Statistical Properties of 2-D Generalized Hyperbolic Attractors

V. S. Afraimovich[1], N. I. Chernov[1] and E.A. Sataev[2]

[1]Center for Dynamical Systems and Nonlinear Studies
Georgia Institute of Technology, Atlanta, GA 30332
[2]Institute of Nuclear Power Engineering
Obninsk 249020, Studgorodok, Russia

Abstract

Recently Ya.B. Pesin introduced a large class of hyperbolic attractors, and for those attractors he established the Smale spectral decomposition. In this paper our main results are a stretched exponential bound on the decay of correlations and the central limit theorem. Also we will obtain conditions under which two well known attractors – those of Belykh and Lozi – are subject to our main results.

1 Introduction

In Ref. [1] Ya.B. Pesin introduced a class of dynamical systems which he called generalized hyperbolic attractors. This class includes three well known attractors: the Lorenz attractor, the generalized Lozi attractor and the Belykh attractor.

Also, in Ref. [1] Pesin established certain hyperbolic, ergodic and topological properties for that same class of dynamical systems. In particular, he obtained the so called Smale spectral decomposition for those attractors. This decomposition yields countably many components, each ergodic with respect to any Gibbs u-measure. In this context Gibbs u-measures are often called Bowen-Ruelle-Sinai measures, or, more briefly, BRS-measures. Furthermore, every ergodic component is decomposed into finitely many subsets which are cyclically permuted, and on each of those subsets the corresponding iteration of the map is of mixing and Bernoulli type.

Recently, Sataev [2] has shown that, under a certain additional assumption, the number of ergodic components is finite.

For certain values of the relevant parameters all the examples cited above satisfy the assumptions required by Pesin and Sataev. Hence, those same examples have all the properties just mentioned.

Furthermore, one may expect that these three attractors are ergodic – even mixing – for some values of their parameters. The ergodicity has been proved for certain Lorenz attractors and for certain Lozi attractors. For the case of Lorenz attractors, the relevant definitions appear in Ref. [3] and the necessary proofs in Ref. [4]. For the case of Lozi attractors, the appropriate definitions and proofs appear in Refs. [5] and [6, 7, 8], respectively.

In this paper our goal is to establish strong statistical properties for all those maps which possess generalized hyperbolic attractors in subcomponents of the attractors where the maps or their iterates are mixing. We will establish a stretched exponential bound on the decay of correlations for Hölder continuous functions on attractors, and we will prove the central limit theorem for these functions. Our reasoning is based on the technique of Markov approximation to hyperbolic dynamical systems developed in Refs. [9, 10, 11, 12, 13]. The same such results have been obtained there for hyperbolic billiards and similar models. The reader will note that the Lorentz gas with an external field studied in Ref. [12] is, in fact, a hyperbolic attractor of a special kind. In an early work [14] the Markov approximation techniques, in a different form, were used to establish good statistical properties for the Lorenz attractor. Here we treat two other examples, the Lozi and Belykh attractors.

In all what follows we shall consider only two-dimensional systems. This will significantly simplify our arguments. Note, however, that the techniques of Markov approximations can work in multidimensional case, too, cf. Ref. [13]. All the examples mentioned above are two-dimensional. We will also impose some additional technical conditions, which, as discussed in Section 6 below, are satisfied for Lozi and Belykh attractors in open regions of parameters.

2 Generalized hyperbolic attractors

Let M be a smooth two-dimensional manifold; $U \subset M$, an open connected subset with compact closure; and $\Gamma \subset U$, a closed subset. We assume that the set $S^+ = \Gamma \cup \partial U$ consists of a finite number of compact smooth curves. The set $U \setminus \Gamma$ consists of a finite number of open connected components. We denote by ρ the Riemannian metric on M and by $\mathrm{Vol}(\cdot)$ the Riemannian volume in M.

Let $f : U \setminus \Gamma \to U$ be a C^2-diffeomorphism of the open set $U \setminus \Gamma$ onto its image $f(U \setminus \Gamma)$. We assume that f is twice differentiable up to the boundary $\partial(U \setminus \Gamma)$. This boundary, $\partial(U \setminus \Gamma)$, coincides with S^+. Note that S^+ is the singularity set for the map f. The boundary $\partial(f(U \setminus \Gamma))$ is then a finite union of compact smooth curves, which we denote by S^-. That union, S^-, is the set of singularities for f^{-1}. The inverse map f^{-1} is twice differentiable up to S^-. Since $\bar{U}$ is compact in M, the first and the second partial derivatives of both f and f^{-1} are uniformly bounded.

The differentiability up to the singularity curves for both f and f^{-1} is the most restrictive assumption here. However, both our examples – the Lozi and Belykh attractors – satisfy that assumption. Certain mild singularities of the first and second derivatives do not prevent the machinery of Markov approximations from working in the case of billiards, cf. Refs. [10, 12].

Let $U^+ = \{x \in U : f^n(x) \notin S^+, n = 0, 1, 2, \ldots\}$ and $D = \cap_{n \geq 0} f^n(U^+)$. The set D is invariant under both f and f^{-1}. Its closure $\Lambda = \bar{D}$ is called the attractor for f.

Remark. We do not exclude the examples when $\mathrm{Vol}(\Lambda) > 0$, for instance, piecewise linear toral automorphisms [11]. Our results are valid in those cases. However, the term *attractor* is commonly applied to the systems with $\mathrm{Vol}(\Lambda) = 0$. In order to assure that $\mathrm{Vol}(\Lambda) = 0$ one usually assumes that $\mathrm{clos}(f(U \setminus \Gamma)) \subset U$, cf. Ref. [2].

Next, we are going to define a hyperbolic structure for the map f. Specifically, consider any point $z \in U$, any line P lying in the tangent plane T_zM and any real number $\alpha > 0$. In all that follows we shall refer to the set $\{v \in T_zM : \angle(v, P) \leq \alpha\}$ as the cone $C(z, \alpha, P)$. Also, we shall assume that for each point $z \in U \setminus S^+$ there are two cones $C^u(z) = C(z, \alpha^u(z), P^u(z))$ and $C^s(z) = C(z, \alpha^s(z), P^s(z))$ having the following three properties:

(1) the angle between $C^u(z)$ and $C^s(z)$ is uniformly bounded away from zero;

(2) $df(C^u(z)) \subset C^u(fz)$ for any $z \in U \setminus S^+$ and $df^{-1}(C^s(z)) \subset C^s(f^{-1}z)$ for any $z \in f(U \setminus S^+)$;

(3) there exist constants $C > 0$ and $\lambda \in (0, 1)$ such that for any integer $n > 0$

(a) if $z \in U^+$ and if $v \in C^u(z)$, then $||df^n v|| \geq C\lambda^{-n}||v||$;

(b) if $z \in f^n(U^+)$ and if $v \in C^s(z)$, then $||df^{-n} v|| \geq C\lambda^{-n}||v||$;

An attractor Λ is called a generalized hyperbolic attractor [1] if these two families of cones exist. One can always find an integer $m \geq 1$ so that f^m enjoys the properties (1)–(3) with $C = 1$, see Ref. [2]. In all that follows we shall assume that $C = 1$.

Let z be any point in D. Standard arguments, cf. Ref. [1], yield families of invariant

subspaces E_z^u and E_z^s in T_zM with the following two properties:
(a) $E_z^u \subset C^u(z)$ and $E_z^s \subset C^s(z)$;
(b) $df E_z^{u,s} = E_{f(z)}^{u,s}$.

In addition to Properties (1)–(3) formulated above, we shall henceforth assume the following property:
(4) The two cones $C^u(z)$ and $C^s(z)$ depend continuously on $z \in U^+$. Furthermore, for any point $z \in \Gamma$ the two limit cones $C^{u,s}(z) = \lim_{z' \to z} C^{u,s}(z')$ exist on both sides of Γ. Also, the angle between the tangent line to Γ at z and the unstable limit cone $C^u(z)$ is uniformly bounded away from zero.

The condition (4) was also assumed in Ref. [1].

Overall, the preceding assumptions guarantee a uniform hyperbolic structure for the map f. The expansion factors in C^u and the contraction factors in C^s are uniformly bounded away from unity. Since f is differentiable up to the boundary of $U \setminus S^+$, the expansion and contraction factors are also uniformly bounded above. The singularities of f and f^{-1} are "mild": they are concentrated on a finite union of smooth compact curves, and the first and second derivatives of f and f^{-1} have one-sided limits there. However, we do need an additional assumption that guarantees, in an appropriate fashion, that expansion and contraction prevail over discontinuities. The following such assumption was formulated in Ref. [1].

Condition A1. There exists an integer $\tau \geq 1$ such that $f^{-k}(\Gamma) \cap \Gamma = \emptyset$ for $k = 1, 2, \ldots, \tau$ and $\lambda^{-\tau} > 2$, where $\lambda^{-1} > 1$ is, as before, the minimal factor of expansion of vectors in the unstable cones $C^u(z)$ at all points $z \in U \setminus S^+$. Moreover, there is a neighborhood of the attractor Λ in which the smooth components of Γ do not intersect one another.

As mentioned in Ref. [1], some (but not all) Lorenz, Lozi and Belykh attractors satisfy Condition A1. For our purposes, a much weaker assumption than A1 is sufficient. In all that follows, S_m^+ denotes the union of the curves on which the map f^m is singular.

Condition A2. There exist constants $C_0 > 0$ and $K_0 < \lambda^{-1}$ such that for any integer $m \geq 1$ no more than $C_0 K_0^m$ smooth components of the union $\cup_{l=0}^m S_l^+$ can meet at any point $z \in U$.

Condition A2, in a more stringent form (with $K_0 = 1$), has been used in Refs. [9, 10, 11, 12]. Note that Condition A1 implies A2 with $K_0 = 2^{1/\tau}$. In fact, for our purposes it is enough that A2 holds for a single, sufficiently large value of $m \geq 1$.

Two more conditions were adduced in Ref. [2]:

Condition A3. There exist constants $B > 0, \beta > 0$ and $\varepsilon_0 > 0$ such that for any integer $n \geq 1$ and any $\varepsilon \in (0, \varepsilon_0)$ one has

$$\nu(f^{-n}\Gamma_\varepsilon) < B\varepsilon^\beta.$$

Henceforth, we will denote by Γ_ε the ε-neighborhood of the set Γ and by ν the Lebesgue measure on M. We call a smooth curve γ in U an unstable curve (a stable curve) if its tangent line belongs in $C^u(z)$ (resp., $C^s(z)$) at any $z \in \gamma$.

Condition A4. There is a constant $\varepsilon_0 > 0$ such that for any unstable curve W^u there exist an integer $n_0 = n_0(W^u)$ and a constant $B_0 = B_0(W^u)$ such that for any $\varepsilon \in (0, \varepsilon_0)$ one has
(a) $\nu^u(W^u \cap f^{-n}\Gamma_\varepsilon) < \varepsilon^\beta \nu^u(W^u)$ for all integers $n > n_0$;
(b) $\nu^u(W^u \cap f^{-n}\Gamma_\varepsilon) < B_0\varepsilon^\beta \nu^u(W^u)$ for all integers $n \geq 1$;

Condition A3 was also assumed in Ref. [1], see Eq. (4) there. Certain condition similar to A4, but essentially weaker, was also assumed in Ref. [1], see (H4) there. Conditions A3 and A4 might not be easy to check in particular examples. Fortunately, we can *prove* that both these conditions follow from our Condition A2.

Proposition 1 *If a generalized hyperbolic attractor* Λ *satisfies Condition A2, then it satisfies Conditions A3 and A4.*

The proof of Proposition 1 is provided in Appendix below.

The following definitions, given $\varepsilon > 0$ and $l = 1, 2, \ldots$, are compiled from Ref. [1]:

$$\hat{D}^+_{\varepsilon,l} = \{z \in U^+ : \rho(f^n(z), S^+) \geq l^{-1}e^{-\varepsilon n}, n = 0, 1, \ldots\}$$

$$D^-_{\varepsilon,l} = \{z \in D : \rho(f^{-n}(z), S^-) \geq l^{-1}e^{-\varepsilon n}, n = 0, 1, \ldots\}$$

$$D^+_{\varepsilon,l} = \hat{D}^+_{\varepsilon,l} \cap \Lambda, \qquad D^0_{\varepsilon,l} = D^+_{\varepsilon,l} \cap D^-_{\varepsilon,l}$$

$$D^\pm_\varepsilon = \cup_{l\geq 1} D^\pm_{\varepsilon,l}, \qquad D^0_\varepsilon = \cup_{l\leq 1} D^0_{\varepsilon,l}.$$

Roughly speaking, $D^+_{\varepsilon,l}$ ($D^-_{\varepsilon,l}$) consists of points that do not approach the singularity set too rapidly in the future (respectively, in the past). It is easy to see that the sets $\hat{D}^+_{\varepsilon,l}$, $D^\pm_{\varepsilon,l}$ and $D^0_{\varepsilon,l}$ are closed; $D^0_\varepsilon = D^+_\varepsilon \cap D^-_\varepsilon$; the set D^+_ε is f-invariant; D^-_ε is f^{-1}-invariant; D^0_ε is both f and f^{-1} invariant. Besides, $D^0_\varepsilon \subset D$ for any $\varepsilon > 0$. The attractor Λ is said to be regular [1] if $D^0_\varepsilon \neq \emptyset$ for all sufficiently small $\varepsilon > 0$. Pesin [1] has proved the regularity under weaker assumptions than A3 and A4, so that under our assumption A2 the attractor Λ is regular.

