Nonuniformly hyperbolic K-systems are Bernoulli

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September 14, 2006

Abstract

We prove that those nonuniformly hyperbolic maps and flows (with singularities) that enjoy the K-property are also Bernoulli. In particular, many billiard systems, including those systems of hard balls and stadia that have the K-property, and hyperbolic billiards, such as the Lorentz gas in any dimension, are Bernoulli. We obtain the Bernoulli property for both the billiard flows and the associated maps on the boundary of the phase space.

1 Introduction

The ergodic properties of dynamical systems, listed in increasing order according to the extent to which they indicate that the systems are chaotic, include ergodicity, weak mixing, multiple mixing, K, and ultimately, Bernoulli. A Bernoulli process is a sequence of independent identically distributed random variables; the prototypical example being a sequence of coin tosses. It is the classical model of a series of completely random, unpredictable tests. A map is said to be Bernoulli if it is measure-theoretically isomorphic to a Bernoulli process, and a flow, $\{\Phi^t\}$, $t \in \mathbb{R}$, is said to be Bernoulli if the map Φ^t is Bernoulli for every $t \neq 0$. Bernoulli dynamical systems enjoy all the other ergodic properties, and in some sense are "completely unpredictable". Indeed they can be modeled to any degree of accuracy by a Markov or semi-Markov process. Moreover, Bernoulli systems typically exhibit strong stochastic stability under random (and often even deterministic) perturbations. See the survey [23] for a detailed account of these issues.

Despite its importance, we remark that certain statistical properties of dynamical systems, like the rate of decay of correlations, the central limit theorem (CLT) and other limit theorems [11, 10], do not follow as a consequence of the Bernoulli property (or conversely). This is because the latter is an extreme chaotic property in the measure-theoretic sense alone, independent of any metric or coordinates in the phase space of the dynamical system. Fast decays of correlations and the CLT require certain degrees of smoothness of the dynamics and phase functions. See [10] for more discussion.

The theory of Bernoulli dynamical systems blossomed in the seventies. It started with pioneering works by Ornstein [19, 20] which proved that any two Bernoulli shifts with equal entropies are isomorphic, thus solving a long standing and celebrated problem. He also introduced [21] the notions of weak Bernoulli and very weak Bernoulli partitions, the key tools used to verify the Bernoulli property for concrete dynamical systems.

The Bernoulli property was established for ergodic toral automorphisms by Katznelson [13] (even in the nonhyperbolic case when some of the eigenvalues lie on the unit circle but none of them is a root of unity) and for ergodic automorphisms of the infinite torus by Lind [17]. Bowen [2] proved that mixing Axiom A diffeomorphisms (in particular Anosov maps) are Bernoulli. Interval maps have been shown to be Bernoulli by Smorodinsky [32] in the case of β-automorphisms, by Bowen [3] and Ratner [29] in the case of mixing piecewise monotone expanding maps, by Ledrappier [16] in the case of quadratic maps with an a.c.i.m., etc.

The first Bernoulli flows were found by Ornstein [21]. Then Ornstein and Weiss [22] proved that geodesic flows on surfaces of negative curvature are Bernoulli. Bunimovich [4] and Ratner [28] extended that result to suspension flows over subshifts of finite type with a class of ceiling functions which is larger than Hölder continuous. Pesin [25, 26] proved that geodesic flows on manifolds of various types are Bernoulli. He also established the Bernoulli property for large classes of smooth nonuniformly hyperbolic maps and flows [26].

The Bernoulli property for billiards, which is the main focus of this paper, has been studied by Gallavotti and Ornstein [12]. They considered dispersing billiard tables with smooth boundaries on the two-torus and showed that both the billiard flow and the associated map on the boundary are Bernoulli. Later, Kubo and Murata [14] extended that result to small perturbations of these billiards. Many more classes of billiard systems are now known to be chaotic and K and are also believed to be Bernoulli: dispersing billiards with corner points, semi-dispersing billiards, Bunimovich-type stadia and flowerlike tables, the multi-dimensional Lorentz gas, systems of hard balls on tori, etc. See Section 3 for definitions and references.

In the eighties, the number of articles devoted solely to the Bernoulli property of dynamical systems sharply decreased. However, nearly every paper which studied the ergodic properties of dynamical systems included a statement on their Bernoulliness as well (often with no explicit proof because the techniques of those proofs were standard).

This tradition may have contributed to the absence of proofs of Bernoulliness for billiards since the first (and only) paper [12].

However, the status of nonuniformly hyperbolic systems with singularities (to which billiards belong) is not clear at all. Katok and Strelcyn in their fundamental monograph [15] on those systems remarked that their estimates were not strong enough to obtain the Bernoulli property (see p. 155 there). Recently, Szasz in his survey [34] reiterated the necessity of an explicit proof of the Bernoulli property for hard ball gases.

This paper provides an explicit proof of the Bernoulli property for nonuniformly hyperbolic maps and flows with singularities under standard general assumptions. In particular, we show that the billiard systems listed above satisfy our assumptions so it follows that they are all Bernoulli.

2 Nonuniformly hyperbolic maps and flows

2.1. *Nonuniformly hyperbolic maps with singularities.* Let M be a smooth (at least C^4) compact d-dimensional Riemannian manifold, possibly with boundary, ∂M. Let Γ be a closed subset of M. We denote $\partial M \cup \Gamma$ by S_1. Let $T : M \setminus S_1 \to M$ be a C^2 diffeomorphism of $M \setminus S_1$ onto its image. Denote $M \setminus T(M \setminus S_1)$ by S_{-1}. Obviously, T^{-1} is a C^2 diffeomorphism of $M \setminus S_{-1}$ onto its image. We think of S_1 as the singularity set for T and S_{-1} as the singularity set for T^{-1}.

Let ρ be the Riemannian metric on M. We denote by $B_r(x)$ the ball of radius r centered at $x \in M$, and by $B_r(A)$ the r-neighborhood of A, $\cup_{x\in A} B_r(x)$. Let ν be an absolutely continuous probability measure on M, invariant under T. We assume that

$$\nu(B_\varepsilon(S_1 \cup S_{-1})) \le c_1 \varepsilon^{a_1} \tag{1}$$

for some constants $c_1, a_1 > 0$. In particular, $\nu(S_1 \cup S_{-1}) = 0$. We also assume that

$$||D^2 T_x|| \le c_2 \rho(x, S_1)^{-a_2} \quad \text{and} \quad ||D^2 T_x^{-1}|| \le c_2 \rho(x, S_{-1})^{-a_2} \tag{2}$$

for some $c_2, a_2 > 0$ and every $x \in M$. Next, we assume that

$$\int_M \ln^+ ||DT_x||\, d\nu < \infty \quad \text{and} \quad \int_M \ln^+ ||DT_x^{-1}||\, d\nu < \infty \tag{3}$$

where $\ln^+(x) = \max\{\ln x, 0\}$.

The maps (1)-(3) are usually called smooth maps with singularities [15, 27]. In particular, under (3) the Oseledec theorem [24] works and ensures the existence of Lyapunov exponents a.e. in M. We assume that all the Lyapunov exponents are different from zero a.e. (in this case the map T is said to be *completely hyperbolic*, as opposed to *partially hyperbolic* maps which have some exponents equal to zero). Under this assumption there are two measurable families of subspaces $E^u_x, E^s_x \subset T_x(M)$ defined at a.e. point $x \in M$. These are invariant under DT:

$$DT^t_x(E^{u,s}_x) = E^{u,s}_{T^t x} \tag{4}$$

with strictly positive Lyapunov exponents in unstable subspaces E^u_x:

$$\lim_{t\to\pm\infty}\frac{1}{t}\ln||DT^t_x(v)|| = \chi^\pm_x(v) > 0 \text{ for all } v\in E^u_x,\ ||v|| = 1 \tag{5}$$

and strictly negative Lyapunov exponents in stable subspaces E^s_x:

$$\lim_{t\to\pm\infty}\frac{1}{t}\ln||DT^t_x(v)|| = \chi^\pm_x(v) < 0 \text{ for all } v\in E^s_x,\ ||v|| = 1 \tag{6}$$

Here $t \in \mathbb{Z}$ is a discrete time parameter. Furthermore, along a.e. trajectory the angle between E^u_x and E^s_x, which we denote by $\sphericalangle\,(E^u_x, E^s_x)$, cannot decrease at any exponential rate:

$$\lim_{t\to\pm\infty}\frac{1}{t}\ln\sphericalangle\,(E^u_{T^tx}, E^s_{T^tx}) = 0 \qquad \text{a.e.} \tag{7}$$

The subspaces $E^{u,s}_x$ are transversal and generate the tangent space to M at x: $E^u_x\oplus E^s_x = T_x(M)$.

Finally, we assume that the map T is ergodic and enjoys the K-property. This is a necessary assumption since systems which are not ergodic or K cannot be Bernoulli. The ergodicity of T implies that the spectrum of Lyapunov exponents is constant a.e. in M and the dimensions $d^u = \dim E^u_x$ and $d^s = \dim E^s_x$ are constant a.e. as well.

