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Generating partitions for two-dimensional hyperbolic maps

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Abstract:

For a class of two-dimensional hyperbolic maps (which includes certain billiard systems) we construct finite generating partitions. Thus trajectories of the map can be labeled uniquely by doubly infinite symbol sequences, where the symbols correspond to the atoms of the partition. It is shown that the corresponding conditions are fulfilled in the case of the cardioid billiard, the stadium billiard (and other Bunimovich billiards), planar dispersing and semi-dispersing billiards.

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1 Introduction

In the last decades much work has been devoted to the investigation of classical dynamical systems possessing strong stochastic properties. In particular billiard systems have been studied thoroughly as prototypical systems exhibiting ergodicity, mixing, K- and Bernoulli-property, see e.g. [**?**]. The quantized version of the classical billiards are also studied intensively in the context of quantum chaos, see e.g. [**?**]. Here the connection between the classical billiard system and the quantum mechanical system is given in terms of trace formulas, which relate the quantum mechanical density of states to a sum over properties of the periodic orbits of the classical dynamical system. This clearly demonstrates the need of a complete classification of trajectories of the classical system by means of symbolic dynamics.

When studying the statistical properties of hyperbolic dynamical systems much insight could be gained by constructing Markov partitions of the phase space. For non-uniformly hyperbolic billiard systems with singularities usually Markov partitions are countably infinite. However, if one is interested in a symbolic description of trajectories, it is already sufficient and even more appropriate to find a finite generating partition (see e.g. [**?**]), without the need for the Markov property. Then trajectories can be labeled by doubly-infinite symbol sequences built from a finite alphabet corresponding to the atoms of the partition, and periodic orbits of the map correspond to periodic symbolic sequences. Thus a symbolic dynamics allows to search for periodic orbits in a systematic way. Of course, there remains the problem of finding numerically the periodic orbits for a given periodic symbolic sequence. For example, in the hyperbola billiard a minimum principle was used to compute a huge number of periodic orbits [**?**]. For certain billiard systems it is proven in [**?**], that at most one local minimum corresponds to a given periodic symbol sequence. And for the cardioid billiard periodic orbits correspond to maxima of the action function [**?**]. In other cases, where a well-ordered symbolic dynamics can be found, one can use the approach described in [**?**]. For examples of coding trajectories with finitely many symbols see e.g. [**?**, **?**, **?**, **?**, **?**, **?**, **?**, **?**] and references therein.

The paper is organized as follows: In section 2 we formulate conditions which imply the existence of a finite generating partition, and give the proof of the main theorem. In section 3 we give examples of billiard systems to which the theorem applies. The examples include the cardioid billiard, the stadium billiard (and other Bunimovich billiards), planar dispersing and semi-dispersing billiards.

2 Generating partitions for two-dimensional hyperbolic maps

We will first define the setting under which we prove the existence of generating partitions for two-dimensional hyperbolic maps and after some remarks give a proof of the theorem.

Let $\mathcal{P}$ be a compact domain in $\mathbf{R}^2$ with a piecewise smooth boundary and $T : \mathcal{P} \to \mathcal{P}$ an (invertible) transformation, such that T and T^{-1} are piecewise C^2 smooth.

Let Γ and Γ^-, the singularity sets for T and T^{-1}, respectively, (i.e. the sets of points where these maps fail to be C^2) consist of a finite number of C^1 smooth compact curves in $\mathcal{P}$. Be $\mathcal{P} \backslash (\Gamma \cup \partial \mathcal{P}) = O_1 \cup \cdots \cup O_r$ and $\mathcal{P} \backslash (\Gamma^- \cup \partial \mathcal{P}) = O_1^- \cup \cdots \cup O_r^-$, where O_i and O_i^- are open connected domains in $\mathcal{P}$ with piecewise smooth boundary. The sets $\{O_i\}_{i=1}^r$ define the r atoms of the partition of $\mathcal{P}$. Likewise, the sets $\{O_j^-\}_{j=1}^r$ are atoms of another partition of $\mathcal{P}$. By construction T is continuous on O_i and T^{-1} is continuous on O_i^-, $i = 1, \ldots, r$. We assume that T preserves a finite measure μ on $\mathcal{P}$ such that the μ-measure of every open set $E \subset \mathcal{P}$ is positive.