Proposition 2 (cf., e.g., Ref. [1]) *There exists an* $\varepsilon > 0$ *such that for any point* $z \in D^+_{\varepsilon,l}$ *(*$z \in D^-_{\varepsilon,l}$*) there is a local stable fiber, LSF, denoted by* $V^s(z)$ *(resp., a local unstable fiber, LUF, denoted by* $V^u(z)$*). An LSF (LUF) is a* C^1*-curve in* M*. It is tangent to the line* E^s_z *(resp., to* E^u_z*) at* z*. The* ρ*-distance of the point* z *from the endpoints of that fiber is at least* $\delta_l = 1/l$*, a quantity determined by* l *and independent of* z*.*

We always denote $V^{s,u}(z)$ the maximal smooth local stable and unstable fibers passing through z. Note that $V^u(z) \subset D^-_\varepsilon$ for any $z \in D^-_\varepsilon$.

Remark. In some examples the LUF's and LSF's may be of an infinite length, as in the case of linear toral automorphisms. If this is the case, we redefine LUF's and LSF's. We simply pick a large $L > 0$ and denote by $V^{u,s}(x)$ a segment of the LUF (LSF) at the point x that has length L and is centered at x.

Next, we define Gibbs u-measures on Λ. Let $J^u(z)$ denote a one-step expansion factor in E^u_z (i.e., the Jacobian of the map $df|E^u_z$ at z). For any $z \in D^-_{\varepsilon,l}$ such that $V^u(z)$ exists, we define

$$\kappa(z,y) = \lim_{n\to\infty} \prod_{j=1}^{n} [J^u(f^{-j}(z))] \cdot [J^u(f^{-j}(y))]^{-1}. \tag{1}$$

for any $y \in V^u(z)$. This limit exists, is positive and continuous on $D^-_{\varepsilon,l}$ [1]. It is easy to show that under our assumption on smoothness of f up to S^+ the above limit is uniformly bounded away from zero and infinity on D^-_ε.

A measure μ on Λ is called a Gibbs u-measure, or a Bowen-Ruelle-Sinai measure (BRS measure) if
(a) it is f-invariant;
(b) $\mu(D^0_\varepsilon) = \mu(\Lambda) = 1$ for some $\varepsilon > 0$;
(c) the conditional measure on LUF's $V^u(z)$ induced by μ has a density with respect to the Lebesgue measure on $V^u(z)$ proportional to $\kappa(z,y)$ (see Ref. [1] for an exact version of the condition (c)).

Gibbs u-measures can be constructed in the following way. Given a $z \in D^-_\varepsilon$ one takes the normalized Lebesgue measure ν^u on $V^u(z)$ and pulls it forward under f_*: $\nu_k = f^k_* \nu^u$ (i.e., for any Borel set $A \subset U$ one takes $\nu_k(A) = \nu^u(f^{-k}A \cap V^u(z))$, as usual). Then the sequence of measures

$$\mu'_n = \frac{1}{n} \sum_{k=0}^{n-1} \nu_k$$

has a limit point (a measure) in the weak topology. That measure is a Gibbs u-measure [1] for any $z \in D^-_\varepsilon$. Instead of the Lebesgue measure on $V^u(z)$, one can take any measure equivalent to the Lebesgue measure on that LUF. Alternatively, one can take a measure on U absolutely continuous with respect to the Lebesgue measure and pull it forward under f_*. Then the time averages defined above will have a weak limit point, which is a Gibbs u-measure on Λ. Of course, a Gibbs u-measure is not unique, in general.

The following proposition is known as the Smale spectral decomposition. It has been proven in Ref. [1]. We formulate it for the sake of completeness, our arguments are not based on it.

Proposition 3 *There are subsets Λ_i $(i = 0, 1, \ldots)$ and Gibbs u-measures μ_i $(i \geq 1)$ such that*
(a) $\Lambda = \cup_{i\geq 0}\Lambda_i$ *and* $\Lambda_i \cap \Lambda_j = \emptyset$ *for* $i \neq j$*;*
(b) *for* $i \geq 1 : \Lambda_i \subset D, f(\Lambda_i) = \Lambda_i, \mu_i(\Lambda_i) = 1$ *and* $f|\Lambda_i$ *is ergodic with respect to* μ_i*;*
(c) *for* $i \geq 1$*: there exists a finite decomposition*

$$\Lambda_i = \cup_{j=1}^{r_i} \Lambda_i^j,$$

where $\Lambda^j_i \cap \Lambda^{j'}_i = \emptyset$ *for* $j \neq j'$, $f(\Lambda^j_i) = \Lambda^{j+1}_i$ *and* $f(\Lambda^{r_i}_i) = \Lambda^1_i$, *and* $f^{r_i}|\Lambda^1_i$ *is a Bernoulli automorphism;*
(d) *any Gibbs u-measure* μ *is a weighted sum* $\mu = \sum_{i\geq 1} \alpha_i \mu_i$ *with some* $\alpha_i \geq 0$ *and* $\sum \alpha_i = 1$. *In particular,* $\mu(\Lambda_0) = 0$.

Sataev proved in Ref. [2] that under Conditions A3 and A4 the number of ergodic components Λ_i is finite. We will briefly explain in Appendix why this is true.

The spectral decomposition is not unique. As shown in Ref. [2] , there is always one that enjoys three additional properties:
(e) for every $l > 0$ the sets $\Lambda_i^j \cap D_{\varepsilon,l}^-$ are closed;
(f) for every $l > 0$, every $i = 1, \ldots, r$ and every open subset $Q \subset U$ such that $Q \cap \Lambda_i^j \cap D_{\varepsilon,l}^- \neq \emptyset$ we have $\mu_i(Q \cap \Lambda_i^j \cap D_{\varepsilon,l}^-) > 0$;
(g) if $z \in \Lambda_i^j$, then $V^u(z) \subset \Lambda_i^j$.

Remark. If $\mathrm{Vol}(\Lambda) = 0$, then any Gibbs u-measure is singular with respect to the Lebesgue measure on U. It is also singular with respect to the Lebesgue measure on any LSF. However, a Gibbs measure μ has no atoms, and, moreover, any particular LUF or LSF has μ measure zero.

3 Statement of results

We now formulate our results. Let H_β denote the class of Hölder continuous (HC) functions on the attractor Λ. A function $F(x)$ is said to be Hölder continuous if

$$|F(x) - F(y)| \leq C(F)[\rho(x, y)]^\beta, \tag{2}$$

where $\beta > 0$ is called the Hölder exponent. More generally, let ξ be a partition of U into a finite number of domains separated by a finite number of compact smooth curves. Then we denote by $H_\beta(\xi)$ the class of functions that are Hölder continuous (with the exponent β) within each of those domains. We say that such functions are piecewise Hölder continuous (PHC) (with respect to the given partition ξ).

We will study an arbitrary subcomponent $\Lambda_* = \Lambda_i^j$ of any ergodic component Λ_i of the attractor. Let $f_* = f^{r_i}|\Lambda_*$ and μ_* be the normalized measure $\mu_i|\Lambda_*$. We put $r_* = r_i$ and denote by $\langle\cdot\rangle$ the expectation with respect to μ_*. According to Proposition 3, the triple (Λ_*, f_*, μ_*) is a Bernoulli dynamical system. In particular, it is mixing. Our results are the next two theorems.

Theorem 1 (Decay of correlations) *Let $F(x)$ and $G(x)$ be two HC or PHC functions on M. Then, for any integer N*

$$|\langle (F \circ f_*^N) \cdot G \rangle - \langle F \rangle \langle G \rangle| \leq c(F, G)\alpha^{\sqrt{|N|}} \tag{3}$$

where $c(F, G) > 0$ depends on F and G and $\alpha < 1$ is determined by the subcomponent $\Lambda_* = \Lambda_i^j$ and the class of HC or PHC functions under consideration.

Theorem 2 (Central limit theorem) *Again, let $F(x)$ be an HC or a PHC function. Assume that $\langle F \rangle = 0$. Then, the quantity*

$$\sigma_F^2 = \sum_{N=-\infty}^{\infty} \langle (F \circ f_*^N) \cdot F \rangle \tag{4}$$

is finite and nonnegative. If $\sigma_F \neq 0$, then the sequence

$$\frac{F(x) + F(f_* x) + \cdots + F(f_*^{N-1} x)}{\sqrt{\sigma_F^2 N}} \tag{5}$$

converges in distribution to the standard normal law as $N \to \infty$.

Remark (see, e.g., Ref. [15]). The sum in Eq. (4) equals zero if and only if the function $F(x)$ is a coboundary, i.e. $F(x) = G(f_* x) - G(x)$ a.e. on Λ_* for another function $G \in L_2(\Lambda_*, \mu_*)$.

Our proofs of Theorems 1 and 2 are based on Markov approximation to the dynamical system (Λ_*, f_*, μ_*). We employ the techniques of Markov sieves developed in Refs. [9, 10, 11, 12, 13]. Next, we define the Markov sieves (MS's).

A MS is a partition of the phase space of a dynamical system (in our case it is Λ_*) satisfying four conditions stated below. It is determined by two integer parameters N and n. Here N is the number of iterates of f_* involved in Eqs. (3) and (5) and $n = [N^\gamma]$ for some fixed $\gamma \in (0,1)$. We denote the MS by $\Re_{N,n}$ and its atoms by $A_0, A_1, \ldots, A_I$ with an $I = I(n,N)$. We call A_0 the marginal set. We denote by $\Im$ the set of indices $\{1, \ldots, I\}$, and so $\Im^k$ is the set of k-tuples of non-zero indices.

The MS $\Re_{N,n}$ is defined by four conditions. Here and further on $\alpha, \alpha_1, \alpha_2, \ldots$ stand for various constants in the open interval $(0,1)$ whose exact values are not relevant in the proofs, and $c, c_1, c_2, \ldots$ stand for various positive constants, usually coefficients. The values of α_i and c_i do not depend on the MS parameters N and n (but may depend on γ).

Condition MS1 (Sizes). $\mathrm{diam} A_i \le c_1 \alpha_1^n$ for all $i \in \Im$.

Condition MS2 (Marginal set). $\mu_*(A_0) \le c_2 \alpha_2^n$.

Condition MS3 (Markov approximation). For any integers $k > l > 1$ and $1 \le i_1 < i_2 < \cdots < i_k \le N$ and for any collection $(j_1, \ldots, j_k) \in \Im^k$ one has

$$\begin{aligned} &\mu_*(f_*^{i_1} A_{j_1} \cap f_*^{i_2} A_{j_2} \cap \cdots \cap f_*^{i_{l-1}} A_{j_{l-1}} / f_*^{i_l} A_{j_l} \cap \cdots \cap f_*^{i_k} A_{j_k}) \\ &\qquad = \mu_*(f_*^{i_1} A_{j_1} \cap \cdots \cap f_*^{i_{l-1}} A_{j_{l-1}} / f_*^{i_l} A_{j_l})(1 + \Delta) \end{aligned} \tag{6}$$

with some $|\Delta| \le c_3 \alpha_3^n$. Here $\mu_*(A'/A'')$ means the conditional measure, i.e. $\mu_*(A' \cap A'')/\mu_*(A'')$, and we always assume that $\mu_*(A'') > 0$ in our equations.

Condition MS4 (Doeblin condition [16]). There are constants $g_0, g_1 > 0$ independent of N and n such that for every $k \ge g_0 n$ and for any pair $(i,j) \in \Im^2$ one has

$$\frac{1}{2} \sum_{l=0}^{I} |\mu_*(A_l / f_*^k(A_i)) - \mu_*(A_l / f_*^k(A_j))| \le 1 - g_1. \tag{7}$$

According to Conditions MS1–MS3, the MS's provide a good approximation to the dynamical system (Λ_*, f_*, μ_*) by a stationary Markov chain on the given finite interval of time $(0, N)$. Condition MS4 is added to assure a rapid mixing in the approximating Markov chain. This is a key property of our Markov approximation. It manifests in the following theorem.

