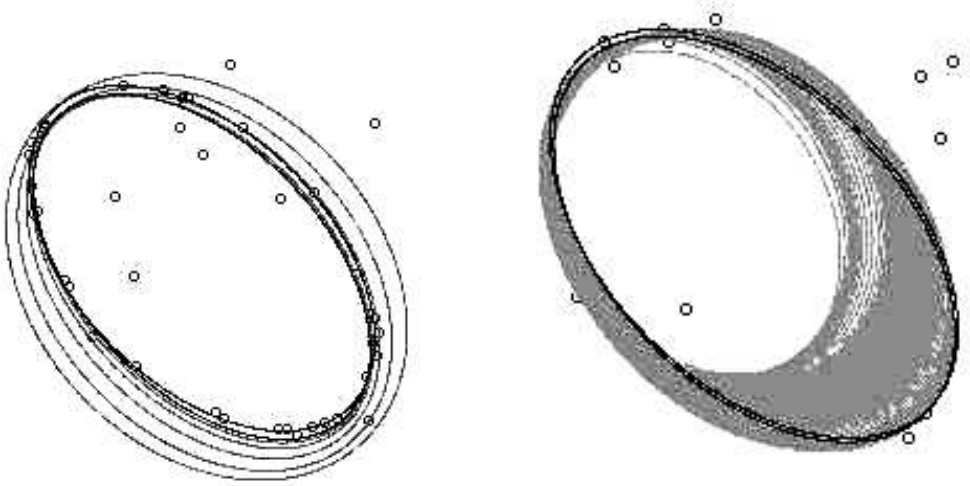
If one differentiates (5) by parameters and equals derivatives to zero, one obtaines a simple iterative scheme (m is the iteration number):

$$\left\{ \begin{array}{lcl} x_{f1}^{(m)} & = & (\sum w_i x_i g_1^{(m-1)})/G_1^{(m-1)}, \\ y_{f1}^{(m)} & = & (\sum w_i y_i g_1^{(m-1)})/G_1^{(m-1)}, \\ x_{f2}^{(m)} & = & (\sum w_i x_i g_2^{(m-1)})/G_2^{(m-1)}, \\ y_{f2}^{(m)} & = & (\sum w_i y_i g_2^{(m-1)})/G_2^{(m-1)}, \\ a^{(m)} & = & (\sum_{i=1}^{n} w_i (d_1^{(m-1)} + d_2^{(m-1)}))/(2\sum_{i=1}^{n} w_i), \end{array} \right. \tag{7}$$

where
$g_1^{(m)} = 1 + (d_2^{(m-1)} - 2a^{(m-1)})/d_1^{(m-1)}$,
$g_2^{(m)} = 1 + (d_1^{(m-1)} - 2a^{(m-1)})/d_2^{(m-1)}$,
$G_1^{m-1} = \sum_{i=1}^{n} w_i g_1^{(m-1)}$, $G_2^{m-1} = \sum_{i=1}^{n} w_i g_2^{(m-1)}$,
$d_1^{(m-1)} = \sqrt{(x_1^{(m-1)} - x_{f1})^2 + (y_i - y_{f1}^{(m-1)})^2}$
($d_2^{(m-1)}$ is obtained similarly).

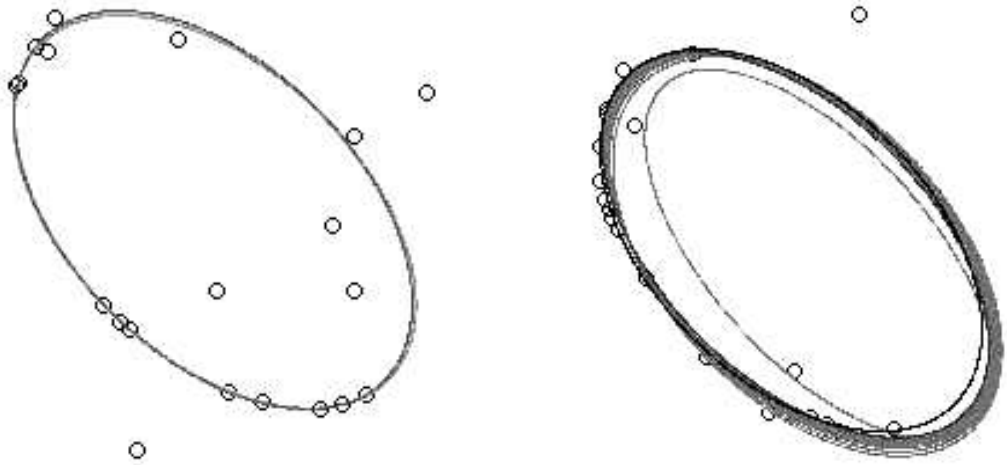
Initial values $x_{f1}^{(0)}, y_{f1}^{(0)}, x_{f2}^{(0)}, y_{f2}^{(0)}, a^{(0)}$ are obtained either by setting $g_1^{(0)} = g_2^{(0)} = 1$ or from physical background, which is as usual expected from out-of-RICH TPC measurements.