Theorem 2.1 *Assume that*

1. *The curves of Γ are increasing and those of Γ^- are decreasing.*

   By increasing or decreasing curves we always mean strictly *increasing or decreasing curves, i.e. those defined by a function $y = f(x)$ such that $f'(x) > 0$ and $f'(x) < 0$, respectively, inside $\mathcal{P}$. On the boundary of $\mathcal{P}$, where our curves terminate, the derivative $f'(x)$ may approach zero or infinity.*

2. *The interior angles of the domains O_i, O_j^- are $\leq \pi$.*

3. *T is hyperbolic at μ-almost every point (Precisely, T is a smooth (nonuniformly) hyperbolic map with singularities in the standard sense [**?**].) In particular, unstable and stable fibers exist almost everywhere, and they are decreasing and increasing curves, respectively.*

4. *T (T^{-1}) takes decreasing (increasing) curves into decreasing (increasing) curves.*

5. *For any vertical or horizontal segment I in the interior of $\mathcal{P}$ there is an $n \in \mathbf{Z}$ such that $T^n I$ contains an increasing or decreasing curve.*

Then the partition of $\mathcal{P}$ into O_i is a generating partition. The same is true for the partition of $\mathcal{P}$ into O_i^-.

Remark 2.2 *Notice, that ergodicity of T is not assumed.*

Remark 2.3 *The monotonicity assumptions 1, 2 and 4 in the theorem can be reversed by interchanging the words decreasing and increasing. Moreover, the theorem can be generalized to maps on regions $\mathcal{P} = \mathcal{P}_1 \cup \cdots \cup \mathcal{P}_m$ such that every $\mathcal{P}_i$ is a domain satisfying the conditions of the theorem and the monotonicity assumption 4 is different on each domain $\mathcal{P}_i$, $1 \leq i \leq m$, see an example in section 3.3.*

Corollary 2.4 *For the topological entropy one has $h_{top}(T) \leq \ln r$, where r is the number of atoms of the partition.*

We will prove the theorem for the partition of $\mathcal{P}$ into O_i. To that end we first need some notations:
Define the singularity set for the map T^n, $n \geq 1$, by

$$\Gamma^n = \bigcup_{i=1}^{n} T^{-i+1}\Gamma \tag{1}$$

and the singularity set for the map T^{-n}, $n \geq 1$, by

$$\Gamma^{-n} = \bigcup_{i=1}^{n} T^{i-1}\Gamma^- \quad . \tag{2}$$

Points of Γ^n, $n \geq 1$, will map under $\leq n$ iterations of T into a singularity, i.e., for any $\xi \in \Gamma^n$ one has $T^i\xi \in \Gamma$ for some $1 \leq i \leq n-1$. Similarly for $\xi \in \Gamma^{-n}$ one has $T^{-i}\xi \in \Gamma^-$ for some $1 \leq i \leq n-1$.
Define the set of points whose forward iterates will never hit a singularity

$$\tilde{\mathcal{P}}^+ = \mathcal{P} \backslash (\Gamma^{+\infty} \cup \partial\mathcal{P}) \ , \tag{3}$$

where $\Gamma^{+\infty}$ is the measure zero set of orbits which will hit a singularity under application of T^n for some $n \geq 1$. For a given point $\xi \in \tilde{\mathcal{P}}^+$ its future is well-defined for any iterate. Similarly one defines

$$\tilde{\mathcal{P}}^- = \mathcal{P} \backslash (\Gamma^{-\infty} \cup \partial\mathcal{P}) \ , \tag{4}$$

where $\Gamma^{-\infty}$ is the measure zero set of orbits which will hit a singularity under application of T^{-n} for some $n \geq 1$. Moreover define the set of initial conditions never hitting a singularity under either forward or backward iterations

$$\tilde{\mathcal{P}} = \tilde{\mathcal{P}}^+ \cap \tilde{\mathcal{P}}^- \quad . \tag{5}$$

The doubly-infinite code $\omega = \ldots \omega_{-2}\omega_{-1}.\omega_0\omega_1\omega_2 \ldots$ generated by a point $\xi \in \tilde{\mathcal{P}}$, is given by ($i \in \mathbf{Z}$)

$$\omega_i = O_k, \quad \text{if } T^i\xi \in O_k, \ \ 1 \leq k \leq r \ . \tag{6}$$