Through a.e. point $x\in M$, there is a local unstable (expanding) manifold, W^u_x, and there is a local stable (contracting) manifold, W^s_x. See [15] for a construction of such manifolds. These manifolds are at least C^1 smooth and at every point $y\in W^{u,s}_x$ the tangent space to $W^{u,s}_x$ coincides with $E^{u,s}_y$. All the iterates T^t are smooth on W^u_x for $t\le 0$ and on W^s_x for $t\ge 0$. The main characteristic property of these manifolds is

$$\rho(T^ty, T^tz) \le C(x)e^{-\chi|t|}\rho(y,z) \tag{8}$$

for all $y,z\in W^u_x$ and $t\le 0$, and for all $y,z\in W^s_x$ and $t\ge 0$. Here $\chi>0$ is a constant and $C(x)$ is a measurable positive function on M.

Another important property of local stable and unstable manifolds is their absolute continuity. See [15] for the exact definition and proof of this property. Combined with the smoothness of the invariant measure ν, the absolute continuity can be characterized by two other properties. The first is that the conditional measure induced by ν on a.e. local stable and unstable manifold is absolutely continuous with respect to the Riemannian measure on that manifold. The second is a property of the canonical isomorphisms also known as holonomy maps between sufficiently close unstable (also stable) manifolds, W^u_x and W^u_y. Such a map takes the point $z\in W^u_x$ to the point $W^s_z\cap W^u_y$ (whenever the latter point exists). They have the property of being absolutely continuous for a.e. W^u_x and W^u_y. That is, the jacobian of the canonical isomorphism with respect to the Riemannian measures on W^u_x and W^u_y is finite and positive at a.e. point z where that isomorphism is defined.

Our main result for nonuniformly hyperbolic K-automorphisms is the following theorem.

Theorem 2.1 *Let (M, T, ν) be a smooth system with singularities satisfying (1)-(3). If the map T is completely hyperbolic and K, then it is Bernoulli.*

Remark. If we relax our assumption of the ergodicity and K property of T, then the general Katok-Strelcyn theory [15] ensures that the map T has ergodic components of positive measure, whose union has measure one. Furthermore, every such ergodic component is a finite union of subcomponents of equal measures which are cyclically permuted by T, and on which the corresponding iterate of T is mixing and K. It is fairly straightforward to apply our arguments from Sections 4-6 to the maps on the subcomponents to show that they are in fact Bernoulli.

2.2. *Nonuniformly hyperbolic flows with singularities.* We will also study hyperbolic systems with singularities that have a continuous time parameter $t \in \mathbb{R}$ (i.e., flows). There is no conventional definition for such flows, which is understandable: the intersection over all $t > 0$ of the singularity sets of the maps Φ^t which comprise the flow is empty, so there is no obvious canonical way to define the singularity set of the flow. The only available general construction for such systems involves suspension flows, also called special flows or Kakutani flows. Even though that construction looks "special", by the Ambrose-Kakutani theorem [1] every flow on a Lebesgue measure space whose fixed points form a set of zero measure is isomorphic to a suspension flow. Moreover, for billiard systems and many other models the construction is quite natural.

Let (M, T, ν) be a nonuniformly hyperbolic map with singularities defined above by (1)-(3) with a smooth invariant measure ν. Let $\varphi(x)$ be a positive integrable C^2 function on $M \setminus S_1$. A suspension flow with a base map T and a ceiling function φ is defined on the manifold $\mathcal{M} = \{(x, s) : x \in M, 0 \le s < \varphi(x)\}$ by the rule

$$\Phi^t(x, s) = \begin{cases} (x, s+t) & \text{for } \ 0 \le t < \varphi(x) - s \\ (Tx, s+t-\varphi(x)) & \text{for } \ \varphi(x) - s \le t < \varphi(Tx) + \varphi(x) - s \end{cases}$$

This flow preserves the smooth probability measure μ on $\mathcal{M}$ defined by $d\mu = c \cdot d\nu \times ds$, where $c^{-1} = \int_M \varphi(x)\, d\nu(x)$.

We again assume complete hyperbolicity of the map T or, equivalently, complete hyperbolicity of the flow Φ^t, which in the latter case means that all the Lyapunov exponents but one are different from zero almost everywhere. (The Lyapunov exponent of the tangent vector to the flow is necessarily zero.) For a point $y = (x, s) \in \mathcal{M}$ we denote the stable and unstable subspaces in $T_y\mathcal{M}$ again by E^s_y and E^u_y, respectively. In the context and from the location of the point y it should be clear whether $E^{s,u}_y$ are subspaces of the tangent space to $\mathcal{M}$ or the tangent space to M. The formulas (4)-(7) hold for the flow Φ^t, one only needs to substitute Φ for T and think of t as a continuous parameter: $t \in \mathbb{R}$.

We assume, as before, that the system $(\mathcal{M}, \Phi^t, \mu)$ is ergodic and enjoys the K-property. As a result, the spectrum of Lyapunov exponents for Φ^t is constant a.e., and the dimensions $d^u = \dim E^u_y$ and $d^s = \dim E^s_y$ are constant a.e. too. Note that $d^u + d^s = \dim \mathcal{M} - 1$.

At almost every point $y \in \mathcal{M}$ there are local unstable and stable manifolds $W^{u,s}_y \in \mathcal{M}$. They are at least C^1 smooth and enjoy the same properties as local manifolds for the map

T, one again only needs to substitute Φ for T and think of t as a continuous parameter (the constant χ and the function $C(x)$ in (8) need not be the same for the map T and the flow Φ^t, of course).

Remark. Let π be the natural projection of $\mathcal{M}$ onto M defined by $\pi(x,s) = x$. It is clear that the projection $\pi(W_y^{u,s})$ of any local manifold in $\mathcal{M}$ is $W_{\pi(y)}^{u,s}$, the local manifold in M (or a subset of it). Conversely, local manifolds in $\mathcal{M}$ can be constructed by lifting local manifolds of the map T from M to $\mathcal{M}$, see [4] for a detailed construction. A more direct way to obtain local manifolds in $\mathcal{M}$ is to apply the Katok-Strelcyn general result [15] to the one-time map Φ^1 (which is a C^2 smooth map with singularities). It is then fairly easy to show that local manifolds for the map Φ^1 will be local manifolds for the flow Φ^t as well. We do not dwell on this anymore.

Denote by $\mathcal{E}_y^{u,s}$ the subspaces in $T_y(\mathcal{M})$, $y \in \mathcal{M}$, spanned by the flow direction and the subspaces $E_y^{u,s}$, respectively. It is now clear that $E_y^u \oplus \mathcal{E}_y^s = T_y(\mathcal{M})$ and $E_y^s \oplus \mathcal{E}_y^u = T_y(\mathcal{M})$. For any $y \in \mathcal{M}$ we denote by $\mathcal{W}_y^{u,s}$ the smooth connected component of the set $\cup_t W_{\Phi_y^t}^{u,s}$ that contains the point y. We call $\mathcal{W}_y^{u,s}$ the weakly unstable (respectively, weakly stable) local manifold through y. Clearly, the tangent space to $\mathcal{W}_y^{u,s}$ at any point $z \in \mathcal{W}_y^{u,s}$ coincides with $\mathcal{E}_z^{u,s}$.

The local manifolds of the flow Φ^t are absolutely continuous, just like those of the map T. We only emphasize here two necessary consequences of the absolute continuity of those manifolds and the smoothness of the measures ν and μ. First, the conditional measure induced by μ on a.e. local manifold $W_x^{u,s}$ and on a.e. local 'weak' manifold $\mathcal{W}_x^{u,s}$ is absolutely continuous with respect to the Riemannian measure. Second, the canonical isomorphisms between any two close unstable manifolds and any two close weakly unstable manifolds, which are defined as for the map T by translating points along weakly stable manifolds and stable manifolds respectively, are absolutely continuous. Both these consequences can be readily obtained directly from the absolute continuity of local manifolds of the base automorphism T and by the smoothness of the natural lifting of those to local manifolds of the flow Φ^t. (Indeed, that lifting is at least C^1, so it preserves the absolute continuity of conditional measures and canonical isomorphisms.)

Our main result for hyperbolic K-flows is the following theorem.

Theorem 2.2 *Let $(\mathcal{M}, \Phi^t, \mu)$ be a suspension flow over a smooth map T with singularities satisfying (1)-(3) and whose ceiling function is C^2. If the flow Φ^t is completely hyperbolic and K, then it is Bernoulli.*

Remark. It is widely believed that the general Pesin-Katok-Strelcyn theory [15, 25] also follows through for flows. In other words if the assumption of ergodicity (and K) is relaxed then the flow has ergodic components of positive measure whose union has measure one. Furthermore, in every such ergodic component the flow is K up to a possible rotation factor. In that case it is fairly straightforward to generalise our arguments and show that in each ergodic component the flow is in fact Bernoulli up to a possible rotation factor.

3 Bernoulli Billiards

This section is devoted to billiard dynamical systems which constitute the main application of our general theorems 2.1 and 2.2. We describe the classes of billiards for which we are able to establish the Bernoulli property.