This algorithm was implemented in C^{++}. Some of results of its application are presented in fig.1 and fig.2. It works in cases of non-complete and contaminated data, but its efficiency depends too much on the initial parameter values, besides on this way we could not meet the best fit requirement.



(a) (b)

Fig. 1. Robust fit of the Kepler's equation. Complete ellipse, contaminated data. (a) $\sigma = 0.1$, 7 iterations (b) $\sigma = 1$, 120 iterations



(a) (b)

Fig. 2. Robust fit of the Kepler's equation. Non-complete (one-half of the ellipse arc) and contaminated data. (a) $\sigma = 0.1$ (of bin-size), good initial parameter values, (b) $\sigma = 1$

4 Robust geometric fitting

Geometrical fit. Under the assumption that all the experimental points are quite close to our ellipse (3) with the equation

$$K(i; x_{f1}, y_{f1}, x_{f2}, y_{f2}, a) = \rho_i = 0$$

the distance from the point (x_i, y_i) to the ellipse is approximately equal to

$$D_i = |K(i; ....)|/||gradK(i; ....)||, \tag{8}$$

where $||gradK(i; ....)||$ is the length of the gradient vector of the function $K(i; ....)$ at the point (x_i, y_i) or on the ellipse near that point. Now, instead of (4) for the best fit we need to minimize

$$\tilde{L}(x_{f1}, y_{f1}, x_{f2}, y_{f2}, a) = \sum_{i=1}^{n} \rho_i^2/||gradK(i; ....)||^2 \tag{9}$$

However, because of the data contamination we have to use one of robust methods. We choose the summed Gaussian weights approach [9].

$$\tilde{\tilde{L}} = -\sum_{i=1}^{n} e^{-\frac{D_i^2}{2\sigma_w^2}} \Rightarrow min_{x_{f1}, y_{f1}, x_{f2}, y_{f2}, a} \tag{10}$$

with D_i from (8) and a suitably chosen σ_w. The fast drop of the double negative exponent for big D_i guarantees that our basic assumption is accomplished. The expansion of $\tilde{\tilde{L}}$

$$\tilde{\tilde{L}} = -n + \sum_{i=1}^{n} \frac{D_i^2}{2\sigma^2} - 1/2 \sum_{i=1}^{n} \frac{D_i^4}{4\sigma^4} + ... \tag{11}$$

shows that for small D_i $\tilde{\tilde{L}}$ can be interpreted as χ^2 for conventional unweighted least square method.

5 FUMIVI and its applications

Unfortunately, the structure of this functional (10) is really disgusting. It is non-convex, has many local minima sometimes at the end of a curved valley. Therefore, its matrix of the second derivatives by parameters is not everywhere positively defined.

An experience with handling such functionals in the simpler problem of robust circle fitting [9] shows that one of the most equipped famous MINUIT minimizig program appeared to be too time consuming.

However the new minimization program
FUMIVI (FUnction MInimization by Valley Investigation) was recently developed by one of authors [13, 14] especially to overcome such type of problems.

The main FUMIVI features are:

- Minimization of arbitrary regular functions with the arbitrary structure.
- Constraints of an arbitrary structure can be applied.
- In the case of the known functional structure the simplified matrix of the second derivatives by parameters is used with a linear expansion of the functional argument as well as the quadratic expansion of the functional itself. It gives the essentional decreasing of the calculation amount. The quality of the above expansions is controlled automatically by the FUMIVI algorithm.
- The FUMIVI is using the quadratic approximation of the minimized function in order to minimize it on an auxilary box in the parameter space that limits parameter steps. It is remarkable that it is possible also for non-positive quadratic functions.
- Special means are provided in FUMIVI to speed up considerably a deccent along curved multidimentional valleys in parameter space.

Thus FUMIVI gives us an unique complex of resources to solve a wide circle of complicated minimization problems, in particular, our robust fitting of ellipses is a suitable object for it.

The FORTRAN-77 version of FUMIVI was used (although a simplified C^{++} version is also available [13, 14].

6 Simulation results and concluding remarks

We use Monte-Carlo simulations to test more in details the accuracy and efficiency of proposed methods and the dependences of their results on such factors as (i) the sample contamination, (ii) measurement uncertainties, (iii) a quality of initial values of parameters.