Theorem 3 (Relaxation to equilibrium distribution) *For any integers $k \geq l > 1$ and $1 \leq i_1 < i_2 < \cdots < i_k \leq N$ there is a subset $R_* = R_*(i_1, \dots, i_k) \subset \Im^{k-l+1}$ of $(k-l+1)$-tuples of indices such that*
(i) *if $(j_l, \dots, j_k) \in R_*$, then*

$$\sum_{j_1,\dots,j_{l-1}=0}^{I} |\mu_*(f_*^{i_1} A_{j_1} \cap \dots \cap f_*^{i_{l-1}} A_{j_{l-1}} / f_*^{i_l} A_{j_l} \cap \cdots$$
$$\cdots \cap f_*^{i_k} A_{j_k}) - \mu_*(f_*^{i_1} A_{j_1} \cap \dots \cap f_*^{i_{l-1}} A_{j_{l-1}})| \leq \Delta;$$

(ii) *one has*

$$\sum_{(j_l,\dots,j_k)\in R_*}^{I} \mu_*(f_*^{i_l} A_{j_l} \cap \dots \cap f_*^{i_k} A_{j_k}) \geq 1 - \Delta,$$

where $\Delta = \max\{c_4 \alpha_4^n, (1-g_1/2)^{[L/2]}\}$ with $L = [(i_l - i_{l-1})/(g_0 n)]$.

Note that Theorem 3 is still true if one reverses "the time", i.e. if $N \geq i_1 > \cdots > i_k \geq 1$. The meaning of Theorem 3 is that the conditional distributions relax to equilibrium exponentially fast in the parameter $|i_l - i_{l-1}|$ (which represents the "interval" between the "future" and the "past"), at least as long as that interval is less than const$\cdot n^2$.

The complete proofs of the stretched exponential bound (Theorem 1) and the central limit theorem (Theorem 2) based on Theorem 3 are provided in Ref. [10]. The proof of Theorem 1 is, however, so short and instructive that we outline it here.

Obviously, it is enough to consider a function F with $\langle F \rangle = 0$. We take the Markov sieve $\Re_{n,N}$ with $n = [\sqrt{N}]$. Let $\tilde{F}$ be a function that on each set $A_i \in \Re_{n,N}$, $i = 0, 1, \dots, I$, it is constant and takes the average value of F on that set:

$$F_i = (\mu_*(A_i))^{-1} \int_{A_i} F(x)\, d\mu_*(x).$$

From Conditions MS1 and MS2 of the Markov sieves one readily gets a bound

$$|\langle (F \circ f_*^N) \cdot G \rangle - \langle (\tilde{F} \circ f_*^N) \cdot \tilde{G} \rangle| \leq c(F, G) \alpha^n.$$

Then one expands

$$\langle (\tilde{F} \circ f_*^N) \cdot \tilde{G} \rangle = \sum_{i,j=0}^{I} F_i G_j \mu_*(A_j / A_i) \mu_*(A_i).$$

By applying Theorem 3 and recalling that $\langle F \rangle = \langle \tilde{F} \rangle = 0$ one obtains the bound in Eq. (3). Theorem 1 is proven.

The proof of Theorem 3 is also elaborated in Ref. [10]. First, as shown in Ref. [10], it is enough to prove it for the simplest case $k = l = 2$. Next, if the Doeblin condition (7) were valid for all the pairs $0 \leq i, j \leq I$, then the $k = l = 2$ version of Theorem 3 would be just a slightly stronger version of a classical theorem in probability theory, cf. Ref. [16], p. 174. For the sake of completeness we provide the proof of Theorem 3 with $k = l = 2$ in Appendix.

Thus, all our arguments boil down to the construction of Markov sieves, which is carried out in the next section.

4 Construction of Markov sieves

Recall that a smooth curve γ is in U an unstable curve (a stable curve) if the tangent line $T_z(\gamma)$ to it belongs in $C^u(z)$ (resp., $C^s(z)$) at any $z \in \gamma$. Given a $k \geq 1$, we call γ a k-unstable curve (k-stable curve) if $f_*^{-k}\gamma$ (resp., $f_*^k\gamma$) is a smooth unstable (stable) curve.

Let $m \geq 1$ be a large integer. The set $U_m = f_*^m(U \setminus S^+) \setminus S_m^+$ is a finite union of domains in M bounded by a finite number of smooth compact curves. The maps f_*^m and f_*^{-m} are smooth on U_m. We stress that the value of m is large but fixed. On the contrary, the MS parameters N and n grow to infinity and all our estimates are asymptotic in N and n.

For any large n we consider a collection of small subdomains in U_m denoted by $Q_1, \ldots, Q_J, J = J(n)$ and satisfying the conditions:

(Q1) Each Q_i is bounded by two m-unstable curves and two m-stable curves; we call such domains quadrilaterals;

(Q2) The sizes of smooth components of ∂Q_i are greater than $c'e^{-n}$ but less than $c''e^{-n}$ with some positive c', c'', which may depend on m, but not on n;

(Q3) The union $\cup Q_i$ covers the entire domain U_m except for the $(c\alpha^n)$-neighborhood of ∂U_m.

For some particular examples, say, for the Belykh attractor, such a collection $\{Q_i\}$ is easy to construct. A procedure of construction that works in general case may be found in Refs. [10] and [11]. For a point $z \in U_m$ we denote by $Q(z)$ the quadrilateral where z belongs.

Next, we need an invariance property for the boundaries of the quadrilaterals Q_i. Denote $\partial^u Q$ ($\partial^s Q$) the union of unstable (stable) curves bounding the quadrilaterals $Q_i, 1 \leq i \leq J$. The invariance property of the boundaries consists of two parts. First,

(Q4-u) $f_*^m(\partial^s Q) \subset \partial^s Q$;

A symmetric property for the unstable boundary $\partial^u Q$ generally fails, because the preimage $f_*^{-m}(\partial^u Q)$ needs not be inside U_m. However, if a quadrilateral Q_i intersects the attractor Λ, then $f_*^{-m}Q_i$ intersects $\Lambda \subset U_m$. In that case, either $f_*^{-m}Q_i \cap U_m$ is located in the $(c\alpha^n)$-neighborhood of ∂U_m, or its u-sides (unstable bounding curves) can be "adjusted". Therefore,

(Q4-s) if $z \in f_*^{-m}(\partial^u Q) \cap U_m$ and $\rho(z, \partial U_m) > c\alpha^n$, then $z \in \partial^u Q$.

We remind the reader that c and α denote various positive constants (α is always less than one). Each of our statements is true for some c and α, whose values need not be the same in all the statements.

The invariance properties can be provided by general arguments developed in [9, 10, 11]. Those arguments are carried out under the assumption that m is large enough.

As a result, we obtain that $\partial^s Q$ consists of a finite number of stable fibers.

Next, for any point $z \in U_m$ such that $f_*^{-l}z \in \cup Q_i$ for all $l = 0, 1, \ldots, m-1$ we define a quadrilateral $\hat{Q}(z) = \{y \in U_m : Q(f_*^{-l}y) = Q(f_*^{-l}z),\ l = 0, 1, \ldots, m-1\}$. The distinct quadrilaterals $\hat{Q}(z)$ clearly have the following properties:

(Q5) Each $\hat{Q}(z)$ is bounded by two m-unstable curves and two stable fibers; we again call such domains quadrilaterals;

(Q6) The sizes of smooth components of $\partial\hat{Q}(z)$ are greater than $\hat{c}'e^{-n}$ but less than $\hat{c}''e^{-n}$ with some positive $\hat{c}', \hat{c}''$, which may depend on m, but not on n;

(Q7) The union $\cup\hat{Q}$ covers the entire domain $U_{2m} = f_*^{2m}U \setminus S_{2m}^+$ except for the $(c\alpha^n)$-neighborhood of ∂U_{2m}.

Denote $\partial^{u,s}\hat{Q}$ the union of unstable and stable curves bounding the quadrilaterals $\hat{Q}(z)$. Then

(Q8-u) $f_*(\partial^s\hat{Q}) \subset \partial^s\hat{Q}$;

(Q8-s) if $z \in f_*^{-1}(\partial^u\hat{Q}) \cap U_{2m}$ and $\rho(z, \partial U_{2m}) > c\alpha^n$, then $z \in \partial^u\hat{Q}$.

Next three lemmas describe the structure of the measure μ_*.

For any point $z \in \Lambda$ denote $r^u(z)$ ($r^s(z)$) the distance from z to the nearest endpoint of the fiber $V^u(z)$ (resp., $V^s(z)$).

Lemma 1 *For any $\varepsilon > 0$ one has $\mu_*\{z \in \Lambda_* : \min\{r^u(z), r^s(z)\} < \varepsilon\} \le c\varepsilon^{\beta_1}$ with some $c > 0$ and $\beta_1 > 0$.*

Lemma 2 *For any $\varepsilon > 0$ the μ_*-measure of the ε-neighborhood of the set $S^+ \cup S^-$ is less than $c\varepsilon^{\beta_2}$ with some $c > 0$ and $\beta_2 > 0$. Consequently, the μ_*-measure of the ε-neighborhood of the boundary ∂U_{2m} is less than $c_m\varepsilon^{\beta_m}$ with some $c_m > 0$ and $\beta_m > 0$.*

Lemma 3 *Let W^u be a k-unstable curve, $k \ge 1$, of length $l > 0$. Let D be the union of all the stable curves of length $\le \varepsilon$ which intersect W^u and on which f_*^{-k} is continuous. Then*

$$\mu_*(D) \le c \min_{0 \le t \le k} \{\lambda_1^t l + \lambda_1^{-t}\varepsilon\}$$

with constants $\lambda_1 = \lambda_1(f_*) < 1$ and $c = c(f_*) > 0$.

In particular, if $k \ge a\ln(l/\varepsilon)$ with an $a > 0$, then

$$\mu_*(D) \le c l^{1-\beta}\varepsilon^{\beta},$$

where $\beta > 0$ is determined by the factor a alone.

The proofs of Lemmas 1-3 are provided in Appendix.

Now let $\hat{Q}_1, \ldots, \hat{Q}_{\hat{j}}$ be all the above quadrilaterals $\hat{Q}$ that have two additional properties:

(Q9) they do not intersect the $(c\alpha^n)$-neighborhood of ∂U_{2m} with the same c and α as in (Q8-s);

(Q10) they do not intersect the singularity curves for $f^{-[dn]}$ (i.e., the union $\cup_{i=1}^{[dn]} f^i(S^-)$) with some $d > 0$ defined below.

The value of d is chosen so that the total μ_*-measure of the quadrilaterals lacking the property (Q10) is $\le c'(\alpha')^n$ with some $c' > 0$ and $\alpha' < 1$. Such d, c' and α' exist in view of Lemma 2.

The next property readily follows from (Q8-u,s) and (Q9):

(Q11) For any $i, j = 1, \dots \hat{J}$ the intersection $f_* \hat{Q}_i \cap \hat{Q}_j$ is a quadrilateral bounded by two stable fibers belonging to $\partial \hat{Q}_j$ and two $(m+1)$-unstable curves belonging to $f_*(\partial \hat{Q}_i)$.

Next, for each quadrilateral $\hat{Q}_i, 1 \leq i \leq \hat{J}$, we call the proper unstable sides of $\hat{Q}_i$ two the most distant LUF's that belong in $\hat{Q}_i$ and intersect both stable sides of $\hat{Q}_i$ (if, of course, such LUF's exist). Denote $\tilde{Q}_i$ the subdomain in $\hat{Q}_i$ bounded by two stable and two proper unstable sides of $\hat{Q}_i$ (if $\hat{Q}_i$ does not have proper unstable sides, then we set $\tilde{Q}_i = \emptyset$). We take the set of points $x \in \tilde{Q}_i$ such that $V^u(x)$ crosses both stable sides of $\hat{Q}_i$ and $V^s(x)$ crosses both proper unstable sides of $\hat{Q}_i$. Denote that set by $B_i = B(\hat{Q}_i)$. Clearly, it has a direct product structure: for any $x, y \in B_i$ the set $V^u(x) \cap V^s(y)$ consists of one point, which also belongs in B_i. Such sets are called parallelograms [17, 9, 10, 11, 12, 13], or, sometimes, rectangles.

Lemma 4 *Every parallelogram $B_l, 1 \leq l \leq \hat{J}$ belongs (mod 0) to one ergodic component Λ_i of f and, moreover, to one "Bernoulli" subcomponent Λ_i^j with some i and j.*

Proof. It is well known in ergodic theory that, given a hyperbolic dynamical system, for almost every point x the LUF $V^u(x)$ and the LSF $V^s(x)$ belong to one ergodic component. We specify this claim and provide its proof in Appendix. From this claim our lemma readily follows.