Recall that a partition is called generating, if for any code-word ω the intersection $\cap_{n\in\mathbf{Z}} T^{-n}\omega_n$ contains at most one point.
Let $-\infty < m \leq n < \infty$. We define the finite truncation of a code word ω by

$$\omega_{m,n} = \omega_m \omega_{m+1} \dots \omega_n \tag{7}$$

and the corresponding intersection of sets by

$$\tilde{\omega}_{m,n} = \bigcap_{i=m}^{n} T^{-i}\omega_i \quad . \tag{8}$$

Lemma 2.5 *For a given $\xi \in \tilde{\mathcal{P}}$ the set $\tilde{\omega}_{m,n}$ is a connected domain with piecewise smooth boundary whose interior angles are $\leq \pi$. The maps T^i, $m \leq i \leq n+1$ are continuous on $\tilde{\omega}_{m,n}$.*

Proof. The proof proceeds by induction:
Let $m = 0$, $n = 0$. There are r different possibilities for $\omega_{0,0} = \omega_0$ (namely: $.O_i$, $i = 1, \dots, r$) giving r sets $\tilde{\omega}_{0,0} = \omega_0$, each of them being a connected domain. Moreover the map T is continuous on the domain $\tilde{\omega}_{0,0}$. Assume now that for a given ω and some $m \leq n$ the set $\tilde{\omega}_{m,n}$ satisfies our lemma. Consider

$$T^{n+1}\tilde{\omega}_{m,n} \cap \omega_{n+1} \quad . \tag{9}$$

By induction, $T^{n+1}\tilde{\omega}_{m,n}$ is a connected set bounded by the curves from $\Gamma^{-\infty} \cup \partial\mathcal{P}$. Assumption 2 gives that any interior angle of the set $T^{n+1}\tilde{\omega}_{m,n}$ whose vertex is on $\partial\mathcal{P}$ is $\leq \pi$. The smoothness of T away from the singularity curves implies, by induction on n, that any other interior angle of $T^{n+1}\tilde{\omega}_{m,n}$ is $\leq \pi$ as well. Next, the set ω_{n+1} is a connected domain (O_{n+1}) bounded by curves of Γ and $\partial\mathcal{P}$. Since Γ is increasing, and all the curves in $\Gamma^{-\infty}$ are decreasing, the set $T^{n+1}\tilde{\omega}_{m,n} \cap \omega_{n+1}$ is connected, and has interior angles $\leq \pi$. Obviously T is continuous on this set. Thus, the lemma holds for the word $\omega_{m,n+1}$.
A similar argument provides the extension of the word $\omega_{m,n}$ to the left. $\square$

Corollary 2.6 (Continuation of the singularity lines)
Let $n \geq 1$. Any smooth singularity curve γ in Γ^n (Γ^{-n}) either terminates on $\partial\mathcal{P}$ or is a part of a larger C^0 continuous decreasing (increasing) curve $\gamma' \subset \Gamma^n$ ($\gamma' \subset \Gamma^{-n}$) that terminates in $\partial\mathcal{P}$.

Lemma 2.7 *The singularity sets $\Gamma^{+\infty}$ and $\Gamma^{-\infty}$ are dense in $\mathcal{P}$.*

Proof. Assume that $\Gamma^{+\infty}$ is not dense, thus there exists an open set $E \subset \mathcal{P}$, which has positive measure, $\mu(E) > 0$, and $E \cap \Gamma^{+\infty} = \emptyset$. Consider a point $\xi \in E$ for which the unstable manifold exists. Thus we locally have a decreasing curve $\gamma^u \subset E$ such that $\gamma^u \cap \Gamma^{\infty} = \emptyset$. Iterating this curve by T^n, $n \geq 1$, will

give connected curves whose lengths grow to infinity, because it is an unstable manifold. Since $\mathcal{P}$ is a compact subset of $\mathbf{R}^2$ and the stable/unstable manifolds are monotone, we get a contradiction.
The proof for $\Gamma^{-\infty}$ proceeds similarly.

$\square$

Observe that, as a result of Lemma 2.7, no atom of the partitions into O_i or O_i^- can be invariant under the map T or T^{-1}.