Let Q be a bounded closed connected domain in $\mathbb{R}^d$, $d \geq 2$, or on the d-torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$. Let the boundary ∂Q consist of a finite number of smooth components $\partial Q = \Gamma_1 \cup \cdots \cup \Gamma_r$, $r \geq 1$, such that every Γ_i is defined by an equation $\varphi_i(x) = 0$, where $\varphi_i(x)$ is a C^5 smooth function with no singular points in the closure $\bar{\Gamma}_i$. Let the set

$$\cup_{i \neq j}(\bar{\Gamma}_i \cap \bar{\Gamma}_j) = \Gamma^*$$

be a finite union of smooth compact submanifolds of dimension $\leq d-2$. At any (regular) point $q \in \partial Q \setminus \Gamma^*$ we denote by $n(q)$ the inward unit normal vector to ∂Q.

The billiard dynamical system in Q is generated by the free motion of a pointlike particle at unit speed in Q with elastic reflections at ∂Q. We call Q the billiard table and ∂Q the wall. If the particle hits the singular set Γ^* (a corner point of the wall), it stops, and its trajectory is no longer defined. The same happens if the collisions of the particle with the wall accumulate at a finite point in time.

The phase space of the billiard system in Q is the $(2d-1)$-dimensional manifold, $Q \times S^{d-1}$, where S^{d-1} is the unit $(d-1)$-dimensional sphere, with a natural identification on the boundary $\partial Q \times S^{d-1}$. We denote the phase space $\mathcal{M}$. The billiard flow, Φ^t, on $\mathcal{M}$ is a smooth flow with singularities. To be precise, the map Φ^t is singular at a point $x \in \mathcal{M}$ if and only if the segment of the trajectory of length t, which starts at x, hits Γ^*, or is tangent to a smooth component of ∂Q, or its collisions with the wall accumulate somewhere in the interval $(0, t]$. The flow Φ^t preserves the Liouville measure $d\mu = c_\mu \, dq \, dv$, where dq and dv are Lebesgue measures on Q and S^{d-1}, respectively, and c_μ is a normalizing factor. For the classes of billiards that we consider below, the measure of singular trajectories of the flow is zero (see [15, 8]) even though they are always dense in $\mathcal{M}$.

Every billiard flow has a cross section naturally constructed at the wall of the table Q, namely

$$M = \{(q, v) \in \mathcal{M} : \, q \in \partial Q \text{ and } (v, n(q)) > 0\}$$

If $Q \subset \mathbb{R}^d$ all nonsingular trajectories cross M within a finite time. If $Q \subset \mathbb{T}^d$ it is not hard to see that the set of nonsingular trajectories that never cross M has measure zero and may therefore be disregarded, and those that cross M once must cross it again. The first return map $T : \, M \to M$, takes a point $x \in M$ to the point on the trajectory of x immediately after its first reflection in ∂Q. The map T preserves the measure $d\nu = c_\nu (v, n(q)) \, dq \, dv$, where dq is now Lebesgue measure on ∂Q and c_ν is a normalizing factor.

It is clear that the flow Φ^t is a suspension flow with the base automorphism (M, T, ν) and ceiling function $\tau(x)$ whose value at any $x \in M$ is equal to the time at which the trajectory of x first collides with ∂Q. In particular, $d\mu = c_\tau \, ds \, d\nu$, where ds is linear

Lebesgue measure along the trajectories of the flow and $c_\tau = (\int_M \tau(x)\, d\nu(x))^{-1}$ is a normalizing factor, see [8]. Note that if $Q \subset \mathbb{T}^d$ the function $\tau(x)$ may be unbounded, but it is always integrable [8].

The map $T : M \to M$ is C^2 except on the closed set $\Gamma \subset M$ consisting of those points whose trajectories hit the wall ∂Q either in Γ^* or at a zero angle (i.e. tangentially to the wall). The ceiling function $\tau(x)$ is C^2 smooth on $M \setminus \Gamma$. Thus, a direct consequence of Theorems 2.1 and 2.2 is the following theorem.

Theorem 3.1 *Let Q be a billiard table whose associated billiard map, T, satisfies the assumptions (1)-(3). Suppose in addition that both the billiard map, T, and the billiard flow, Φ^t, are completely hyperbolic (i.e. their Lyapunov exponents are nonzero) and are K-systems. Then they are both Bernoulli.*

The assumptions (1)-(3) were proven for very large classes of planar billiards (i.e. when $d = 2$) in [15]. In higher dimensions, $d \geq 3$, the only general class of billiards for which these assumptions have been carefully checked is that of semidispersing billiards (see [8]). Billiards are called semidispersing if the hypersurfaces Γ_i, $1 \leq i \leq r$, are concave outward. In this case the curvature operator of the wall ∂Q at every regular point $q \in \partial Q \setminus \Gamma^*$ is nonnegative with respect to the normal $n(q)$. If the boundary ∂Q is strictly concave outward (so that the above curvature operator is strictly positive), then the billiard system is said to be dispersing.

Given a billiard system that satisfies the assumptions (1)-(3) it is usually hard to determine whether it is completely hyperbolic or not. Once the complete hyperbolicity is established, the K-property can be obtained using the so called Sinai fundamental theorem [30, 31, 33] or some variations of it, see references in [34]. We now describe those classes of billiards for which these two properties have been established already. It follows from Theorem 3.1 that they are all Bernoulli.

1. *Semidispersing planar billiards.* Let Q be a billiard table in $\mathbb{R}^2$ or $\mathbb{T}^2$ with semidispersing walls. The wall ∂Q may contain rectilinear segments, whose union is called the neutral part of ∂Q. Assume that the trajectories that reflect solely in the neutral part of ∂Q form a set of zero μ-measure[1]. Complete hyperbolicity and the K-property were established for such billiards by Sinai [30] (see also [31]). It therefore follows that they are also Bernoulli. Our result here is an extension of the direct proof of the Bernoulli property provided in [12] for the subclass consisting of planar dispersing billiards with no corner points (i.e. $\Gamma^* = \emptyset$). (Note that at corner points the smooth components of ∂Q are allowed to make zero angles, so that semidispersing billiards with cusps are still K and Bernoulli).

2. *Planar billiards with focusing components of special types.* Assume that the wall ∂Q of a planar billiard table Q consists of a finite number of smooth curves of three

[1] If the neutral part of ∂Q is a rectangle, this assumption is easy to verify. For general polygons it is equivalent to the still open problem of whether or not a.e. trajectory in a polygonal billiard table is everywhere dense.

types: strictly concave outward (*dispersing*), flat (*neutral*) and strictly convex outward (*focusing*). Generally, billiards with focusing components are not hyperbolic or ergodic, with certain remarkable exceptions. Bunimovich proved [5] that if every focusing component is a circular arc such that the corresponding circle wholly lies in Q, then the billiard is hyperbolic and K. The most celebrated example of that kind is a stadium [5]. Much more general classes of hyperbolic billiards with focusing components of the wall were discovered by Wojtkowski [35] and Markarian [18]. In some cases, those billiards have been shown to be K also [33]. Theorem 3.1 says that all such billiards that are K are Bernoulli as well.

3. *Systems of hard balls (disks).* It is well known that a system of a finite number of identical hard balls (or disks) on a torus or in a reservoir, that collide elastically both with each other and with the walls of the container (if there are any), generates a billiard flow in the corresponding configuration space, see e.g., [30, 31]. The walls of the billiard configuration space consist of surfaces which are generated by the walls of the original container and cylindrical surfaces corresponding to the collision surfaces between the balls. If the walls of the container are flat or concave outward, then the generated billiard in the configuration space is semidispersing.

The classical Boltzmann hypothesis says that systems of hard balls on tori or in rectangular boxes are ergodic[2]. It was proposed by L. Boltzmann in an attempt to develop a mathematical foundation for statistical physics. This hypothesis is now known as the Boltzmann-Sinai ergodic conjecture, see [31, 34] and references therein. It has been proven only in a few particular cases: two, three or four balls on a torus, any number of disks in certain special containers, and any number of balls on sufficiently high-dimensional tori, see references in [34]. In these cases, not only ergodicity, but also complete hyperbolicity and the K-property have been established. In other cases virtually none of these properties is known. For instance nothing is known about disks in a rectangular box. It is, however, a common belief that the Boltzmann-Sinai conjecture is true for any system of hard balls on any torus or in any container with flat walls. Our result says that any system of hard balls or disks on a torus or in any container is Bernoulli provided it is completely hyperbolic and K.

4. *The periodic Lorentz gas (in any dimension).* Another billiard system which is very popular among physicists is the Lorentz gas. The Lorentz gas consists of a collection of fixed, immovable obstacles in space and pointlike particles which move freely in between them and bounce off them elastically (moving particles do not interact with one another). This is a classical model of electron gases in metals. Since the moving particles are independent, it is customary to study a single particle instead of many (or infinitely many). Furthermore, if the obstacles are periodically situated in space the Lorentz gas is said to be periodic. In that case the system can be projected onto a fundamental cube in space and a billiard system on a torus is obtained.