Now, let $B_1, \dots, B_{I'}$ be all the above parallelograms that belong in Λ_*. We then estimate their total measure. The measure of the set $\Lambda \setminus (\cup \hat{Q}_i)$ can be estimated by Lemma 2. The measure of the set $\cup(\tilde{Q}_i \setminus B_i)$ can be estimated by Lemma 1. It remains to consider the set $\cup(\hat{Q}_i \setminus \tilde{Q}_i)$. For some values of i there is a preimage $f_*^{-l}(\hat{Q}_i \setminus \tilde{Q}_i)$ with some $l = 1, \dots, n$ that belongs to the $(c\alpha^n)$-neighborhood of U_{2m}. In that case one can easily apply Lemma 2. For the other values of i the maps $f_*^{-l}, 1 \leq l \leq n$ are smooth on $\hat{Q}_i \setminus \tilde{Q}_i$ and due to (Q8-s) the unstable sides of $\hat{Q}_i$ are n-unstable curves. Therefore, they deviate from the proper unstable sides of $\hat{Q}_i$ by no more than $c\alpha^n e^{-n}$ with some $c > 0$ and $\alpha < 1$. Hence we can apply Lemma 3 to each such a $\hat{Q}_i$ and then make use of the fact that $I' \leq$const$\cdot e^n$. As a result, one obtains the bound

$$\mu_*(\cup B_i) \geq 1 - c\alpha^n \tag{8}$$

For every $i = 1, \dots, I'$ we denote

$$A_i = B_i \cap (\cap_{l=-N}^{N} f_*^l(\cup_1^{\hat{J}} \hat{Q}_i)).$$

In other words, the set A_i consists of the points of B_i whose trajectories stay within the union $\cup \hat{Q}_i$ during N iterates in the future and N iterates in the past. In view of (Q11) every A_i is also parallelogram. Obviously, (Q7) and Eq. (8) imply

$$\mu_*(\cup A_i) \geq 1 - c\alpha^n. \tag{9}$$

Recall that c and α denote various constants independent of n and N, so their values, say, in Eqs. (8) and (9) need not be the same.

Next, we consider intersections $f_*^k B_i \cap B_j$ with $1 \le i, j \le I'$ and $|k| \le N$. We will use certain notions introduced in Ref. [10]. Let A' and A'' be two parallelograms in Λ_* and $k \ge 1$. A subset of $f_*^k A' \cap A''$ defined as

$$\Re(f_*^k A' \cap A'') = \{z : V^u_{A''}(z) \subset f_*^k A' \text{ and } V^s_{A'}(f_*^{-k} z) \subset f_*^{-k} A''\}$$

is called the regular part of $f_*^k A' \cap A''$ (here and on we denote $V^{u,s}_A = V^{u,s} \cap A$). The other part, $\Im(f_*^k A' \cap A'') = (f_*^k A' \cap A'') \setminus \Re(f_*^k A' \cap A'')$ is called the irregular part of $f_*^k A' \cap A''$. For any parallelogram A a subparallelogram $B \subset A$ is said to be u-inscribed (s-inscribed) in A if $V^u_A(z) = V^u_B(z)$ (resp., $V^s_A(z) = V^s_B(z)$) for every $z \in B$. It is easily seen that for any $k > 0$ the regular part of $f_*^k A' \cap A''$ is a parallelogram u-inscribed in A'' and its preimage (under f_*^{-k}) is a parallelogram s-inscribed in A'. For any $k < 0$ the regular part of $f_*^k A' \cap A''$ is simply defined as $f_*^k(\Re(f_*^{-k} A'' \cap A'))$.

The idea of regularity goes back to the notion of Markov partitions. Indeed, the elements of Markov partitions for hyperbolic systems with singularities are parallelograms, and their characteristic property is nothing but the regularity of all the intersections of their images and preimages.

Lemma 5 *For any $|k| \le N$ and any $i, j = 1, \dots, I'$ the irregular part $\Im(f_*^k A_i \cap A_j)$ is empty.*

The proof of Lemma 5 can be easily obtained from (Q11) by induction in k. Let us emphasize that our parallelograms behave like elements of Markov partition, but only during the first N steps. After the Nth step irregular intersections may occur, but this is not a nuisance since we work only within the first N iterates of f_*.

Finally, we discard some parallelograms A_i that are "not dense enough" on LUF's. Specifically, we retain a parallelogram A_i iff

(Q12) there is a point $x \in A_i$ such that

$$\nu^u(V^u(x) \cap A_i)/\nu^u(V^u(x) \cap \hat{Q}_i) \ge 1 - c_d \alpha_d^n$$

with some $c_d > 0$ and $\alpha_d < 1$ specified below. Here ν^u stands for the Lebesgue measure on V^u.

If the above inequality fails for every $x \in A$, then $\nu^u(A_i)/\nu^u(\hat{Q}_i) < 1 - c_d \alpha_d^n$. Hence, the total measure of the removed parallelograms is $\le c\alpha^n c_d^{-1} \alpha_d^{-n}$. If α_d is sufficiently close to one, then $\alpha/\alpha_d < 1$ and the remaining parallelograms satisfy the inequality (9) with some other $c > 0$ and $\alpha < 1$.

Let $A_1, \dots, A_I$ be all the remaining parallelograms and $\hat{Q}_1, \dots, \hat{Q}_I$ be the corresponding quadrilaterals. The Markov sieve $\Re_{n,N} = \{A_0, A_1, \dots, A_I\}$ is now constructed. Here A_0 is simply $\Lambda_* \setminus \cup_{i=1}^I A_i$.

5 Proof of the properties MS1-MS4

In this section we prove the characteristic properties MS1-MS4 of the Markov sieve.

The property MS1 follows from (Q6), and MS2 follows from Eq. (9).

To prove MS3 we consider the parallelogram $A = A_{j_l}$ entering Eq. (6). Due to Lemma 5 the intersection

$$B = f_*^{i_1 - i_l} A_{j_1} \cap \cdots \cap f_*^{i_{l-1} - i_l} A_{j_{l-1}} \cap A_{j_l}$$

is a subparallelogram s-inscribed in A. Likewise, the intersection

$$C = A_{j_l} \cap f_*^{i_{l+1} - i_l} A_{j_{l+1}} \cap \cdots \cap f_*^{i_k - i_l} A_{j_k}$$

is a subparallelogram u-inscribed in A. Eq. (6) can be rewritten as

$$\frac{\mu_*(B \cap C)}{\mu_*(C)} = \frac{\mu_*(B)}{\mu_*(A)}(1 + \Delta). \tag{10}$$

The parallelogram A has a direct product structure in the topological sense. If it had such a structure in the measure-theoretic sense (i.e. if the measure μ_* on A were a direct product of two linear measures on stable and unstable fibers, respectively), then Eq. (10) would hold with no error term, i.e. with $\Delta = 0$. Our next goal is to show that μ_* on A has approximately a direct product structure. We only have to show that the deviation of μ_* on A from a direct-product measure is, roughly speaking, exponentially small in n.

For any LUF V^u intersecting A we put $V_A^u = V^u \cap A$ and $\mu_{V^u,A}^u$ the normalized conditional measure induced by μ_* on V_A^u. Then for any subparallelogram $D \subset A$ one has

$$\mu_*(D) = \int \mu_{V^u,A}^u (D \cap V^u)\, d\mu_A^s \tag{11}$$

with a factor measure μ_A^s on the collection of all the nonempty sets V_A^u.

Now, recall that the density of the conditional measure $\mu_{V^u,A}^u$ is proportional to $\kappa(z, y)$, cf. Eq. (1), where z is a fixed point of V^u and $y \in V_A^u$ is a variable.

Lemma 6 *There are constants $c > 0$ and $a > 0$ such that*

$$|\kappa(z, y) - 1| \leq c \cdot [\mathrm{dist}(z, y)]^a.$$

In other words, the density of a Gibbs u-measure on any LUF is a Hölder continuous function.

Proof. It is enough to prove that if z and y belong in one LUF and are close enough, then

$$\mathrm{dist}(E_z^u, E_y^u) \leq c' \cdot [\mathrm{dist}(z, y)]^{a'} \tag{12}$$

for some constants $c' > 0$ and $a' > 0$. To define the distance between E_z^u and E_y^u one should translate E_y^u along the closest geodesic between z and y into a subspace $E_{z,1}^y \subset T_z M$ and then take [2]

$$\mathrm{dist}(E_z^u, E_y^u) = \mathrm{dist}(z, y) + \angle(E_{z,1}^u, E_z^u).$$

Let us postpone the proof of Eq. (12). In fact, Eq. (12) follows from the estimate 4.4.7 in Ref. [2], but we give another proof here.

Now, based on Eq. (12) and the C^2-smoothness of the map f one can readily obtain that $|J^u(z) - J^u(y)| \le c''[\mathrm{dist}(z,y)]^{a''}$ with some constants $c'' > 0$ and $a'' > 0$. Applying the limit formula (1) then completes the proof of Lemma 6.

The meaning of Lemma 6 is that the conditional measures $\mu^u_{V^u,A}$ in Eq. (11) are approximately uniform for each LUF V^u intersecting A. The next lemma establishes a relation between the conditional measures on different LUF's within A.

Consider two sufficiently close LUF's V_1^u and V_2^u. Denote γ_1 the set of points $x \in V_1^u$ such that $V^s(x) \cap V_2^u \neq \emptyset$. The map $\varphi : \gamma_1 \to V_2^u$ defined as $\varphi(x) = V^s(x) \cap V_2^u$ is called the canonical isomorphism. The set γ_1 and its image $\varphi(\gamma_1)$ are closed Cantor-like subsets of V_1^u and V_2^u, respectively. The Jacobian of φ with respect to the Lebesgue measures (lengths) on V_1^u and V_2^u at almost every point $x \in \gamma_1$ is well-known (see, e.g., Ref. [10]) to be

$$J^s(x) = \lim_{n\to\infty} \prod_{j=0}^{n} [J^u(f^j(x))] \cdot [J^u(f^j(\varphi(x))]^{-1} \tag{13}$$

Lemma 7 *If f^{-n} is defined and continuous on the part of $V^s(x)$ between $x \in V_1^u$ and $\varphi(x) \in V_2^u$, then*

$$|J^s(x) - 1| \le c\alpha^n$$

with some $c > 0$ and $\alpha < 1$ determined by the map f alone.

Note that Lemma 7 is a stronger version of Theorem 4.8 from Ref. [2].

Proof. As in the previous lemma, it is enough to prove that if two points z and y belong in one LUF V^s and f^{-n} is continuous on the part of V^s between z and y, then

$$\mathrm{dist}(E_z^u, E_y^u) \le c'(\alpha')^n \tag{14}$$

with some constants $c' > 0$ and $\alpha' < 1$. After that the proof can be accomplished by employing the expansion (13) and the C^2-smoothness of the map f.

We now prove Eqs. (12) and (14). The key point of our arguments is that in both cases the preimages $z_i = f_*^{-i}(z)$ and $y_i = f_*^{-i}(y), i \ge 1$, stay close long enough "in the past". In case of Eq. (12) the points z_i, y_i are getting closer as i grows, and so $\mathrm{dist}(z_i, y_i) \le \varepsilon = \mathrm{dist}(z,y)$ for all $i \ge 1$. In case of Eq. (14) the points z_i, y_i are getting more distant as i grows, but during the first $n/2$ steps they are still exponentially close in n: $\mathrm{dist}(z_i, y_i) \le c\lambda^{n/2}$ for all $i = 1, \dots, [n/2]$.

We fix the value $i_1 = -\beta_1 \ln \varepsilon$ in case of Eq. (12) and $i_1 = \beta_1 n$ in case of Eq. (14) with a sufficiently small $\beta_1 > 0$ specified below. We then take the line $E_1 = E^u_{z_{i_1}}$ and translate it along the closest geodesic between z_{i_1} and y_{i_1} into a line $E_2 \subset T_y M$. The map df is smooth on the compact 3-D manifold of linear one-dimensional subspaces of TM, and so

$$\mathrm{dist}(df_*^{i_1}(E_1), df_*^{i_1}(E_2)) \le D_1^{i_1} \cdot \mathrm{dist}(E_1, E_2) = D_1^{i_1} \cdot \mathrm{dist}(z_{i_1}, y_{i_1}) \tag{15}$$

with some constant $D_1 > 1$ determined by the map f_* alone. Besides, $df_*^{i_1}(E_1) = E_z^u$, and, due to the uniform hyperbolicity of f_*, one has $\angle(f_*^{i_1}(E_2), E_y^u) \le c_1\lambda^{i_1}$ with some

$c_1 > 0$. We now choose the constant $\beta_1 > 0$ so small that the RHS of Eq. (15) is $\leq c\varepsilon^a$ in case of Eq. (12) and $\leq c\alpha^n$ in case of Eq. (14) with some $c, a > 0$ and $\alpha < 1$. The bounds (12) and (14) then easily follow.

We now turn back to the proof of the property MS3. Due to (Q10) the map $f_*^{-[dn]}$ is continuous on the quadrilaterals $\hat{Q}_1, \ldots, \hat{Q}_{\hat{j}}$, and so Lemma 7 applies to the parallelogram A. Summarizing the results of Lemmas 6 and 7, one can say that the conditional measures $\mu^u_{V^u,A}$ are almost uniform with respect to the Lebesgue measures on LUF's (up to an exponentially small in n error term) and the Jacobians of the canonical isomorphisms between different sets V^u_A within A equal one up to an exponentially small in n error term. Then it is an easy calculation to derive Eq. (10) from Eq. (11) with $|\Delta| \leq c\alpha^n$. The property MS3 is now proven.