Lemma 2.8 *For any decreasing (increasing) C^1 curve $\gamma \subset \mathcal{P}$ there is a C^0 continuous curve $\gamma' \subset \Gamma^\infty$ (resp., $\gamma' \subset \Gamma^{-\infty}$) that terminates on $\partial\mathcal{P}$ and crosses γ.*

Proof. Let γ be a decreasing curve. Denote by ξ its midpoint. There is a curve $\gamma'' \subset \Gamma^n$, $n \geq 1$, arbitrary close to ξ. Due to corollary 2.6, its continuation in Γ^n will cross γ. $\square$

Corollary 2.9 *Let $\xi_1, \xi_2 \in \tilde{\mathcal{P}}$ be two points such that the segment $\xi_1\xi_2 \subset \mathbf{R}^2$ is neither vertical nor horizontal. Then the domain $\mathcal{P}$ can be partitioned by one or more C^0 continuous curves $\gamma' \subset \Gamma^n$, $n \in \mathbf{Z}$, that terminate on $\partial\mathcal{P}$, so that ξ_1 and ξ_2 lie in different atoms of that partition.*

Remark 2.10 *Note that if the domain $\mathcal{P}$ is simply connected, then one curve γ' would suffice, and the points ξ_1 and ξ_2 would lie on the opposite sides of γ'.*

After this considerations we now come to the proof of theorem 1.

Proof. Let $\xi_1, \xi_2 \in \tilde{\mathcal{P}}$, $\xi_1 \neq \xi_2$, and the code words they generate be identical, $\omega = \omega_1 = \omega_2$. We shall show, that this implies a contradiction so that $\xi_1 = \xi_2$, giving the uniqueness of the coding.
Every finite sequence $\omega_{-n,n}$ corresponds, according to lemma 1, to an open, connected set $\tilde{\omega}_{-n,n}$. Consider the intersection

$$\overline{\omega}_{-\infty,\infty} = \bigcap_{n=1}^{\infty} \mathrm{cl}(\tilde{\omega}_{-n,n}) \ , \tag{10}$$

where $\mathrm{cl}(\cdot)$ means the closure of a domain. The set $\overline{\omega}_{-\infty,\infty}$ is a non-empty connected set, containing ξ_1 and ξ_2. We now show, that this set consists of only one point. Assume first that the set $\overline{\omega}_{-\infty,\infty}$ is neither a vertical nor a horizontal segment in $\mathbf{R}^2$. Then, according to corollary 2.9, this connected set is crossed by a singularity curve $\gamma' \subset \Gamma^n$, $n \in \mathbf{Z}$, and we get a contradiction. If $\overline{\omega}_{-\infty,\infty}$ is a vertical or horizontal segment, we apply assumption 5 and complete the proof.

$\square$

Corollary 2.11 *The set $\tilde{\omega}_{0,\infty}$ is a local stable fiber and the set $\tilde{\omega}_{-\infty,-1}$ is a local unstable fiber.*

3 Examples

3.1 Cardioid billiard

The boundary $\partial\Omega$ of the cardioid billiard is given in polar coordinates by

$$\rho(\varphi) = 1 + \cos\varphi \ , \qquad \varphi \in [-\pi, \pi] \ . \tag{11}$$

Inside Ω a point particle moves with unit velocity along straight lines until it reaches the boundary, where it is reflected elastically. The cardioid billiard is the limiting case of a family introduced in [?]. Later it has been proven, that the cardioid billiard has non-vanishing Lyapunov exponents almost everywhere, it is ergodic, mixing, a K-system and a Bernoulli system [?, ?, ?, ?, ?]. The symbolic dynamics and periodic orbits are investigated in detail in [?], see also [?]. For studies of the cardioid billiard in the context of quantum chaos, see [?, ?, ?, ?] and references therein.

Figure: fig1.ps

Figure 1: Boundary curve for the cardioid billiard. For a given point s its tangent vector $t(s)$ and the velocity vector v are shown.

As a Poincaré section $\mathcal{P}$ we define

$$\mathcal{P} = \{\xi = (s,p) \ | \ s \in [-4,4], \ p \in [-1,1]\} \ , \tag{12}$$

where s is a point on the boundary $\partial\Omega$ in the arclength representation ($s = 4\sin(\varphi/2)$) and p is the projection of the (unit) velocity on the normalized tangent vector (orientated counterclockwise) after the reflection, see fig. 1. The billiard ball map $T : \mathcal{P} \to \mathcal{P}$ is now obtained by starting at the point s in the direction defined by p and looking for the first intersection s' with the boundary $\partial\Omega$. Then we have $T\xi = (s', p')$, with $p' = \langle t(s'), v\rangle$, where $t(s')$ is the unit tangent vector at s' and v the velocity vector of the particle after the reflection at s', see fig. 1. The invariant measure is given by $\mathrm{d}\mu = \frac{1}{2|\partial\Omega|}\mathrm{d}s\mathrm{d}p$. Note that the intervals $\{(s,p) \ | \ s \in]-4,4[, p = \pm 1\}$ are invariant under the map T and its inverse T^{-1}.