[2]on any surface in the phase space where all the obvious integrals of motion are fixed.

We consider a periodic Lorentz gas and assume that all the obstacles are strictly convex solids with smooth boundary, see, e.g., [9] for detail. We then have a dispersing billiard system on a torus. The complete hyperbolicity and K-property were proven in [30] in two dimensions and in [31] in any dimension. The Bernoulli property was established in [12] only in two dimensions. The Bernoulli property in all dimensions follows from our Theorem 3.1.

This is a practically complete list of billiard dynamical systems which are known so far to be hyperbolic and K.

Our last remark concerns two subclasses of the above four classes: dispersing billiards in two dimensions and the periodic Lorentz gas in any dimension but with finite horizon (this means that the distance the particle can cover without collisions with obstacles is uniformly bounded above). These are the only two subclasses of billiards for which the map T is not only completely hyperbolic, but also *uniformly* hyperbolic. This means, in the notations of Section 2, that the function $C(x)$ in (8) is bounded above and the angle $\sphericalangle\,(E^u_x, E^s_x)$ is bounded away from zero on M.

In the case of uniformly hyperbolic billiard ball maps T specified above, there are finer and more powerful systems of rectangles than our ε-regular coverings constructed below in Section 5. Those are called Markov sieves. They have been constructed in [7] for two dimensional dispersing billiards and in [9] for multidimensional Lorentz gases with finite horizon. Markov sieves then have been used in [7, 9] to estimate the rate of mixing of the map T, and, as a result, the rate of decay of correlations (the estimates in those works are stretched exponentials in t, i.e. $\text{const}\cdot e^{-\text{const}\cdot t^\gamma}$ for a certain $\gamma \in (0,1)$). It is also possible to use Markov sieves, instead of ε-regular coverings, to prove that any smooth partition of the space M is not only very weak Bernoulli, but also weak Bernoulli, and, moreover, that the rate of weak Bernoulliness is bounded below by a stretched exponential. An explicit proof of this claim might be the subject for a separate paper, and we state it here as just a conjecture.

4 The $\bar{d}$-metric and very weak Bernoulli partitions

The rest of the paper is devoted to the proofs of Theorems 2.1 and 2.2. Our proofs closely follow that first used by Ornstein and Weiss in [22] to prove that certain toral automorphisms and geodesic flows on surfaces of negative curvature are Bernoullian. Throughout, $\varepsilon > 0$ is a small parameter, and c denotes various positive constants that depend on the system alone (and not on ε). Thus, for example, the formula $(1+\varepsilon)c\varepsilon < c\varepsilon$ is correct since the value of c on the right can be taken to be twice as big as the value on the left, and ε is small.

Notice first of all that a flow Φ^t is said to be Bernoulli if, for every $t \neq 0$, the *map* Φ^t is Bernoulli. If the map Φ^t is Bernoulli for some value of t then it is Bernoulli for all values of t. Thus, to conclude that the flow is Bernoulli, it suffices to prove that the map Φ^1 is Bernoulli. For this reason our proofs only involve maps, in particular Φ^1 and T.

Ornstein characterized Bernoulli systems using the notion of very weak Bernoulli partitions [21]. In this section we explain what it means for a partition to be very weak Bernoullian and formulate and prove a general lemma that we will need to verify the Bernoulli property of the maps and flows described in Section 2.

Consider the non-atomic Lebesgue probability spaces (X,μ) and (Y,ν). A *joining*, λ, of X and Y is a measure on $X \oplus Y$ whose marginals are μ and ν respectively. In other words, for any measurable sets $A \subset X$, $B \subset Y$, $\lambda(A \oplus Y) = \mu(A)$ and $\lambda(X \oplus B) = \nu(B)$.

Consider partitions $\alpha = \{A_1, A_2, \dots A_k\}$ of X and $\beta = \{B_1, B_2, \dots B_k\}$ of Y. For a point $x \in X$, denote by $\alpha(x)$ the atom in α in which x lies (similarly for Y). If $x \in A_j$ and $y \in B_j$ for the same value of j, then we will abuse this notation and say $\alpha(x) = \beta(y)$. Notice that this property depends on the order in which the atoms appear in α and in β.

We think of the two partitions as being close if there is a joining of X and Y in which most of the measure lies on pairs of points (x,y) with $\alpha(x) = \beta(y)$. To be precise, the $\bar{d}$-distance between α and β is defined to be:

$$\bar{d}(\alpha,\beta) = \inf_{\lambda} \lambda(\{(x,y)\colon \alpha(x) \neq \beta(y)\})$$

where λ is a joining of X and Y.

In the language of probability theory, consider the random variable $W_X\colon X \to \{1,2,\dots k\}$ that maps each atom A_j to the corresponding integer j, and the random variable W_Y which is defined similarly on Y. The measures μ and ν determine the distributions of W_X and W_Y respectively. The $\bar{d}$-distance between α and β is equal to the distance in variation between W_X and W_Y, i.e.

$$\bar{d}(\alpha,\beta) = \frac{1}{2}\sum_{i=1}^{k} |\mu(A_i) - \nu(B_i)|$$

Other useful formulas for the $\bar{d}$-distance between partitions may be found in [22].

Now consider two sequences of partitions $\{\alpha_i\}_{i=1}^n$, and $\{\beta_i\}_{i=1}^n$ of X and Y respectively. By the $\bar{d}$-distance between the two sequences of partitions we mean,

$$\bar{d}\left(\{\alpha_i\},\{\beta_i\}\right) = \inf_{\lambda} \int_{X \times Y} h(x,y)\ d\lambda$$

where λ is a joining of the two measure spaces X and Y and h measures how far apart the 'name' of x is from that of y, or, more specifically

$$h(x,y) = (1/n) \sum_{i:\alpha_i(x) \neq \beta_i(y)} 1 \tag{9}$$

Notice that if $n = 1$ this definition is identical to the previous one.

It might seem more natural to define the $\bar{d}$-distance between the sequences of partitions to be $\bar{d}(\vee_{i=1}^n \alpha_i, \vee_{i=1}^n \beta_i)$. However, this is typically too large. It is easy to see this by observing that the latter expression is obtained when $h(x,y)$ in the definition above is replaced with the larger function $g(x, y)$ which is equal to 1 whenever there is any integer i for which $\alpha_i(x) \neq \beta_i(y)$ and is equal to 0 otherwise.

In the language of probability theory, consider two discrete time stochastic processes $\{W_X^i\}_{i=1}^n$ and $\{W_Y^i\}_{i=1}^n$ defined on X and Y respectively, with corresponding sequences of partitions $\{\alpha_i\}_{i=1}^n$ and $\{\beta_i\}_{i=1}^n$. The measures μ and ν determine the distributions of their respective sample paths. A joining of X and Y determines the joint distribution of sample paths. The two processes are close in the $\bar{d}$-metric if there is a joining under which most pairs of sample paths are close most of the time.

If a property holds for all atoms of α except for a collection of atoms whose union has measure less than ε, then we will say that the property holds for ε-a.e. atom of α. If $E \subset X$, then α/E (alpha conditioned on E) denotes the partition of E into sets of the form $A \cap E$, $A \in \alpha$, and $\mu(\cdot/E)$ denotes the measure μ conditioned on E.

Let $f\colon X \to X$ be an invertible measure preserving transformation. (We can think of X as either M or $\mathcal{M}$ and of f as T or Φ^1, respectively). The partition α of X is said to be *very weak Bernoullian* (vwB) if for every $\varepsilon > 0$ there is an integer $N > 0$ such that for every $n > 0$ and N_0, N_1 satisfying $N < N_0 < N_1$, and for ε-a.e. atom, A, of $\bigvee_{N_0}^{N_1} f^i\alpha$,

$$\bar{d}\left(\{f^{-i}\alpha\}_{i=1}^n, \{f^{-i}\alpha/A\}_{i=1}^n\right) < \varepsilon \tag{10}$$

Notice that, since f is measure preserving, to establish that α is vwB it is enough to show that for every $\varepsilon > 0$ there are integers m and $N > 0$ such that for every n, N_0, and N_1 as above, and for ε-a.e. atom, A, of $\bigvee_{N_0-m}^{N_1-m} f^i\alpha$,

$$\bar{d}\left(\{f^{-i}\alpha\}_{i=1+m}^{n+m}, \{f^{-i}\alpha/A\}_{i=1+m}^{n+m}\right) < \varepsilon \tag{11}$$

The following two theorems provide a characterisation of the Bernoulli property in terms of very weak Bernoulli partitions, see [22].