The proof of the property MS4 consists of three steps.

Step I (Expansion). At this step we pull the parallelograms A_i and A_j involved in Eq. (7) forward, so that their images will expand and become "long enough". Consider a point $x \in A_i$ and the part V^u_1 of $V^u(x)$ confined between two LSF's bounding the quadrilateral $\hat{Q}_i$ where A_i belongs.

For any LUF $V^u, D > 0$ and $k > 0$ let $V^u_{k,D}$ denote the set of points $y \in V^u$ such that $f_*^k(y)$ belongs to a smooth component of $f_*^k(V^u)$ of length $\geq D$. Let $l_1^u(\cdot)$ denote the Lebesgue measure (length) on V^u.

Lemma 8 *There are $D > 0, c > 0$ and $\alpha < 1$, independent of the LUF V^u, such that for any $k \geq 1$*

$$l_1^u(V^u \setminus \cup_{j=1}^k V^u_{j,D}) \leq c\alpha^k.$$

In other words, if k is large enough, then the majority of points $y \in V^u$ have images in long components (of length $\geq D$) during the first k iterates of f_* in the future.

Proof. We outline a short and elegant proof that goes back to Bunimovich and Sinai [17], see also Ref. [10].

For any $k \geq 1$ denote by L_k the number of smooth components of $f_*^k(V^u)$ that are shorter than D and such that their preimages under f_*^{-i} for every $i = 1, \ldots, k$, also belong to smooth components of $f_*^{k-i}(V^u)$ of length $\leq D$. Obviously, one has

$$l_1^u(V^u \setminus \cup_{j=1}^k V^u_{j,D}) \leq L_k D \lambda^{r_* k} \tag{16}$$

where λ^{-1} is the lower bound on the one-step expansion factor in LUF's under the map f_* and r_* stands for the lowest power of f that coincides with f_* on Λ_*

We now estimate L_k. Due to Condition 2 in Sect. 2 for any $m \geq 1$ there is a $D_m > 0$ such that any LUF V^u_0 of length $\leq D_m$ intersects no more than $C_0 K_0^m$ smooth components of the union $\cup_{i=0}^m S_i^+$. Each of those components can intersect any LUF at most once, and so $f_*^m V_0^u$ consists of $\leq (C_0 K_0^m + 1)$ smooth components. Therefore, if $D \leq D_m$, then $L_k \leq (C_0 K_0^m + 1)^{[kr_*/m]+1}$. Recall that $K_0 < \lambda^{-1}$. Therefore, by choosing m large enough one can make the RHS of Eq. (16) exponentially small in k, thus obtaining Lemma 8.

Recall that the length of V_1^u is about const$\cdot\alpha^n$, cf. (Q6). Therefore, if $c_1 > 0$ is a large enough integer (independent of n), then the following relative Lebesgue measure is exponentially small in n:

$$l_1^u(V_1^u \setminus \cup_{j=1}^{c_1 n} V_{j,D}^u)/l_1^u(V_1^u) \leq cD\alpha^n \tag{17}$$

with some $c > 0$ and $\alpha < 1$ independent of n.

Step 2 (Connection). At this step we pull the "long" components of the images of A_i and A_j further on, so that they become close enough and can be connected by LSF's.

Let V_2^u be an arbitrary LUF of length $\geq D$.

Lemma 9 *For any $D > 0$ there is a subset $\Lambda_*(D) \subset \Lambda_*$ such that*
(a) *for any point $x \in \Lambda_*(D)$ the LUF $V^u(x)$ has a length $\geq D$ and wholly belongs to $\Lambda_*(D)$;*
(b) *the union of LUF's of length $\geq D$ lying apart from $\Lambda_*(D)$ has zero μ_*-measure;*
(c) *the set of LUF's of which $\Lambda_*(D)$ consists is compact in C^1-topology;*
(d) *for every LUF $V^u(x) \subset \Lambda_*(D)$ the intersection of $\Lambda_*(D)$ with any neighborhood $U(y)$ of any point $y \in V^u(x)$ has a positive μ_*-measure.*

Proof. Let $\Lambda_*(D, 0)$ be the union of all the LUF's in Λ_* of length $\geq D$. We define $\Lambda_*(D)$ as the union of all the LUF's V^u of length $\geq D$ such that the intersection of $\Lambda_*(D, 0)$ with any neighborhood of any point of V^u has a positive μ_*-measure. Obviously, $\mu_*(\Lambda_*(D)) = \mu_*(\Lambda_*(D, 0))$.

One can easily show by inspection that if a sequence of LUF's of length $\geq D > 0$ converges in the C^0-topology, then the limit curve is an LUF, too. Besides, the convergence takes place in the C^1-topology as well. Therefore, the set of all the LUF's of length $\geq D$ is compact in C^1 topology. It is now easy to see that the clause (c) holds. Lemma 9 is proven.

We say that a LUF $V^u \in \Lambda_*(D)$ is one-sided if only one-sided neighborhoods of the points of V^u intersect $\Lambda_*(D)$ by subsets of positive measures. One-sided LUF may, for instance, intersect or touch the boundary $\partial(f^k U), k \geq 0$.

From now on we denote by D the constant involved in Lemma 8. We also assume that V_2^u and all the smooth components of its images $f_*^n V^u, n \geq 1$ that have lengths $\geq D$ belong in $\Lambda_*(D)$. The LUF's that do not enjoy this last property clearly form a subset of zero measure in Λ_*.

Lemma 10 *There is a closed maximal parallelogram $\tilde{A} = \tilde{A}(V_2^u)$ of positive μ_*-measure such that*
(i) *$V_{\tilde{A}}^u(z) \subset V_2^u$ for some $z \in \tilde{A}$;*
(ii) *$\tilde{A}$ has some nonempty parts on both sides of V_2^u unless V_2^u is a one-sided LUF;*
(iii) *the endpoints of V_2^u do not belong to $\tilde{A}$.*

Proof. For almost every point $x \in V_2^u$ (with respect to the Lebesgue measure in that LUF) there is an LSF $V^s(x)$. This follows, for instance, from Lemma 3.4 and

Theorem 4.4 in Ref. [2]. This property and Lemma 9 readily give a parallelogram $\tilde{A}$ with the properties (i)-(iii). Lastly, any parallelogram can be easily completed to a closed and maximal one, and so Lemma 10 is proven.

Let us consider all the LUF's $V_2^u \subset \Lambda_*(D)$. The collection of such LUF's is compact in C^1-topology. Furthermore, the parallelogram $\tilde{A}(V_2^u)$ defined in Lemma 10 satisfies the conditions (i)-(iii) of the lemma for any other LUF in $\Lambda_*(D)$ sufficiently close to V_2^u in C^1-topology. Therefore, there is a finite collection of parallelograms $\tilde{A}_1, \dots, \tilde{A}_r$ such that for any LUF $V_2^u \in \Lambda_*(D)$ there is an $i = i(V_2^u)$ such that $\tilde{A} = \tilde{A}_i$ satisfies (i)-(iii). The collection $\tilde{A}_1, \dots, \tilde{A}_r$ depends on D alone.

Next, we fix another maximal parallelogram $\tilde{A}_0 \subset \Lambda_*$ of positive measure. Due to the mixing of f_* there is an $k_0 = k_0(\tilde{A}_0, D)$ such that for any $k \geq k_0$ and $i = 1, \dots, r$ one has

$$\mu_*(f_*^k \tilde{A}_i \cap \tilde{A}_0) \geq \frac{1}{2}\mu_*(\tilde{A}_i)\mu_*(\tilde{A}_0). \tag{18}$$

As we know, the intersection $f_*^n \tilde{A}_i \cap \tilde{A}_0$ consists of regular and irregular parts. The next lemma bounds the measure of the irregular part.

Lemma 11 *For any two maximal parallelograms A and B and any $k \geq 1$ one has $\mu_*(\Im(f_*^k A \cap B)) \leq c\alpha^k$ with some $c > 0$ and $\alpha < 1$ determined by the map f_* alone.*

Proof. Our proof is almost compiled from Ref. [11]. First, for any parallelogram A we denote by $Q(A)$ the minimal quadrilateral containing A and bounded by two LUF's and two LSF's. We call the LUF's (LSF's) bounding $Q(A)$ the u-sides (s-sides) of $Q(A)$. The intersection $f_*^n Q(A) \cap Q(B)$ consists of a finite number of closed domains. If such a domain is a quadrilateral bounded by two s-sides of $Q(B)$ and the images of two u-sides of $Q(A)$, then the part of $f_*^k A \cap B$ within that domain is regular (this is an easy consequence of the maximality of both A and B). The other domains of $f_*^k Q(A) \cap Q(B)$ are of two types:

a) adjacent to the set $S_l^- = f^l S^-$ for some $l = 1, \dots, r_* k$ (recall that r_* is the minimal integer such that $f^{r_*} = f_*$ on Λ_*);

b) bounded by LUF's and LSF's only but adjacent to either an u-side of $Q(B)$ or to the image of an s-side of $Q(A)$.

There are at most four domains of the type (b), and their width in the direction of E^s is $\leq c\lambda^k$. In virtue of Lemma 3 the measure of the parts of $f_*^k A$ in those domains is $\leq c\alpha^k$.

Next, we fix an $l = 1, \dots, r_* k$ and collect all the domains of the type (b) that are adjacent to S_l^- and intersect the set $f_*^k A \cap B$. We denote those domains by $D_1^{(l)}, \dots, D_{r(l)}^{(l)}$. Note that the domain $\tilde{D}_i^{(l)} = f^{-l} D_i^{(l)}$ is adjacent to S^-. Due to Lemma 3 $\mu_*(D_i^{(l)} \cap f_*^k A) \leq c_0 \alpha_0^k$ for any i and l with some $c_0 > 0$ and $\alpha_0 < 1$ independent of A, B and k. We consider three cases:

(i) Let $1 \leq l \leq [\delta r_* k]$ with some small $\delta > 0$ specified below. There are at most $\Lambda_0^{\delta r_* k}$ smooth components of the set $S_1^- \cup \dots \cup S_{[\delta r_* k]}^-$, where $\Lambda_0 > 1$ is a constant determined by the map f. Each of those components can touch no more than two domains $D_i^{(l)}$. Therefore, $r(l) \le \Lambda_0^{[\delta r_* k]}$ for every $l = 1, \dots, [\delta r_* k]$. Hence,

$$\sum_{l=1}^{[\delta r_* k]} \sum_{i=1}^{r(l)} \mu_*(D_i^{(l)} \cap f_*^k A) \le c_0 r_* n \Lambda_0^{\delta r_* k} \alpha_0^k.$$

(ii) Let $[(1-\delta) r_* k] \le l \le r_* k$. Then every domain $\tilde{D}_i^{(l)}$ belongs in $f^{r_* k - l} Q(B)$ and is adjacent to S^-. Obviously, there are no more than $\Lambda_0^{[\delta r_* k]}$ of such domains $D_i^{(l)}$, and one again has

$$\sum_{l=[(1-\delta) r_* k]}^{r_* k} \sum_{i=1}^{r(l)} \mu_*(D_i^{(l)} \cap f_*^k A) \le c_0 r_* k \Lambda_0^{\delta r_* k} \alpha_0^k.$$

(iii) Let $[\delta r_* k] \le l \le [(1-\delta) r_* k]$. For each such an l the domains $\tilde{D}_i^{(l)}$ are adjacent to S^-, disjoint and belong in the $(c\lambda^{\delta r_* k})$-neighborhood of S^-. Due to Lemma 2 one has

$$\sum_{l=[\delta r_* k]}^{[(1-\delta) r_* k]} \sum_{i=1}^{r(l)} \mu_*(D_i^{(l)} \cap f_*^k A) \le c r_* k \lambda^{\beta \delta k}.$$

Choosing δ small enough and summarizing the estimates in the cases (i)-(iii) complete the proof of Lemma 11.

In view of Lemma 11 the bound (18) yields

$$\mu_*(\Re(f_*^k \tilde{A}_i \cap \tilde{A}_0)) \ge \frac{1}{4} \mu_*(\tilde{A}_i) \mu_*(\tilde{A}_0) \tag{19}$$

for every $k \ge k_0'(D, \tilde{A}_0)$.