For the cardioid billiard the singularity of the map T occurs when a trajectory hits the cusp of the cardioid, $s = \pm 4$. The set of initial conditions, which will hit the cusp at the next iteration is given by

$$\Gamma = \{(s,p) \ | \ p = s/4, \ s \in [-4,4]\} \tag{13}$$

and the set of initial conditions which will hit the cusp under the application of the inverse map T^{-1} is given by

$$\Gamma^{-} = \{(s,p) \ | \ p = -s/4, \ s \in [-4,4]\} \ . \tag{14}$$

The segment Γ partitions $\mathcal{P}$ into two triangles

$$O_1 = A \quad = \{(s,p) \ | \ s \in]-4,4[\ ,\ -1 < p < s/4\} \tag{15}$$
$$O_2 = B \quad = \{(s,p) \ | \ s \in]-4,4[\ ,\ s/4 < p < 1\}\ . \tag{16}$$

In [**?**, **?**] it was conjectured that this partition is generating, i.e. it gives a symbolic dynamics for the cardioid billiard. The map T is a diffeomorphism in the interior of A and B.
In order to apply theorem 1, we have to verify assumptions 1–5. Obviously, assumptions 1 and 2 hold for the triangles A and B and their images under T,

$$TA \quad = \{(s,p) \ | \ s \in]-4,4[\ ,\ -1 < p < -s/4\} \tag{17}$$
$$TB \quad = \{(s,p) \ | \ s \in]-4,4[\ ,\ -s/4 < p < 1\}\ . \tag{18}$$

Due to [**?**] the cardioid billiard has nonvanishing Lyapunov exponents almost everywhere, thus assumption 3 holds. Assumption 4 holds due to the monotonicity of the map, see [**?**, **?**] and assumption 5 is obviously fulfilled in $\mathcal{P}\backslash\partial\mathcal{P}$.
Therefore all the conditions for the application of theorem 1 are fulfilled, giving

Theorem 3.1 *The partition of the Poincaré section $\mathcal{P}$ for the cardioid billiard into two open regions A and B is a generating partition, i.e. for a given doubly-infinite symbol sequence ω there is at most one physical trajectory.*

Remark 3.2 *Using the above proposition one can show that orbits starting in the cusp can be labeled uniquely by a one-sided symbol sequence, which either terminates, if this orbit hits the cusp, or otherwise is infinite.*

Using the billiard map of the cardioid billiard, one can give the following example which shows that assumption 5 of theorem 2.1 is necessary to obtain a generating partition:
Take two copies of the Poincaré section $\mathcal{P}$ of the cardioid billiard. Glue these two rectangles along their two horizontal sides, making a bigger rectangle (twice as high). Now, the map is discontinuous on two parallel slanted segments, but the line of contact of the two original rectangles is not a discontinuity line, but a fixed line and therefore the map is continuous on it. It is also possible to change T in the vicinity of that fixed line so that T will be C^2 smooth on it. This map satisfies all the assumptions of our theorem but the last one, and, as a result, the theorem fails: the entire line where the original rectangles come in contact has the same symbolic sequence.

3.2 Dispersing billiards and semidispersing billiards

Let Ω be a compact domain in $\mathbf{R}^2$ or $\mathbf{T}^2$ (a two-dimensional torus) with a piecewise smooth boundary $\partial\Omega = \Delta_1 \cup \cdots \cup \Delta_l$, where Δ_i are smooth compact curves, Figure: fig2.eps

Figure 2: Notation used in the case of the dispersing billiards.