Theorem 4.1 *If a partition α of X is vwB then $(X, \bigvee_{-\infty}^{\infty} f^{-n}\alpha, \mu, f)$ is a Bernoulli shift.*

Theorem 4.2 *If $\alpha_1 \leq \alpha_2 \leq \dots$ is an increasing sequence of partitions of X such that $\bigvee_{i=1}^{\infty} \bigvee_{n=-\infty}^{\infty} f^{-n}\alpha_i$ generates the whole σ-algebra, and for each i, $(X, \bigvee_{n=-\infty}^{\infty} f^{-n}\alpha_i, \mu, f)$ is a Bernoulli shift, then (X, μ, f) is a Bernoulli shift.*

In our cases X is a manifold, so it is clear that there is an increasing sequence of partitions which has the properties that each partition in the sequence consists of a finite number of atoms with piecewise smooth boundaries, and the infinite join of all the partitions generates the complete σ-algebra in X. Thus by Theorems 4.1 and 4.2 to conclude that f is Bernoulli it suffices to prove that such partitions are vwB. Technically, this will involve showing that two sequences of partitions are close in the $\bar{d}$-metric. The following general lemma indicates the technique we will use to do this (cf lemma 1.3 in [22]).

Lemma 4.3 *Let (X, μ) and (Y, ν) be two nonatomic Lebesgue probability spaces. Let $\{\alpha_i\}$ and $\{\beta_i\}$, $1 \leq i \leq n$, be two sequences of partitions of X and Y, respectively. Suppose there is a map $\psi \colon X \to Y$ such that*

1. *There is a set, $E_1 \subset X$, whose measure is less than ε, outside of which*
$$h(x, \psi x) < \varepsilon$$

2. *There is a set, $E_2 \subset X$, whose measure is less than ε, such that for any measurable set $A \subset X \backslash E_2$*
$$\left| \frac{\mu(A)}{\nu(\psi A)} - 1 \right| < \varepsilon$$

Then $\bar{d}\left(\{\alpha_i\}, \{\beta_i\}\right) < c\varepsilon$.

Proof: We construct a joining of X and Y, λ, which has the property
$$\lambda\left(\{(x, y) \colon h(x, y) < \varepsilon\}\right) > 1 - c\varepsilon$$

The existence of such a joining guarantees the lemma. Let A_1, A_2, ... denote atoms of $\bigvee_{i=1}^{n} \alpha_i$.

Claim: The map ψ can be modified to produce a map $\bar{\psi}$ which has the following properties:

1. The image of every atom A_j is a set of the same measure, and these images are all mutually disjoint.

2. The set $E_3 := \{x \colon \bar{\psi} x \neq \psi x\}$ has measure less than $c\varepsilon$.

Proof of Claim: Given any measurable set $S \subset X$ let $\bar{S}$ denote $S \backslash E_2$. It follows from the second hypothesis of the lemma that $|\mu(\bar{S}) - \nu(\psi \bar{S})| < c\varepsilon\mu(\bar{S})$. Thus,

$$\begin{aligned} \sum_j \nu(\psi \bar{A}_j) &< 1 + c\varepsilon \\ \text{and} \quad \nu\left(\bigcup_j \psi \bar{A}_j\right) &> 1 - c\varepsilon \end{aligned} \tag{12}$$

Let $B := \psi^{-1}(\cup_{j \neq k}(\psi \bar{A}_j \cap \psi \bar{A}_k))$. It follows from (12) that $\nu(\cup_{j \neq k}(\psi \bar{A}_j \cap \psi \bar{A}_k)) < c\varepsilon$, and thus, by the second hypothesis of the lemma, $\mu(\bar{B}) < (1 + \varepsilon)c\varepsilon < c\varepsilon$. Now consider the sets $\tilde{A}_j := A_j \backslash (B \cup E_2)$. Notice that $\psi \tilde{A}_j \cap \psi \tilde{A}_k = \emptyset$, for all $A_j \neq A_k$, and that $\mu(\cup_j \tilde{A}_j) > 1 - c\varepsilon$. To construct the map $\bar{\psi}$ we modify ψ on a small proportion of each of the sets $\tilde{A}_j$ to ensure that, under $\bar{\psi}$, that set is mapped to a set of the same measure disjoint from all the others. The remainder of X, $B \cup E_2$, is then mapped in any measure-preserving way onto the remainder of Y.

First consider those sets where $\nu(\psi\tilde{A}_j) > \mu(\tilde{A}_j)$. Consider a set $G_j \subset \tilde{A}_j$ which has the property that $\nu(\psi G_j) = \mu(\tilde{A}_j)$. Notice that $\mu(G_j) > (1-\varepsilon)\nu(\psi G_j) = (1-\varepsilon)\mu(\tilde{A}_j)$. Define $\bar{\psi}$ to be equal to ψ in G_j, and to map $\tilde{A}_j \backslash G_j$ to any set of measure zero.

Now consider those sets where $\nu(\psi\tilde{A}_j) < \mu(\tilde{A}_j)$. Consider a set $G_j \subset \tilde{A}_j$ whose measure is greater than $(1-\varepsilon)\mu(\tilde{A}_j)$. Define $\bar{\psi}$ to be equal to ψ in G_j, and to map $\tilde{A}_j \backslash G_j$ to any set of measure $\mu(\tilde{A}_j) - \nu(\psi G_j)$ which does not intersect with the images of any of the other $\tilde{A}_j$'s.

Clearly $\bar{\psi}$ satisfies the first property of the claim. Furthermore, $E_3 \subset X \backslash \cup_j G_j$ so $\mu(E_3) < 1 - (1-\varepsilon)\mu(\cup_j \tilde{A}_j) < c\varepsilon$.

This completes the proof of the claim. Q.E.D.

Let λ be that measure on $X \times Y$ which is supported on the sets $A_j \times \bar{\psi}A_j$ and on each such set it is the product of μ and ν normalized to have total measure equal to $\mu(A_j) = \nu(\bar{\psi}A_j)$. Projecting λ onto X it is clear that the marginal obtained is μ. The fact that the other marginal is ν follows from property 1 of the claim. Thus λ is a joining of X and Y. Furthermore,

$$
\begin{aligned}
\lambda\left(\{(x,y): h(x,y) < \varepsilon\}\right) &= \sum_j \lambda\left(A_j \times \left\{y \in \bar{\psi}A_j: h(x,y) < \varepsilon, x \in A_j\right\}\right) \\
&= \nu\left(\left\{y = \bar{\psi}x: h(x,y) < \varepsilon\right\}\right) \\
&\geq \nu\left(\bar{\psi}(X \backslash (E_1 \cup E_2 \cup E_3))\right) \\
&> (1-\varepsilon)(1-c\varepsilon) \\
&> 1 - c\varepsilon
\end{aligned}
$$

This completes the proof of Lemma 4.3. Q.E.D.

5 ε-regular coverings for maps and flows

In this section we prepare certain tools that we will need to prove Theorems 2.1 and 2.2. In particular we prove that all of the phase space M (or $\mathcal{M}$), except for a subset of arbitrarily small measure, can be covered by rectangles built up of stable and unstable manifolds such that the invariant measure within each rectangle is arbitrarily close to a product measure which we will define. We call such coverings ε-regular coverings, where $\varepsilon > 0$ is a small parameter.

5.1. *ε-regular coverings for maps.* Consider a map (M, T, ν) satisfying the hypotheses of Theorem 2.1. For the construction performed here we do not need the K-property of T.

Definition. *A rectangle in M is a measurable set, $R \subset M$, equipped with a distinguished point $z \in R$. The set R has the property that for all points $x, y \in R$ the local manifolds W^u_x and W^s_y intersect each other at a single point which lies in R.*

Notice that a rectangle, R, can be thought of as the cartesian product of $W^u_z \cap R$ and $W^s_z \cap R$ (where a point $y \in R$ is given by $(W^s_y \cap W^u_z, W^u_y \cap W^s_z)$). It can be endowed with the product measure $\nu_R^p = \nu_z^u \times \nu_f^s$, where ν_z^u is the measure ν conditioned on $W_z^u \cap R$ and ν_f^s is a factor measure on the set of leaves $W_x^u \cap R$, $x \in R$. (This set of leaves can be identified with W_z^s, so we can think of ν_f^s as a measure on W_z^s.) It follows from the absolute continuity of the stable and unstable foliations that $\nu_R^p << \nu$.

Definition. *Given any $\varepsilon > 0$, an ε-regular covering of the phase space M is a finite collection of disjoint rectangles $\mathcal{R} = \mathcal{R}_\varepsilon$ for which*
(a) $\nu(\cup_{R\in\mathcal{R}} R) > 1 - \varepsilon$
(b) Given any two points $x, y \in R \in \mathcal{R}$, which lie in the same unstable or stable manifold, there is a smooth curve on that manifold that connects x and y and has length less than $8 \cdot \operatorname{diam} R$.
(c) For every $R \in \mathcal{R}$ we have

$$\left| \frac{\nu_R^p(R)}{\nu(R)} - 1 \right| < \varepsilon$$

and, moreover, R contains a subset, G, with $\nu(G) > (1-\varepsilon)\nu(R)$ which has the property that for all points in G,

$$\left| \frac{d\nu_R^p}{d\nu} - 1 \right| < \varepsilon$$

The next section is devoted to the proof of the following lemma.