Next, we consider the parallelogram $B = f_*^{-k}(\Re(f_*^k \tilde{A}_i \cap \tilde{A}_0))$, which is s-inscribed in $\tilde{A}_i$. For each point $x \in B$ let $\varphi(x) = V^s(x) \cap V_2^u$. Then $\varphi(B)$ is a Cantor-like subset of V_2^u. The bound (19) along with Lemmas 6 and 7 implies

$$\nu^u(\varphi(B)) > c_0,$$

where ν^u is the Lebesgue measure on V_2^u and $c_0 > 0$ depends on D and $\tilde{A}_0$ only. Furthermore, for any $y \in \varphi(B)$ the component of $f_*^k V_2^u$ containing $f_*^k y$ intersects both s-sides of the quadrilateral $Q(\tilde{A}_0)$ and covers the set $V_{\tilde{A}_0}^u(f_*^k y)$. We summarize our conclusions in the following lemma:

Lemma 12 *For any $D > 0$ and any maximal parallelogram $\tilde{A}_0$ there are a real $l_2 = l_2(D, \tilde{A}_0) > 0$ and an integer $c_2 = c_2(D, \tilde{A}_0) > 0$ such that for every LUF V_2^u of length $\ge D$ and every $k \ge c_2$ there is a subset $V_{2,k}^u \subset V_2^u$ whose Lebesgue measure is $\ge l_2$ and such that*

$$f_*^k V_{2,k}^u = \cup_p V_{\tilde{A}_0}^u(x_p)$$

for some points $x_p \in \tilde{A}_0$.

We now combine Lemma 12 with the previous estimate (17). As a result, for any $k \geq c_1 n + c_2$ there is a subset $V^u_{1,k} \subset V^u_1$ whose relative Lebesgue measure in V^u_1 is $\geq l_1 = l_1(l_2) > 0$ and such that

$$f^k_* V^u_{1,k} = \cup_p V^u_{\tilde{A}_0}(x'_p)$$

for some points $x'_p \in \tilde{A}_0$. The constant l_1 can be taken, for instance, as

$$l_1 = l_2 / \max_{z \in \Lambda, y \in V^u(z)} \kappa(z, y).$$

Recall that V^u_1 is an LUF within the quadrilateral $\hat{Q}_i$ where the parallelogram A_i belongs. Since $l_1 > 0$ is an absolute constant for the given map f, the "density" condition (Q12) implies that the relative measure of the subset $f^k_*(V^u_{1,k} \cap A_i)$ in $f^k_* V^u_{1,k}$ is $\geq 1 - c\alpha^n$ with some $c > 0$ and $\alpha < 1$.

Corollary 1 *For each parallelogram $A_i \in \Re_{n,N}$ and every $k \geq c_1 n + c_2$ there is an s-inscribed subparallelogram $A_{i,k} \subset A_i$ such that*
(i) $\mu_*(A_{i,k})/\mu_*(A_i) \geq l_1$*;*
(ii) $f^k_* A_{i,k} \subset \tilde{A}_0$*;*
(iii) *for every point $x \in f^k_* A_{i,k}$ one has*

$$\nu^u(V^u(x) \cap f^k_* A_{i,k}) / \nu^u(V^u(x) \cap \tilde{A}_0) \geq 1 - c\alpha^n,$$

where ν^u is the Lebesgue measure on $V^u(x)$.

Lastly, note that all the sets $V^u_{\tilde{A}_0}(x), x \in \tilde{A}_0$ are connected by LSF's within the quadrilateral $Q(\tilde{A}_0)$ circumscribing the parallelogram $\tilde{A}_0$. Note also that the sets $f^k_* A_{i,k}$ and $f^k_* A_{j,k}$ for any $k \geq c_1 n + c_2$ are foliated by LUF's that are mapped onto each other by canonical isomorphisms. The Jacobians of those isomorphisms with respect to both Lebesgue and BRS measures are uniformly bounded away from zero and infinity due to Lemmas 6 and 7.

Step III (Contraction). At this, last, step we pull the sets $f^{k_2}_* A_{i,k_2}$ and $f^{k_2}_* A_{j,k_2}, k_2 = c_1 n + c_2$, obtained at Step II further on, so that they become exponentially (in n) close to each other. Their future images become close since they are connected by LSF's.

For any $k > 0$ we denote by $\tilde{A}_{0,k} \subset \tilde{A}_0$ the set of points $x \in \tilde{A}_0$ such that $f^k_* V^s_{\tilde{A}_0}(x)$ wholly belongs to one of the "proper" quadrilaterals $\tilde{Q}_1, \ldots, \tilde{Q}_I$ (cf. Section 4). Obviously, $\tilde{A}_{0,k}$ is an s-inscribed subparallelogram in $\tilde{A}_0$.

Lemma 13 *There is an integer $c_3 > 0$ such that for any $k \geq c_3 n$ one has $\mu_*(\tilde{A}_0 \setminus \tilde{A}_{0,k}) \leq c\alpha^n$ with some $c > 0$ and $\alpha < 1$ independent of n and k.*

Proof. Evidently, the set $f^k_*(\tilde{A}_0 \setminus \tilde{A}_{0,k})$ belongs to the union of the λ^k-neighborhood of all the proper unstable sides of $\hat{Q}_1, \ldots, \hat{Q}_I$ (cf. Section 2) and the "remainder" set $\Lambda_* \setminus (\cup \tilde{Q}_i)$. Lemma 3 and the bound (8) now yield Lemma 13 for all large enough c_3.

The meaning of Lemma 13 is that in $k \geq c_3 n$ steps the images of all the LSF's intersecting the parallelogram $\tilde{A}_0$ become so short that they typically fall into individual quadrilaterals $\hat{Q}_i$, and only a small fraction of them cross the boundaries $\cup \partial \hat{Q}_i$. One can say that f_*^k then sends the sets $f_*^{k_2} A_{i,k_2}$ and $f_*^{k_2} A_{j,k_2}$ into the same parallelograms A_l, and this is essentially equivalent to the Doeblin condition MS4.

For each $A_i \in \Re_{n,N}$ and $k \geq 1$ we consider $A_{i,k_2,k} = A_{i,k_2} \cap f_*^{-k_2} \tilde{A}_{0,k}$. This is also an s-inscribed subparallelogram of A_i. Corollary 1 and Lemma 13 imply the following

Corollary 2 *For each $A_i \in \Re_{n,N}$ and every $k \geq c_3 n$ one has*
(i) $\mu_*(A_{i,k_2,k})/\mu_*(A_i) \geq l_1/2$;
(ii) $f_*^{k_2} A_{i,k_2,k} \subset \tilde{A}_{0,k}$;
(iii) *for every point $x \in f_*^{k_2} A_{i,k_2,k}$ one has*

$$\nu^u(V^u(x) \cap f_*^{k_2} A_{i,k_2,k})/\nu^u(V^u(x) \cap \tilde{A}_0) \geq 1 - c\alpha^n,$$

where ν^u is the Lebesgue measure on $V^u(x)$.

Recall that for any $A_l \in \Re_{n,N}$ the intersections $f_*^t A_i \cap A_l$ and $f_*^t A_j \cap A_l$ are regular for $|t| \leq N$. Let $k \geq c_3 n$ and $f_*^{k_2+k} A_{i,k_2,k} \cap A_l \neq \emptyset$. Then, due to Corollary 2 and Lemmas 6 and 7 one has

$$\mu_*(A_l/f_*^{k_2+k} A_{i,k_2,k})/\mu_*(A_l/f_*^{k_2+k} A_{j,k_2,k}) \geq l_3$$

with some constant $l_3 > 0$ determined by the map f_* alone. Applying the part (i) of Corollary 2 and summing over l give the property MS4.

Let us finally discuss another aspect of the problem.

In typical examples, cf. Section 6, the map f depends on some parameters. It is often important to find values of the parameters for which the attractor Λ has one ergodic (and one mixing) component, i.e. for which $r = 1$ and $r_1 = 1$ in terms of Proposition 3. For some examples those values of parameters form open sets [18], i.e. the ergodicity and mixing are stable under certain small C^2-perturbations. We claim here one more stability result.

Let f_0 be a map $U \setminus \Gamma_0 \to U$ satisfying all the assumptions of Section 2, plus it is an "onto" map, i.e. clos($f_0(U \setminus \Gamma_0)$) =clos($U$). We assume that f_0 preserves an absolutely continuous invariant measure and is ergodic and mixing. For example, f_0 may be a dispersing billiard ball map [17, 9, 10] or the baker transformation.

Proposition 4 *Let f be a sufficiently small C^2-perturbation of f_0 such that the singularity curves Γ of the map f are close enough to the singularity curves Γ_0 in C^1-metric. Then the attractor Λ generated by f has only one ergodic component and f is mixing on Λ.*

The proof is essentially outlined in Ref. [12]. We only note that the only argument in our proofs that is based on the mixing of f_* is the inequality (18) in the proof of Lemma 12. Another proof of Lemma 12 based on the mixing of f_0 instead of f_* may be found in Ref. [12] (see Lemma 13 there). We do not go into detail.

6 Examples

Belykh attractor. One of the famous systems possessing a generalized hyperbolic attractor is the Belykh map [19].

It is defined on a square $\bar{U}$ =clos $U = \{(x, y) : 0 \le x \le 1, 0 \le y \le 1\}$. We cut it in two parts by a line $\Gamma = \{(x, y) : y - 1/2 = k(x - 1/2)\}$. The map $f : \bar{U} \setminus \Gamma \to \bar{U}$ is defined by $f(x, y) = (\lambda x, \gamma y)$ on the lower part (where $y - 1/2 < k(x - 1/2)$) and $f(x, y) = (\lambda(x - 1) + 1, \gamma(y - 1) + 1)$ on the upper part of the square $\bar{U}$. The parameters λ, γ, k of the Belykh map satisfy three conditions:

$$\lambda < 1/2, \quad |k| < 1, \quad 1 < \gamma < \frac{2}{1 + |k|}. \tag{20}$$

It has been proved in Refs. [1, 2] that under the conditions (20) the map f has a generalized hyperbolic attractor. This map was introduced by the Russian physicist Belykh and was used as the simplest model in the so-called phase synchronization theory. He showed that the map f is an adequate model of a digital system of phase synchronization.

It follows immediately from the results of Refs. [1, 2] that the Belykh map f satisfies all our assumptions except for, possibly, Condition A2. We now prove that it satisfies A2 under additional restrictions on the parameters λ, γ and k. Let us stress that γ coincides with the constant λ^{-1} in Condition A2.

There are two approaches to ensure Condition A2.

First approach. Consider an arbitrary trajectory $\{z_k\}_{k \in \mathbb{Z}}, z_k = f^k z_0$ and denote by $z_k = (x_k, y_k)^T$, so that z_k is a column vector. The following equation follows from the definition of f:

$$z_n = A^n(z_0 - a_0 \mathbb{1}) + \sum_{j=1}^{n-1} A^{n-j}(a_{j-1} - a_j)\mathbb{1} + a_{n-1}\mathbb{1}, \tag{21}$$

where A =diag(λ, γ), $\mathbb{1} = (1, 1)^T$ and $a_i = 1$ if the point $z_i = (x_i, y_i)^T$ belongs in the upper part of the square (i.e. $y_i - 1/2 > k(x_i - 1/2)$) and $a_i = 0$ otherwise. In what follows we assume that $k < 0$ without loss of generality.

The vector equation (21) can be rewritten as a system

$$x_n = \lambda^n(x_0 - a_0) + \lambda^{n-1}a_0 + (\lambda^{-1} - 1)\sum_{j=1}^{n-1} \lambda^{n-j}a_j$$

$$y_n = \gamma^n(y_0 - a_0) + \gamma^{n-1}a_0 + (\gamma^{-1} - 1)\sum_{j=1}^{n-1} \gamma^{n-j}a_j. \tag{22}$$

Indeed,

$$x_n = \lambda^n(x_0 - a_0) + \sum_{j=1}^{n-1} \lambda^{n-j}(a_{j-1} - a_j) + a_{n-1}$$

$$= \lambda^n (x_0 - a_0) - \sum_{j=1}^{n-1} \lambda^{n-j} a_j + \lambda^{-1} \sum_{j=1}^{n-1} \lambda^{n-j+1} a_{j-1} + a_{n-1}$$

$$= \lambda^n (x_0 - a_0) - \sum_{j=1}^{n-1} \lambda^{n-j} a_j + \lambda^{-1} \sum_{j'=1}^{n-1} \lambda^{n-j'} a_{j'} - \lambda^{-1} \lambda a_{n-1} + a_{n-1} + \lambda^{n-1} a_0$$

$$= \lambda^n (x_0 - a_0) + \lambda^{n-1} a_0 + (\lambda^{-1} - 1) \sum_{j=1}^{n-1} \lambda^{n-j} a_j.$$

We now suppose that $y_0 = kx_0$ and $y_n = kx_n$ with some "admissible" word $(a_0, a_1, \ldots, a_{n-1})$. The system (22) yields the equation

$$\gamma^n k x_0 - a_0 \gamma^n + \gamma^{n-1} a_0 + \frac{1-\gamma}{\gamma} \sum_{j=1}^{n-1} \gamma^{n-j} a_j$$

$$= \lambda^n k x_0 - k a_0 \lambda^n + k \lambda^{n-1} a_0 + k \frac{1-\lambda}{\lambda} \sum_{j=1}^{n-1} \lambda^{n-j} a_j.$$

Hence,

$$\sum_{j=1}^{n-1} \gamma^{-j} a_j = \Big\{ -kx_0 + a_0 - \frac{1}{\gamma} a_0 + \left(\frac{\lambda}{\gamma}\right)^n \left[kx_0 - ka_0 + \frac{k}{\lambda} a_0 \right]$$

$$+ \gamma^{-n} k \frac{1-\lambda}{\lambda} \sum_{j=1}^{n-1} \lambda^{n-j} a_j \Big\} \frac{\gamma}{1-\gamma}. \tag{23}$$

Any other "admissible" word $(a_0, a'_1, \ldots, a'_{n-1})$ for which $y_0 = kx_0$ and $y_n = kx_n$ also satisfies Eq. (23). Subtracting that equation for the word $(a_0, a'_1, \ldots, a'_{n-1})$ from Eq. (23) gives

$$\left| \sum_{j=1}^{n-1} \gamma^{-j} (a_j - a'_j) \right| = \gamma^{-n} \cdot |k| \cdot \left| \sum_{j=1}^{n-1} (a_j - a'_j) \lambda^{n-j} \right| \frac{\gamma}{\gamma - 1} \cdot \frac{1-\lambda}{\lambda} \leq \gamma^{-(n-1)} L_0 \tag{24}$$

where

$$L_0 = \frac{|k|}{\gamma - 1}.$$

We will use the notion of GE-numbers. A real number $\beta \in (0,1)$ is said to satisfy the GE condition (after Garsia-Erdös, cf. Refs. [20, 21]) if there is a constant $C = C(\beta) > 0$ such that for every $x > 0$ and $n \geq 1$ one has

$$\mathrm{Card}\{(i_0, \ldots, i_{n-1}) : \pi(i_0, \ldots, i_{n-1}) \in [x, x + \beta^n)\} \leq C(2\beta)^n,$$

where $\pi(i_0, \ldots, i_{k-1}) = \sum_{k=0}^{n-1} i_k \beta^k$ and the indices i_k take values 0 and 1.