The measured data points were simulated as follows. n points were distributed along an ellipse according to the Poisson law with normally distributed $\mathcal{N}(0, \sigma_0)$ deviations from the ellipse.

A contamination was simulated by N_{cont} points spread uniformly in the area surrounding the ellipse. N_{cont} was used in percents of the number n (0%, 50%, 100%). We used the pad-size as the basic unit, so the typical ellipse diameter was chosen orienting to COMPASS preliminary data as 30. For non-comlete ellipses n points were distributed randomly along one-half of the ellipse angle arc.

One essential factor should be stressed saying about robust weight methods. The width of the weight function, i.e. coefficient σ_w in (10) must decrease stepwisely in full accordance with the famous simulated annealing approach [11] There is also a deep similarity of our robust weighted method with the elastic arm or deformable template approach (see, for instance [12]). Dependently on the initial approximation of parameters the value of σ_w must be choosen big enough to allow to capture the majority of ellipse points. Then after the iteration convergence σ_w must be taken much smaller to allow to remove noise points etc until σ_w would become of the same order of magnitude as the scatter σ_0 of the ellipse points.

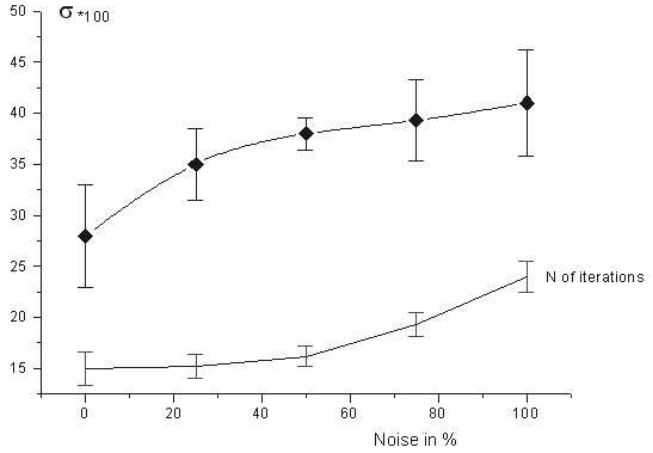


Fig. 3. Fitting results for complete ellipse. Upper curve presents a dependence of parameter RMS (magnified by 100) on the contamination level N_{cont}. Lower curve depicted the average number of FBEA iterations on all three stages of the elastic algorithm.

Thus an FUMIVI-based elastic algorithm (FBEA) was proposed. It calls the FUMIVI program modified to use the summed Gaussian weights method (10) three times with σ_w equal correspondingly to $10\sigma_0, 3\sigma_0$ and σ_0. We choose the following characteristics of the fitting results: RMS of restored parameters, the total number of iterations on all three stages of our elastic algorithm and the FBEA efficiency, i.e. relative number of events which parameter estimations were within the unit sphere around the simulated values. Results showing the dependence of the first two of these criteria on the contamination level N_{cont} for $\sigma_0 = 0.3$ and good inial parameter values are depicted in fig.3. The upper curve presents parameter RMS (magnified by 100) versus N_{cont}. Vertical bars show the RMS range for all five parameters a, b, x_c, y_c, α. Lower curve depicted the average number of FBEA iterations on all three stages of elastic algorithm reflecting its speed. The FBEA efficiency was on the satisfactory level 98.5-99% decreasing to 97% for $N_{cont} = 100\%$. The same level of accuracy we obtained also for non-complete ellipse, although the efficiency fell down to 92-93%.

We tested also the FBEA stability to the growth of the point scattering around the ellipse. As expected, it leads to a linear growth of parameter RMS, but already for $\sigma_0 = 0.4$ the iteration number increased drastically and the efficiency fell down to 90%.

A remarkable fact was that the FBEA kept converging to the right solution (known due to Monte-Carlo simulations) in 92% of events even for randomly chosen initial values of ellipse parameters. However, it took sometime up to 220 iterations, besides we could qualify some of solutions as wrong only because of knowing the right one, what would be excluded in the real life.

Relative slowness of the FBEA in comparison to the robust version of the algebraic fit, or to the straightforward iterations (7) or, at least, to the simplified FUMIVI fit without including the gradient brings one to an idea of a combined method, in which some of these faster methods is used on the first two of the simulated annealing stages. The obtained parameter values are used then on the concluding stage for the final fit. We tested some of these combinations. They are much faster sometimes, but the important observation was that for the level of contamination higher than 50% it leads to inadmissible high persentage of cases when iterations converge to one of local minima and stuck in it.

Thus for cases of a heavy contaminated samples we would recommend the FBEA method as a very reliable and accurate. There are stil some reserves to speed it up. We obtained already the factor four in speed by optimizing the program and, especially, by rewritting the matrix inverse subroutine including in it special index operations.

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