Lemma 5.1 *For any $\varepsilon > 0$ there exist ε-regular coverings $\mathcal{R}_\varepsilon$ with arbitrary small rectangles.*

In other words, the lemma says that up to a subset of measure $< \varepsilon$, the phase space, M, consists of rectangles which have product structure not only in a topological sense, but in a measure-theoretical sense also (approximately, with an arbitrarily small error). Our coverings are substantially different from Markov partitions [6] and Markov sieves [9]. First of all, we do not impose any requirements on intersections $T^tR \cap R'$ for $R, R' \in \mathcal{R}$. Secondly, we do not control how quickly the image T^tR, for a rectangle $R \in \mathcal{R}$, becomes "uniformly" distributed over all the atoms of $\mathcal{R}$ as t increases. As a result, our construction of ε-regular coverings is much simpler than those of Markov partitions or Markov sieves, and we can carry it out in a very general context.

5.2. *Construction of ε-regular coverings for maps.* Let $\varepsilon > 0$ be given. Cover M (up to measure 0) by a finite number of open sets (which we call charts) that are separated by a finite number of smooth compact hypersurfaces and in each of which there is one coordinate system which induces an isomorphism between a bounded domain in $\mathbb{R}^d$ and that chart. The construction of ε-regular coverings will be performed primarily in each chart separately. Given any two points, x and y, which lie in the same chart, the coordinates in that chart induce an identification of $T_x(M)$ and $T_y(M)$. This is the identification we will use when measuring the angle, $\measuredangle\,(L_x, L_y)$, between two subspaces $L_x \subset T_x(M)$ and $L_y \subset T_y(M)$. Also, Lebesgue measure in $\mathbb{R}^d$ can be pushed forward by the coordinate functions to a measure, λ, in each chart. This measure is equivalent to the Riemannian volume which comes from the metric in M. Thus $\nu << \lambda$. In particular, there is a constant, $\delta > 0$, such that any set whose λ-measure is less than δ has ν-measure less than $\epsilon/4$. Similarly, the Euclidean metric in $\mathbb{R}^d$ can be pulled back by the inverse coordinate functions to a metric in each chart. We call the latter the Euclidean metric in the chart. The Euclidean metric and the Riemannian metric are equivalent. In other words, there is a constant, $c \geq 1$, such that given any two points in the same chart, the ratio of the Riemannian distance between them and the Euclidean distance between them is between $1/c$ and c. By choosing the charts and coordinates with some care we can ensure that $c < 2$.

For any $x \in M$ we denote by $r_x^{u,s}$ the Euclidean distance of x to the boundary $\partial W_x^{u,s}$ measured along the manifold $W_x^{u,s}$ (if that manifold does not exist, we set that distance to zero). For any $\alpha > 0$ and $x \in M$ let $r_x^{u,s}(\alpha)$ be the smaller of $r_x^{u,s}$ and the Euclidean distance from x to the nearest point $y \in W_x^{u,s}$ (measured along the manifold $W_x^{u,s}$) for which that $\sphericalangle\,(E_y^{u,s}, E_x^{u,s}) \geq \alpha$ (if there is no such point then $r_x^{u,s}(\alpha) = r_x^{u,s}$). Since $W_x^{u,s}$ are C^1 smooth submanifolds, $r_x^{u,s}(\alpha) > 0$ for any $\alpha > 0$ and any $x \in M$ for which $W_x^{u,s}$ exist.

Pick a compact subset $M_\varepsilon \subset M$ such that
(i) $\nu(M_\varepsilon) > 1 - \varepsilon/4$.
(ii) The subsets E_x^u and E_x^s depend on $x \in M_\varepsilon$ continuously.
(iii) The angle between the above subsets is bounded away from zero on M_ε:

$$\bar{\alpha} = \min_{x \in M_\varepsilon} \sphericalangle\,(E_x^u, E_x^s) > 0$$

(iv) We also require that

$$\bar{r} = \min_{x \in M_\varepsilon} r_x^{u,s}(\beta) > 0$$

where β is the smaller of $\pi/3$ and $\delta\bar{\alpha}/(8d\lambda(M))$, and d is the dimension of M. The existence of a subset of M satisfying (i) and (ii) follows from the Lusin theorem. To fulfill the requirements (iii) and (iv) it is enough to remove from the above subset those parts on which the corresponding measurable functions are too small.

Next, we can cover M_ε (up to a subset of zero measure) by a finite collection of open sets $\mathcal{U}$ with two properties. First, each set lies in one chart, which defines a coordinate system in it. Second, the angles $\sphericalangle\,(E_x^u, E_y^u)$ and $\sphericalangle\,(E_x^s, E_y^s)$ for any $x, y \in M_\varepsilon \cap U$, $U \in \mathcal{U}$, do not exceed β. We can easily ensure that the open sets $U \in \mathcal{U}$ be disjoint and separated by a finite number of smooth compact hypersurfaces.

Now, in each open set $U \in \mathcal{U}$ we pick a point $z \in U$ and fix a new coordinate system so that $d^u = \dim E_z^u$ coordinate axes are mutually orthogonal and their tangents are parallel to E_z^u and the other d^s axes are also mutually orthogonal and their tangents are parallel to E_z^s. In this new coordinate system we partition U into a lattice of d-dimensional boxes whose sides have length $r > 0$, where r is chosen so small that (i) $r < \bar{r}/2d$ and (ii) the union of all the boxes that lie entirely in U has measure $> (1 - \varepsilon/4)\nu(U)$. Note that by decreasing r, if necessary, we can make these boxes as small as we wish. Denote by $\mathcal{B}$ the collection of all the boxes, B, such that $B \subset U$, for some $U \in \mathcal{U}$. Obviously, the boxes $B \in \mathcal{B}$ are disjoint, and

$$\nu(\cup_{B\in\mathcal{B}} B) > 1 - \varepsilon/2$$

Furthermore, since $\sphericalangle\,(E_y^{u,s}, E_x^{u,s}) < \beta \leq \pi/3$ for all points $x, y \in B$ that lie on the same unstable (resp. stable) manifold, it follows that the Euclidean distance between x and y measured along the manifold, is less than twice the actual Euclidean distance. Thus, the Riemannian distance measured along the manifold is less than 8 times the actual Riemannian distance. It follows that every subset of a box, B, will have the property (b) of ε-regular coverings.

We call a face of a box $B \in \mathcal{B}$, $B \subset U \in \mathcal{U}$, a u-face if it is parallel to E_z^u and an s-face if it is parallel to E_z^s (the point $z \in U$ was specified above). Each face of any box $B \in \mathcal{B}$ is either a u-face or an s-face.

Next, in each box $B \in \mathcal{B}$ we take all the points $x \in B \cap M_\varepsilon$ for which the local manifold W_x^u does not cross any u-face of B and the local manifold W_x^s does not cross any s-face of B. Note that these manifolds have length at least $\bar{r}$ so, by our choice of r and since $\beta < \pi/3$, they are long enough to stretch across B completely. Thus our requirements actually mean that the set $\partial(W_x^u \cap B)$ lies on the s-faces of B and the set $\partial(W_x^s \cap B)$ lies on the u-faces of B. We now complete the set of above points $x \in B$ to a rectangle (by adding all the intersections $W_x^u \cap W_{x'}^s$ for all $x, x' \in B \cap M_\varepsilon$ satisfying our requirements). That rectangle, $\tilde{R}$, lies inside B due to our requirements. Denote the collection of all those rectangles in all $B \in \mathcal{B}$ by $\tilde{\mathcal{R}}$.

The rectangles $\tilde{R} \in \tilde{\mathcal{R}}$ do not cover the set $M_\varepsilon \cap (\cup_{B\in\mathcal{B}} B)$ (even mod 0): the points in a tiny neighborhood of the faces of the boxes $B \in \tilde{B}$ are left out. By virtue of the property (iv) of the subset M_ε and our definition of the sets $U \in \mathcal{U}$, what is left out is contained in a $2r\beta$-neighborhood of the faces. Since each box B has $2d$ faces and the λ-volume of a box is greater than $r^d\bar{\alpha}/2$, it follows that the λ-volume of these neighborhoods of the faces of B is less than $8d\beta/\bar{\alpha}$ times the λ-volume of the box B. It follows that the λ-measure of the points in all the boxes that were left out is less than $8d\beta\lambda(M)/\bar{\alpha} = \delta$. As a result, we get

$$\nu(\cup_{\tilde{R}\in\tilde{\mathcal{R}}} \tilde{R}) > 1 - 3\varepsilon/4$$

The rectangles $\tilde{R} \in \tilde{\mathcal{R}}$ are disjoint, cover the phase space M up to a tiny subset of measure $< \varepsilon/4$, and in fact they are very nice – for any $x, y \in \tilde{R}$ and $\tilde{R} \in \tilde{\mathcal{R}}$ the "unstable leaves," $W_x^u \cap \tilde{R}$ and $W_y^u \cap \tilde{R}$, are nearly parallel (the same is, of course, true for stable leaves). This is not enough, however, to ensure the "product property" (c) of the desired ε-regular covering $\mathcal{R}_\varepsilon$. We need to partition the rectangles further to do this. This is our next goal.