either closed or meeting one another at their endpoints. The Poincaré section $\mathcal{P}$ of the billiard system in Ω is coordinatized by (s, ψ), where s is the arclength parameter on $\partial\Omega$ and ψ is the angle between the outgoing velocity vector and the inward normal vector to $\partial\Omega$, so that $-\pi/2 \leq \psi \leq \pi/2$, see fig. 2. The section $\mathcal{P}$ consists of a finite number of cylinders (corresponding to closed curves $\Delta_i \subset \partial\Omega$) and rectangles (for nonclosed curves $\Delta_j \subset \partial\Omega$). A billiard in Ω is said to be dispersing if $\partial\Omega$ is strictly concave outward and semidispersing if $\partial\Omega$ is concave outward or flat (linear). It is known that such billiards are hyperbolic [**?**, **?**, **?**]. Moreover, the billiard ball map T takes increasing curves in $\mathcal{P}$ into increasing curves, while T^{-1} takes decreasing curves into decreasing curves. From the above cited papers the validity of assumption 5 immediately follows. Assuming additionally that the billiard has finite horizon, we obtain that the singularity sets Γ and Γ^- (for the maps T and T^{-1}, respectively) consist of a finite number of decreasing and increasing curves, respectively [**?**]. To apply our theorem, we have to cut every closed curve $\Delta_i \subset \partial\Omega$ at an arbitrary point, thus transforming the cylinder $\Delta_i \times [-\pi/2, \pi/2] \subset \mathcal{P}$ into a rectangle. Therefore, $\mathcal{P}$ consists of rectangles only. Thus we get

Theorem 3.3 *For dispersing and semi-dispersing billiards the partitions of $\mathcal{P}$ into the connected components of $\mathcal{P}\backslash\Gamma$ and $\mathcal{P}\backslash\Gamma^-$ are generating.*

3.3 Stadium billiard and other Bunimovich–billiards

L.A. Bunimovich was first to construct two-dimensional billiard tables with convex (focusing) components of the boundary $\partial\Omega$ with hyperbolic behaviour. Let again Ω be a compact billiard table in $\mathbf{R}^2$ with boundary $\partial\Omega = \Delta_1 \cup \cdots \cup \Delta_l$. Assume that each Δ_i is either concave outward (dispersing), flat (neutral), or convex (focusing). Assume that every focusing component $\Delta_i \subset \partial\Omega$ is a circular arc. Denote by N the set of trajectories where hyperbolicity cannot be enforced: (i) those bouncing off neutral sides of Ω only, and (ii) periodic trajectories with all the reflection points on one circular arc $\Delta_i \subset \partial\Omega$.

Theorem 3.4 (Bunimovich [?], cf. also [?]) *Assume that every focusing component $\Delta_i \subset \partial\Omega$ is an arc of a circle K_i such that its interior lies wholly in Ω. Assume that the measure of N is zero. Then almost every point in $\mathcal{P}$ is hyperbolic.* ∎

The most famous example is the stadium-billiard, a table Ω bounded by two semicircles and two parallel segments [**?**, **?**].

The trajectories of points in N normally make families such that at every reflection the angle of incidence is the same for the entire family, cf. [**?**]. We assume, additionally, that N consists of a finite number of such families. Stadia and convex polygons with pockets satisfy this assumption [**?**]. For such billiards, one can prove ergodicity, K-mixing and Bernoully property [**?**].
We now parametrize the Poincaré section $\mathcal{P}$ of a Bunimovich billiard by (s, ψ) as before. The section $\mathcal{P}$ consists of a finite number of rectangles (after cutting every closed curve $\Delta_i \subset \partial\Omega$ as described in section 3.2). We say that a smooth curve $\gamma \subset \mathcal{P}$ is expanding (contracting) if $\gamma \subset \Delta_i \times [-\pi/2, \pi/2]$ is a monotonous curve that is increasing (decreasing) in the case of a dispersing and neutral component Δ_i and decreasing (increasing) in the case of a focusing component Δ_i. It is known [**?**, **?**] that the billiard ball map T takes expanding curves into expanding curves and T^{-1} takes contracting curves into contracting curves. Moreover, Γ consists of contracting curves, while Γ^- consists of expanding curves. The assumption 5 is fulfilled everywhere except the set N, which consists of a finite number of horizontal segments. Thus we get

Theorem 3.5 *For Bunimovich billiards the partitions of the set $\mathcal{P} \setminus N$ into the connected components of $\mathcal{P}\backslash\Gamma$ and $\mathcal{P}\backslash\Gamma^-$ are generating.*

Note that here we use the generalization of our main theorem given by remark 2.3. In particular for the stadium $\mathcal{P}$ consists of four rectangles with different monotonicity of curves. This gives a partition of $\mathcal{P}$ into 16 atoms, where the set N corresponds to the bouncing ball orbits. In [**?**] a symbolic dynamics with 6 symbols is conjectured. One can easily show that our symbolic dynamics with 16 symbols is equivalent to this and therefore we obtain a proof of the symbolic dynamics proposed in [**?**]. For other studies of the symbolic dynamics in the stadium billiard see [**?**, **?**, **?**] and references therein.

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