It has been proved in Lemma 6 of Ref. [22] that if γ^{-1} is a GE-number, then the number N_{n-1} of all the words $(a'_1, \dots, a'_{n-1})$ for which

$$\left| \sum_{j=1}^{n-1} \gamma^{-j}(a_j - a'_j) \right| < \gamma^{-(n-1)} L_0$$

is bounded above by

$$N_{n-1} \le L \cdot (2\gamma^{-1})^{n-1} \tag{25}$$

with a constant $L > 0$. Of course, the bound (25) holds for the number of admissible words.

It is also known [22] that there exists a $\delta > 0$ such that for almost all $\beta \in [1-\delta, 1)$ the GE condition holds.

In fact, we have shown that if γ^{-1} is a GE-number, then the number of smooth components of the union $\cup_{l=0}^{m} S_l^+$ which meet at the point $z_0 = (x_0, y_0)$ does not exceed

$$2L \cdot [(2\gamma^{-1}) + \dots + (2\gamma^{-1})^m] \le \frac{2L[(2\gamma^{-1})^m - 1](2\gamma^{-1})}{(2\gamma^{-1}) - 1} \le (2\gamma^{-1})^m \cdot C_0,$$

where

$$C_0 = \frac{4L\gamma^{-1}}{2\gamma^{-1} - 1}.$$

In the above bound the factor $2L$ is added because a_0 can take either of two values 0 and 1. Therefore, if $\gamma > \sqrt{2}$, then $2\gamma^{-1} < \gamma$ and Condition A2 is satisfied.

Second approach. It is sometimes simpler to check Condition A1 instead of A2. Let us assume again that $k < 0$ and consider the point $x_0 = 1, y_0 = 1/2 + k/2$ with its trajectory (x_i, y_i). We fix a $\tau \in \mathbb{Z}_+$ such that $\gamma^\tau > 2$ and $\gamma^i \le 2$ for $i = 1, \dots, \tau - 1$.

If $\tau = 2$, then we only need to require that $y_1 - 1/2 > k(x_1 - 1/2)$, where, of course, $x_1 = \lambda$ and $y_1 = \gamma(1+k)/2$. This requirement is equivalent to the inequality

$$\gamma > \frac{2k\lambda - k + 1}{1 + k}. \tag{26}$$

If $\tau > 2$, then we suppose that $a_j = 1$ for $j = 1, \dots, \tau - 1$ and $a_0 = 0$. In virtue of Eq. (22) one has

$$x_i = \lambda^i + (\lambda^{-1} - 1) \sum_{j=1}^{i-1} \lambda^{i-j} = \lambda^i - \lambda^{i-1} + 1$$

$$y_i = \gamma^i (1/2 + k/2) - \gamma^{i-1} + 1$$

for any $i = 1, \dots, \tau$. It is enough to require, in addition to Eq. (26), one more inequality holds:

$$y_i - \frac{1}{2} > k\left(x_i - \frac{1}{2}\right).$$

for every $i = 2, \ldots, \tau$. This last inequality is equivalent to

$$\gamma^i \left(\frac{1}{2} + \frac{k}{2}\right) - \gamma^{i-1} + \frac{1}{2} > k\left(\lambda^i - \lambda^{i-1} + \frac{1}{2}\right) \tag{27}$$

Condition A1 then holds under the assumptions (26) and (27).

Lozi map. R. Lozi introduced in Ref. [5] a map

$$(x, y) \to (by, 1 - a|y| + x),$$

which was a simplified model for the famous Hennon map

$$(x, y) \to (by, 1 - ay^2 + x).$$

It is easy to check that the rectangle $U = \{(x, y) : |y| < 1 + \alpha, |x| < \alpha\}$ is semi-invariant under the Lozi map (invariant under the positive iterates of the map) provided

$$|b|(1 + \alpha) < \alpha < \frac{2 - a}{a}, \quad \text{and} \quad 1 < a < 2. \tag{28}$$

In order to verify hyperbolicity of the Lozi map we invoke a general theorem from Ref. [23]. A map $(x, y) \to (f(x, y), g(x, y))$ is shown there to be hyperbolic if the following conditions hold:

$$||f_x|| < 1; \quad ||g_y^{-1}|| < 1; \text{ and } 1 - ||g_y^{-1}|| \cdot ||f_x|| > 2\sqrt{||g_y^{-1}|| \cdot ||g_x|| \cdot ||g_y^{-1} f_y||};$$

$$||g_x|| \cdot ||f_y g_y^{-1}|| < (1 - ||f_x||)(1 - ||g_y^{-1}||). \tag{29}$$

Here and further on $|| \cdot ||$ denotes the C^0-norm of a function on $U \setminus \Gamma$ (i.e., the supremum of its absolute value). In our case $f_x = 0$, $|f_y| = |b|$, $|g_y^{-1}| = a^{-1}$ and $|g_x| = 1$. Therefore, the conditions (29) hold if one assumes, in addition to Eq. (28), the following:

$$a > 1, \quad a > 2\sqrt{|b|}, \quad |b| < a - 1 \tag{30}$$

The Lozi map has the only singularity curve, the line $\Gamma = \{(x, y) : y = 0, |x| \le \alpha\}$.

We now verify Condition A1. Certain results in this direction have been announced in Refs. [1, 2]. We use the techniques of Ref. [24]. The constant λ^{-1} involved in Conditions A1-A2 can be bounded as $\lambda^{-1} > q$, where

$$q = \frac{1 + ||f_x|| \cdot ||g_y^{-1}|| + \sqrt{(1 - ||f_x|| \cdot ||g_y^{-1}||)^2 - 4||g_y^{-1}|| \cdot ||g_x|| \cdot ||f_y g_y^{-1}||}}{2||g_y^{-1}||}.$$

The third inequality in Eq. (30) implies that

$$q = \frac{a + \sqrt{a^2 - 4|b|}}{2} > 1.$$

We now fix an integer τ such that $q^\tau > 2$ and $q^i \leq 2$ for $i = 0, 1, \ldots, \tau - 1$. Let (x_0, y_0) be a point on the line Γ and let $(x_i, y_i), i \geq 1$ be its trajectory. Notice that $x_1 = 0$ and $y_1 = 1 + x_0$, $x_0 \in (-\alpha, \alpha)$.

If $\tau = 2$, then the assumption (28) implies $1 - \alpha > 0$ and we immediately obtain Condition A1.

If $\tau = 3$, then we have to require two more inequalities:

$$Y_2 := 1 - a(1 - \alpha) < 0 \tag{31}$$

and

$$Y_3 := 1 - a|1 - a(1 + \alpha)| - |b|(1 + \alpha) > 0. \tag{32}$$

It is easy to check that $y_2 < Y_2$ and $y_3 > Y_3$. Therefore, the first three images of the line Γ do not intersect Γ and A1 follows.

In a similar fashion one can find conditions under which A1 holds if $\tau = 4, 5, \ldots$. It was announced in Ref. [2] that A1 is valid for an open set of parameters dense in the region $\{(a, b) : |b| < \min(a - 1, 2 - a)\}$.

Remark. M. Rychlik has informed us that Condition A2 is always satisfied for the Lozi map. In fact, the number of smooth components of the singular curves for f^m that can meet at a point in U grows at most linearly in m. This can be shown by using the continuity of the map f on U. The proof can be obtained along the lines of Section 8 in Ref. [9]. We do not go into details here.

Appendix

Here we provide the proofs of Proposition 1, Lemmas 1–3 and Theorem 3.

Our proofs of Conditions A3 and A4 basically follows the lines of the proof of Theorem 14 in Ref. [1], but we introduce a new useful lemma, see below.

Let W^u be an arbitrary unstable curve and ν^u be the normalized Lebesgue measure on it. For any $n \geq 0$ and $x \in W^u$ denote by $r_n(x)$ the distance from $f^n x$ to the nearest endpoint of the smooth component of $f^n W^u$ containing the point $f^n x$.

Lemma 14 (Distribution of lengths of unstable curves) *There are constants $\varepsilon_0 > 0$ and $C_1 > 0$ such that for any $n \geq 1$ and any unstable curve W^u of length $\geq \varepsilon_0$ one has*

$$\nu^u\{x : r_n(x) < \varepsilon\} < C_1 \varepsilon$$

for any $\varepsilon > 0$.

Note that a somewhat stronger version of this lemma was recently published in Ref. [18] (see Theorem 2 there).

Proof. Note that the meaning of this lemma is close to that of Lemma 8. To prove Lemma 14 we fix a sufficiently large m, specified by $C_0 K_0^m + 2 < d_0 \lambda^{-m}$, where

$d_0 > 0$ is a sufficiently small real defined below. As in the proof of Lemma 8, there is an $\varepsilon_0 = \varepsilon_0(m) > 0$ such that the image of any unstable curve of length $< \varepsilon_0$ under the map f^m consists of no more than $C_0 K_0^m + 1$ smooth components.

It is enough to prove the statement of Lemma 14 for all $\varepsilon \leq \varepsilon_0$. It is obvious for $n = 0$ with any $C_1 \geq 2$.

Recall that the image of the Lebesgue measure on an unstable curve W^u pulled onto the smooth components of $f^n W^u, n \geq 1$ has a density $\rho(x)$ with respect to the Lebesgue measure on those components satisfying the bounds $\bar{c} \leq \rho(x)/\rho(y) \leq \bar{c}^{-1}$ for any x and y in the same smooth component, where $\bar{c} = \bar{c}(f)$ is an absolute constant.

Next, let W_1^u be an arbitrary unstable curve of length $l_1 > 0$ with a normalized measure ν_1^u on it such that the density of ν_1^u with respect to the normalized Lebesgue measure is confined between c and c^{-1}. If $l_1 > \varepsilon_0$, we cut W_1^u into smaller curves of length less than ε_0 but greater than $\varepsilon_0/2$. Let N denote the collection of endpoints of those smaller curves. We set $d_1 = 1$ if $l_1 \leq \varepsilon_0$ and $d_1 = 2 l_1 \varepsilon_0^{-1}$ otherwise. Obviously, d_1 is uniformly bounded: $d_1 \leq d_3 = d_3(f)$. Let $r_m^*(x), x \in W_1^u$ be the distance from x to the set $W_1^u \cap (\cup_{i=0}^m S_i^+ \cup N)$. Since that set consists of no more than $d_1(C_0 K^m + 2)$ points, it is easy to obtain the bound

$$\nu_1^u\{r_m^*(x) \leq \varepsilon\} \leq l_1^{-1} d_2 d_1 (C_0 K_0^m + 2)\varepsilon \tag{33}$$

for any $\varepsilon > 0$. Here $d_2 = d_2(\bar{c})$ is an absolute constant (determined by the map f alone). In particular, the statement of Lemma 14 is true for every $n = 1, \ldots, m$ with any $C_1 \geq \varepsilon_0^{-1} d_2 d_3 (C_0 K_0^m + 2)$.