First of all, notice that we can partition any $\tilde{R} \in \tilde{\mathcal{R}}$ into arbitrarily small subrectangles without any further losses. Indeed, for any $\tilde{R}$ fix a point $z \in \tilde{R}$, and consider any partition of the sets $W_z^{u,s} \cap \tilde{R}$ into a finite number of measurable subsets:

$$W_z^u \cap \tilde{R} = \cup_{i=1}^{k^u} V_z^u(i) \quad \text{and} \quad W_z^s \cap \tilde{R} = \cup_{j=1}^{k^s} V_z^s(j)$$

For any $1 \le i \le k^u$ and $1 \le j \le k^s$ the set $\tilde R_z(i,j) = \{W^s_x \cap W^u_y,\ x \in V^u_z(i),\ y \in V^s_z(j)\}$ is a rectangle, and we have a decomposition

$$\tilde R = \cup_{i=1}^{k^u} \cup_{j=1}^{k^s} \tilde R_z(i,j) \tag{13}$$

of $\tilde R$ into a finite number of disjoint subrectangles. We call such decompositions proper partitions (of $\tilde R$ into subrectangles).

Now, in order to ensure the product property (c) for rectangles in ε-regular coverings, we consider the jacobian of canonical isomorphisms. For a rectangle $\tilde R \in \tilde{\mathcal R}$, a fixed point $z \in \tilde R$ and an arbitrary point $x \in \tilde R$ the map from $W^u_x \cap \tilde R$ to $W^u_z \cap \tilde R$ that takes any point $y \in W^u_x \cap \tilde R$ to $W^s_y \cap W^u_z$ is the canonical isomorphism between the two leaves in a rectangle, see Section 2. The measure ν induces a conditional measure, ν^u_x, on a.e. leaf $W^u_x \cap \tilde R$. The canonical isomorphism carries the measure ν^u_x over to $W^u_z \cap \tilde R$, and so the jacobian $J^u_z(x) = d\nu^u_z/d\nu^u_x(x)$ is defined at a.e. point $x \in \tilde R$. Due to the absolute continuity of local stable and unstable manifolds, see Section 2, that jacobian is an a.e. finite and strictly positive measurable function in x on $\tilde R$ for a.e. $z \in \tilde R$.

In virtue of the Lusin theorem, for any $\varepsilon > 0$, in any rectangle $\tilde R \in \tilde{\mathcal R}$ there is a compact subset $P_\varepsilon \subset \tilde R$ of measure $\nu(P_\varepsilon) > (1 - \varepsilon^4/10000)\nu(\tilde R)$ on which the function $J^u_z(x)$ is continuous (in x). We can easily ensure that $J^u_z(x)$ is also bounded on P_ε:

$$0 < a_\varepsilon \le J^u_z(x) \le A_\varepsilon < \infty$$

for some constants a_ε and A_ε and all $x \in P_\varepsilon$.

Clearly, there is a proper partition (13) of $\tilde R$ such that for every $\tilde R_z(i,j) \subset \tilde R$ and any $x, y \in \tilde R_z(i,j) \cap P_\varepsilon$ we have $|J^u_z(x) - J^u_z(y)| \le a_\varepsilon \varepsilon/100$, and, therefore,

$$\left| \frac{J^u_z(x)}{J^u_z(y)} - 1 \right| \le \frac{\varepsilon}{100} \tag{14}$$

Note that, if $y \in W^s_x$ then

$$\frac{J^u_z(x)}{J^u_z(y)} = J^u_y(x) \tag{15}$$

where $J^u_y(x)$ is the jacobian of the canonical isomorphism between $W^u_x \cap \tilde R$ and $W^u_y \cap \tilde R$ at x.

For any $\tilde R \in \tilde{\mathcal R}$ denote by $\mathcal P_\varepsilon$ the collection of subrectangles $\tilde R_z(i,j)$ for which

$$\nu(\tilde R_z(i,j) \cap P_\varepsilon) \ge (1 - \varepsilon^2/100)\nu(\tilde R_z(i,j)) \tag{16}$$

It is then an easy calculation that

$$\nu(\cup_{\tilde R_z(i,j) \in \mathcal P_\varepsilon} \tilde R_z(i,j)) \ge (1 - \varepsilon^2/100)\nu(\tilde R)$$

so that we can disregard subrectangles $\tilde R_z(i,j)$ that fail to satisfy (16).

Lastly, due to (14) and (15), for any $\tilde{R} \in \tilde{\mathcal{R}}$ and any $\tilde{R}_z(i,j) \in \mathcal{P}_\varepsilon$ there is a point $z(i,j) \in \tilde{R}_z(i,j)$ such that the jacobian $J^u_{z(i,j)}(x)$ (now defined *inside* $\tilde{R}_z(i,j)$) is sufficiently close to unity:

$$|J^u_{z(i,j)}(x) - 1| \le \varepsilon/10$$

on a subset of points $x \in \tilde{R}_z(i,j)$ whose measure is at least $(1-\varepsilon/10)\nu(\tilde{R}_z(i,j))$ in virtue of (16).

Integrating the jacobian $J^u_{z(i,j)}(x)$ within any rectangle $\tilde{R}_z(i,j)$ that belongs to $\mathcal{P}_\varepsilon$ shows that each such rectangle satisfies the product property (c) of ε-regular coverings. The total measure of all those rectangles is $> 1-\varepsilon$. Thus, we obtain an ε-regular covering in M. The rectangles in $\mathcal{R}_\varepsilon$ can be made arbitrarily small by decreasing the parameter r if necessary. The lemma is proved.

5.3. *ε-regular coverings for flows.* Since any flow necessarily has a zero Lyapunov exponent, we have to modify our definition of ε-regular coverings. As before, we will not need the K-property of the flow to construct these coverings.

We introduce the functions $r^{u,s}_y$ for $y \in \mathcal{M}$ and $r^{u,s}_y(\alpha)$ for $\alpha > 0$ in literally the same way as we did in Subsection 5.1 for the local manifolds of T. In the like manner we introduce functions $\tilde{r}^{u,s}_y$ and $\tilde{r}^{u,s}_y(\alpha)$ for weakly unstable and stable manifolds, $\mathcal{W}^{u,s}_y$, respectively, and $\alpha > 0$.

Definition. *A rectangle in $\mathcal{M}$ is a measurable set, $R \subset \mathcal{M}$, equipped with a distinguished point $z \in R$. The set R has the property that for all points $x, y \in R$ the local unstable manifold W^u_x and the local weakly stable manifold $\mathcal{W}^s_y$ intersect each other at a single point which lies in R.*

Notice that, since the foliations into unstable and stable manifolds need not commute, any rectangle may contain some pairs of points x and y, for which the intersection of the local stable manifold W^s_x and the local weakly unstable manifold $\mathcal{W}^u_y$ does not lie in the rectangle.

As before, a rectangle R can be thought of as the cartesian product of $W^u_z \cap R$ and $\mathcal{W}^s_z \cap R$ (where a point $y \in R$ is given by $(\mathcal{W}^s_y \cap W^u_z, W^u_y \cap \mathcal{W}^s_z)$). Since μ induces conditional measures on each of these two sets, R can thus be endowed with a product measure. To be specific, we consider the product measure $\mu^p_R = \mu^u_z \times \mu^s_f$, where μ^u_z is the conditional measure on $W^u_z \cap R$ and μ^s_f is a factor measure on the set of leaves $W^u_x \cap R$, $x \in R$ (this set of leaves can be identified with $\mathcal{W}^s_z$, so that we can think of μ^s_f as a measure on $\mathcal{W}^s_z$ also). As before it follows from the absolute continuity of the weakly stable and unstable foliations that $\mu^p_R << \mu$.

Definition. *Given any $\varepsilon > 0$, an ε-regular covering of the phase space $\mathcal{M}$ is a finite collection of disjoint rectangles $\mathcal{R} = \mathcal{R}_\varepsilon$ such that*
(a) $\mu(\cup_{R\in\mathcal{R}} R) > 1 - \varepsilon$
(b) Given any two points $x, y \in R \in \mathcal{R}$, which lie in the same unstable or weakly stable manifold, there is a smooth curve on that manifold which connects x and y and has length less than $100 \cdot \operatorname{diam} R$

(c) For every $R \in \mathcal{R}$ we have

$$\left| \frac{\mu_R^p(R)}{\mu(R)} - 1 \right| < \varepsilon$$

and, moreover, R contains a subset, G, with $\mu(G) > (1-\varepsilon)\mu(R)$ which has the property that for all points in G,

$$\left| \frac{d\mu_R^p}{d\mu} - 1 \right| < \varepsilon$$

Lemma 5.2 *For any $\varepsilon > 0$ there exist ε-regular coverings $\mathcal{R}_\varepsilon$ of $\mathcal{M}$ with arbitrary small rectangles.*

Our construction of ε-regular coverings for maps can be carried over to flows almost word for word. Indeed, in our construction we never used the fact that the manifolds $W_x^u, x \in M$, were unstable (expanding) and the manifolds $W_x^s, x \in M$, were stable (contracting). Nor did we use the fact that they commuted. We only used their C^1 smoothness, measurable dependence on $x \in M$, transversality and absolute continuity.