Obviously, for any $x \in W_1^u$ one has $r_m(x) \geq \lambda^{-m} r_m^*(x)$. Therefore, one obtains

$$\nu_1^u\{r_m(x) \leq \varepsilon\} \leq l_1^{-1} d_2 d_1 (C_0 K_0^m + 2)\lambda^m \varepsilon \leq l_1^{-1} d_2 d_1 d_0 \varepsilon. \tag{34}$$

We now assume that the constant d_0 is so small that $d_2(\bar{c})^{-1} d_0 \leq 0.1$. Denote $c_2 = l_1^{-1} d_2 d_3 d_0$. The bound (34) can be rewritten for three different cases as follows:
(i) if $l_1 > \varepsilon_0$, then $d_1 \leq d_3$, and so

$$\nu_1^u\{r_m(x) \leq \varepsilon\} \leq c_2 \varepsilon; \tag{35}$$

(ii) if $l_1 \in [\varepsilon, \varepsilon_0]$, then $d_1 = 1, \nu_1^u\{r_0(x) < \varepsilon\} \geq l_1^{-1} \bar{c} \varepsilon$, and so

$$\nu_1^u\{r_m(x) \leq \varepsilon\} \leq d_2(\bar{c})^{-1} d_0 \nu_1^u\{r_0(x) \leq \varepsilon\} \leq 0.1 \nu_1^u\{r_0(x) \leq \varepsilon\}; \tag{36}$$

(iii) otherwise, $l_1 < \varepsilon$ and one has, obviously,

$$\nu_1^u\{r_m(x) \leq \varepsilon\} \leq \nu_1^u(W^u) = \nu_1^u\{r_0(x) \leq \varepsilon/2\}. \tag{37}$$

Let $C_1 \geq 10 c_2$. We now assume that the statement of Lemma 14 is true for $n = n_0$. We apply the inequalities (35)-(37) to every component of $f^n W^u$ with ν_1^u being the conditional measure on that component induced by $\nu^u \circ f^{-n}$. Adding Eqs. (35)-(37) for all the components of $f^n W^u$ gives

$$\nu^u\{r_{n_0+m}(x) \leq \varepsilon\} \leq c_2 \varepsilon + 0.1 C_1 \varepsilon + C_1 \varepsilon/2 \leq C_1 \varepsilon$$

By the inequality (33) and the remark following it we extend the last bound to all $n = n_0 + 1, \dots, n_0 + m - 1$ with any $C_1 \geq \max\{10c_2, \varepsilon_0^{-1} d_2 d_3 (C_0 K_0^m + 2)\}$. Lemma 14 is proven.

Conditions A3 and A4 easily follow from Lemma 14 in view of the transversality of Γ and unstable curves.

We now provide a short proof of the finiteness of ergodic components in the Smale spectral decomposition (Proposition 3) under our assumptions. This follows from Sataev's paper [2], but our proof is shorter and simpler. It consists of three steps. We assume Proposition 3 with an infinite number of components $\{\Lambda_i\}$, and pick a Gibbs measure μ which is positive on every ergodic component.

Step 1. Classical Hopf's idea is that for a.e. point its stable and unstable fibers belong to one ergodic component. Although this idea is well known, we cannot refer to any published theorem that covers our model. Instead, we outline a proof of this claim. Let $x \in \Lambda$ be a point and $y \in V^u(x)$. Let F be a continuous function on the manifold M. Since the closure of U is compact (see Section 2), F is uniformly continuous on U. Therefore, the time average over the past trajectory

$$\tilde{F}(y) := \lim_{n \to \infty} n^{-1} \cdot \sum_{i=0}^{n-1} f^{-i}(y)$$

is constant on $V^u(x)$ provided the limit exists at least for one $y \in V^u(x)$. Birkhoff's ergodic theorem says that $\tilde{F}$ does exist almost everywhere and is an invariant function on Λ. Moreover, on each ergodic component of Λ that function, $\tilde{F}$, is a constant, which is equal to the average value of F on that component. Next, let Λ_i and Λ_j, $i \neq j$, be two ergodic components of Λ. Continuous functions are dense in $L^2(\Lambda)$, and so the characteristic function (indicator) of Λ_i can be approximated in L^2 by continuous functions. For those continuous functions, that are sufficiently close in the L^2-metric to the indicator of Λ_i, their average values on Λ_i are close to one, while their average values on Λ_j are close to zero, so that $\tilde{F}$ takes different values on Λ_i and Λ_j. Therefore, the set of unstable fibers that intersect both Λ_i and Λ_j has zero measure. This proves Hopf's claim. The argument for stable fibers is the same, provided one averages the function F over the future trajectory of y.

Step 2. Since any ergodic component is invariant under f, not only the LUF $V^u(x)$, but also all its images belong to the same ergodic component for almost every point x. Due to Lemma 14, such images necessarily become long enough, sooner or later. That is, there is a positive constant $c > 0$ such that among all the images of any LUF there is a smooth component of length $> c$. Thus, every ergodic component contains unstable fibers of length $> c$, and their union is a set of positive measure. We now need to show that there can be only a finite number of ergodic components with unstable fibers of length $> c$.

Step 3. Let the number of ergodic components be infinite. In every component Λ_i we pick an unstable fiber V_i^u of length $> c$, which is not an isolated one, i.e. the intersection of any neighborhood of any point of V_i^u with all the other unstable fibers of length $> c$ in the component Λ_i has positive measure. (Such fibers were constructed in Lemma 9.) The sequence V_i^u has a limit point in C^1 topology, and that limit point is also an unstable fiber, V_∞^u, see also the proof of Lemma 9. Finally, through almost every point of V_∞^u (with respect to the length on that curve) we can draw a stable fiber of a positive length, as mentioned in the proof of Lemma 10. Obviously, those stable fibers form a set of positive measure, and they cross infinitely many unstable fibers V_i^u. We apply Hopf's claim again, this time to stable fibers, and deduce that infinitely many V_i^u's must belong to one ergodic component. Q.E.D.

Lemma 14 has one more corollary:

Corollary 3 *Let W be a smooth compact curve in M transversal to unstable cone $C^u(z)$ at every point $z \in W$. Then*

$$\mu(U_\varepsilon(W)) \le c\varepsilon$$

for any $\varepsilon > 0$ and any Gibbs measure μ. Here $U_\varepsilon(W)$ denotes the ε-neighborhood of W and $c = c(W) > 0$.

For generalized hyperbolic attractors satisfying A3 and A4 Pesin [1] and Sataev [2] have proved certain properties from which our Lemma 1 follows immediately. In particular, Sataev [2] proved that there is an $\varepsilon > 0$ such that for any $z \in D^+_{\varepsilon,l}$ ($z \in D^-_{\varepsilon,l}$) the distance of z from the nearest endpoint of the LSF $V^s(z)$ (resp., the LUF $V^u(z)$) is greater than const$\cdot l^{-\gamma}$ with a constant $\gamma = \gamma(f) > 0$. For the μ-measure of the sets $D^\pm_{\varepsilon,l}$ Sataev obtained the estimate

$$\mu(D^\pm_{\varepsilon,l}) \ge 1 - \text{const} \cdot l^{-\beta}$$

with a constant $\beta = \beta(f) > 0$ for any Gibbs measure μ. This estimate implies Lemma 1.

We now turn to the proof of Lemma 2. The set $S^+ \cup S^-$ consists of a finite number of smooth compact curves. For the curves (or their parts) transversal to unstable cones, in particular, for Γ, we can apply Corollary 3. We then consider the other smooth parts of $S^+ \cup S^-$, which are then unstable curves. For any point x belonging in their ε-neighborhoods one has either $r^s(x) <$const$\cdot\varepsilon$ (if x belongs in the ε-neighborhood of ∂U) or $r^s(f^{-1}x) <$const$\cdot\varepsilon$ (if x belongs in the ε-neighborhood of $f(\partial U)$). In both cases applying Lemma 1 completes the proof of Lemma 2.

In order to prove Lemma 3 we consider a set $f_*^{-t}D$ for any $t \le k$ (D and k were defined in Lemma 3). The diameter of that set is obviously less than $\lambda_1^t l + \lambda_1^{-t}\varepsilon$, where $\lambda_1 = \lambda^{r_*}$ and $\lambda < 1$ is the constant involved in the definition of cones $C^{u,s}$ in Section 2. Due to Lemma 14 the μ_*-measure of that set is $\le r_*\varepsilon_0^{-1}C_1(\lambda_1^t l + \lambda_1^{-t}\varepsilon)$ and Lemma 3 follows.

Finally, we prove Theorem 3 in case $k = l = 2$. Denote $t_0 = i_1, t_L = i_2$ and chose some integers $t_1 < t_2 < \cdots < t_{L-1}$ so that $t_0 < t_1, t_{L-1} < t_L$ and $\min_{1\le i\le L}\{t_i - t_{i-1}\} \ge g_0 n$. For convenience, we introduce probabilistic notations. Denote $\pi_{ij}^{(l)} = \mu_*(f_*^{t_{l-1}}A_j / f_*^{t_l}A_i)$ for $1 \le l \le L$ and $0 \le i, j \le I$. Obviously, the matrices $\Pi^{(l)} = ||\pi_{ij}^{(l)}||$ are stochastic for

$1 \leq l \leq L$ and have a common stationary distribution $P = ||p_i||$ with $p_i = \mu_*(A_i)$. The product $\Pi^{(1)} \cdots \Pi^{(l)}$ is denoted by $\Pi^{(1,l)} = ||\pi_{ij}^{(1,l)}||, 1 \leq l \leq L$. In other words, we have a nonstationary Markov chain with a discrete time l and an equilibrium distribution P.

For any $i \in [0, I]$ and $l \in [1, L]$ denote

$$d_l(i) = \sum_{j=0}^{I} |\pi_{ij}^{(1,l)} - p_j|.$$

In the language of probability theory, $d_l(i)$ is twice the distance in variation between the stationary distribution P and the distribution specified by the ith row of the matrix $\Pi^{(1,l)}$. It is well known in probability theory [16] that the sequence $d_l(i)$ is monotonically decreasing in l.

By using Conditions MS3 and MS2, it is easy to show [10] that the inequality in the statement (i) of Theorem 3 is equivalent to the following one:

$$d_L(i) \leq \Delta, \tag{38}$$

where $i = j_1$.

Lemma 15 *For any $l = 0, \ldots, L-1$ and $i \in \Im$ one has*

$$d_{l+1}(i) \leq (1 - g_1/2)d_l(i) + p_0 + \pi_{i0}^{(1,l)}.$$

Proof. We have

$$d_{l+1}(i) = \Sigma_j^+(\pi_{ij}^{(1,l+1)} - p_j) = \Sigma_j^+\Sigma_k(\pi_{ik}^{(1,l)} - p_k)\pi_{kj}^{(l+1)}$$

$$\leq \Sigma_k^+(\pi_{ik}^{(1,l)} - p_k)\Sigma_j^+\pi_{kj}^{(l+1)}. \tag{39}$$

Here and further on Σ_j^+ (Σ_j^-) denotes the summation over values of j for which $\pi_{ij}^{(1,l+1)} > p_j$ (resp., $\pi_{ij}^{(1,l+1)} < p_j$), and Σ_k^+ (Σ_k^-) denotes the summation over values of k for which $\pi_{ik}^{(1,l)} > p_k$ (resp., $\pi_{ik}^{(1,l)} < p_k$). Likewise,

$$d_{l+1}(i) \leq -\Sigma_k^-(\pi_{ik}^{(1,l)} - p_k)\Sigma_j^-\pi_{kj}^{(l+1)}. \tag{40}$$

Lemma 16 *For any $l = 1, \ldots, L-1$ and $i \in \Im$ one has either*

$$\max_{1 \leq k \leq I} \Sigma_j^+\pi_{kj}^{(l+1)} < 1 - g_1/2, \tag{41}$$

or

$$\max_{1 \leq k \leq I} \Sigma_j^-\pi_{kj}^{(l+1)} < 1 - g_1/2. \tag{42}$$

Proof. Denote

$$g' = 1 - \max_{1 \leq k \leq I} \Sigma_j^+ \pi_{kj}^{(l+1)}.$$

If $1 - g' < 1 - g_1/2$, then Eq. (41) holds. If not, then

$$\Sigma_j^- \pi_{kj}^{(l+1)} = g' < g_1/2$$

for some $k \in \Im$. The Doeblin condition MS4 then implies that

$$\max_{1 \leq r \leq I} \Sigma_j^- \pi_{rj}^{(l+1)} < 1 - g_1/2,$$

and so Eq. (42) holds. Lemma 16 is proven.

Combining Eq. (39) with Eq. (41) or Eq. (40) with Eq. (42) completes the proof of Lemma 15.

Lemma 15 readily implies that

$$d_L(i) \leq (1 - g_1/2)^L + \sum_{l=1}^{L-1} \pi_{i0}^{(1,l)} + Lp_0.$$

Eq. (38) is then obtained, as in Ref. [10], by defining the set $R_* \subset \Im$ so that $i \in R_*$ iff

$$\sum_{l=1}^{L-1} \pi_{i0}^{(1,l)} \leq \alpha^n$$

for some $\alpha \in (\alpha_2, 1)$ which then determines the value of α_4 in Theorem 3.

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