In the case of flows, the manifolds W_y^u and $\mathcal{W}_y^s$ for $y \in \mathcal{M}$ are C^1 smooth, transversal to each other, absolutely continuous and depend measurably on y. Thus, ε-regular coverings for the flow Φ^t exist.

6 The proof of the Bernoulli property

We are now ready to prove that any finite partition of the phase space M (or $\mathcal{M}$) with piecewise smooth boundary is very weak Bernoulli (vwB). As remarked in Section 4, this is sufficient to prove Theorems 2.1 and 2.2.

6.1. *Very weak Bernoulli partitions for maps.* Our maps have stronger hyperbolic properties than our flows since all of their Lyapunov exponents are non-zero. This makes them conceptually easier to handle, and is no doubt partially responsible for the tendency in the literature to prove statistical properties of maps in full and to only indicate briefly how to modify those proofs to obtain proofs of similar properties of flows. To address this imbalance we have chosen in this last section to construct very weak Bernoulli partitions for flows and just to mention a few words here about how to modify this construction to obtain very weak Bernoulli partitions for maps.

Replacing weakly stable leaves by stable leaves, both the flow Φ^t and the map Φ^1 by the map T, the phase space $\mathcal{M}$ by M and the measure μ by ν, the arguments for flows can be repeated word for word to obtain a proof for maps.

As suggested in the comment above, the arguments can be made conceptually simpler by observing that, unlike weakly stable leaves, which don't contract in the flow direction, stable leaves actually do contract under the map. In this case the set F_4 (see notations below) can be defined instead to be the set of all points, $x \in M \backslash R_0$, for which there is some $y \in W^s_x \cap \pi(x)$ with the property that $h(x, y) > 0$. Arguing exactly as for the set F_3 it follows that $\nu(F_4) < c\delta$. The definition of $\hat{F}_4$ remains the same and it follows that $\nu(\hat{F}_4) < c\delta^{1/2}$. The rest of the proof remains unchanged. The point is not that any of the calculations become simpler but that, with this choice of F_4, what we produce is a joining of $\{T^{-i}\alpha\}_{i=1+m}^{n+m}$ and $\{T^{-i}\alpha/A\}_{i=1+m}^{n+m}$ in which $c\varepsilon$-a.e. pair of points (x, y) have 'names' that are actually identical. It follows that, not only is the $\bar{d}$-distance between the two sequences of partitions small, but also the typically larger distance discussed in Section 4 after equation (9) is small.

6.2. *Very weak Bernoulli partitions for flows.* Consider a partition, α of $\mathcal{M}$, which consists of a finite number of atoms. Assume that the d-dimensional measure of the boundaries of the atoms is finite (recall that $\dim \mathcal{M} = d + 1$), and that these boundaries are piecewise smooth. It follows that there is a constant, D_0, such that for any $\varepsilon > 0$ the measure of the ε-neighbourhood of the union of these boundaries is $< D_0\varepsilon$. We will show that such a partition is vwB under Φ^1. Recall from Section 4 that this is sufficient to conclude that the flow Φ^t is Bernoulli.

Remark: By making an appropriate choice of δ below, our proof that α is vwB would also work for a more general class of partitions, namely those for which there is some constant $a > 0$ such that the measure of an ε-neighbourhood of the union of the boundaries of their atoms is bounded by a constant times ε^a.

Let $\varepsilon > 0$ be given and let $\delta = \varepsilon^4$. Let $\pi = \{R_0, R_1, \dots R_k\}$ be a partition of $\mathcal{M}$ such that $\{R_1, \dots R_k\}$ is a δ-regular covering of $\mathcal{M}$ consisting of rectangles with diameters less than δ/D_0 and $R_0 = \mathcal{M} \setminus \cup_{i=1}^k R_i$. For each $i = 1, \dots k$, denote the subset of R_i in which $|d\mu^p_{R_i}/d\mu - 1| < \delta$ by G_i. In what follows, when the rectangle R_i is apparent, we will simply write μ^p instead of $\mu^p_{R_i}$.

Let C be chosen so that the set of all points x with $C(x) > C$ has measure less than δ. (Recall that the function $C(x)$ appears in the expression for the flow analogous to (8) which describes the characteristic property of the stable and unstable manifolds.) Since the flow Φ^t has the K-property, there exists an (even) integer, $N = 2m$, such that for any other integers $N_0 < N_1$ which are both greater than N, δ-a.e. atom, A, of $\bigvee_{N_0-m}^{N_1-m} \Phi^i\alpha$ has the property that for all $R \in \pi$

$$\left| \frac{\mu(R/A)}{\mu(R)} - 1 \right| < \delta \tag{17}$$

(recall that $\mu(\cdot/A)$ means the measure μ conditioned on A). By increasing the size of N if necessary, we guarantee in addition that $Ce^{-\chi m}/(1 - e^{-\chi}) < 1$. (Recall that the constant χ appears in the expression for the flow analogous to (8) which describes the characteristic property of the stable and unstable manifolds.)

Let N_0 and N_1 with $N < N_0 < N_1$ and $n > 0$ be given. Let ω be the partition $\bigvee_{N_0-m}^{N_1-m} \Phi^i\alpha$. To show that α is vwB we shall show that $c\varepsilon$-a.e. atom of ω satisfies (11) with ε replaced by $c\varepsilon$. We start with identifying the set of 'bad' atoms, whose union will have measure less than $c\varepsilon$.

Let $\hat{F}_1$ denote the union of all those atoms in ω which do not satisfy (17). By virtue of our choice of N, $\mu(\hat{F}_1) < \delta$.

Let $F_2 = \cup_{i=1}^k R_i \backslash G_i$. By our definition of regular coverings, $\mu(F_2) < \delta$, and

$$\begin{aligned}\sum_{i=1}^k \mu^p(F_2 \cap R_i) &= \sum_{i=1}^k \mu^p(R_i) - \mu^p(G_i) \\ &\leq \sum_{i=1}^k \mu(R_i)(1+\delta) - \mu(G_i)(1-\delta) \\ &\leq \sum_{i=1}^k \mu(R_i)(1+\delta) - \mu(R_i)(1-\delta)^2 \\ &\leq c\delta\end{aligned}$$

Let $\hat{F}_2$ denote the union of all those atoms, A in ω, for which either $\mu(F_2/A) > \delta^{1/2}$ or

$$\sum_{i=1}^k \frac{\mu^p_{R_i}(A \cap F_2)}{\mu(A)} > \delta^{1/2}.$$

It follows from the considerations above that $\mu(\hat{F}_2) < c\delta^{1/2}$.

Let F_3 denote the set of all points, $x \in \mathcal{M}\backslash R_0$, which lie in a 'bad' part of the atom $\omega(x)$ in the sense that the unstable manifold through x intersects the boundary of $\omega(x)$ before it completely stretches across the rectangle $\pi(x)$, i.e.

$$F_3 = \{x \in \mathcal{M}\backslash R_0 \colon W_x^u \cap \pi(x) \not\subset W_x^u \cap \omega(x)\}$$

Since the unstable manifolds expand exponentially under the flow, most atoms in ω are long in the directions of the unstable manifolds (and short in the directions of the stable manifolds). Thus we expect F_3 to be a set of small measure. More specifically, if $x \in F_3$, then there is a curve, $\gamma \subset W_x^u$, whose length is less than δ/D_0, which extends from x to the boundary of $(\Phi^i\alpha)(x)$, for some i in the range $N_0 - m \leq i \leq N_1 - m$. If x is such that $C(x) < C$, then, due to (8), the point $\Phi^{-i}(x)$ lies within a distance $Ce^{-\chi i}(\delta/D_0)$ of the boundary of α. The total measure of points that satisfy such a condition is less than

$$\sum_{i=N_0-m}^{N_1-m} Ce^{-\chi i}\delta < \frac{Ce^{-\chi m}}{1-e^{-\chi}}\delta < \delta$$

by our choice of N. Since the measure of points, x, for which $C(x) > C$ is less than δ it follows that $\mu(F_3) < c\delta$. Let $\hat{F}_3$ denote the union of all those atoms, A in ω, for which $\mu(F_3/A) > \delta^{1/2}$. It follows that $\mu(\hat{F}_3) < c\delta^{1/2}$.

Let F_4 denote the set of all points, $x \in \mathcal{M}\backslash R_0$, for which there is some $y \in \mathcal{W}_x^s \cap \pi(x)$ with the property that $h(x,y) > \delta^{1/2}$. If $x \in F_4$ then for at least $n\delta^{1/2}$ values of i in the interval $1 + m \leq i \leq n + m$, there must be a curve $\gamma_i \subset \mathcal{W}_x^s$ whose length is less than δ/D_0 which extends from x to the boundary of $(\Phi^{-i}\alpha)(x)$. If $C(x) < C$ then under Φ^i any component of the curve which lies in W_x^s contracts by a factor of at least $Ce^{-\chi i} < 1$, and in the flow direction there is also no expansion. It follows that $\Phi^i(x)$ lies within a distance δ/D_0 of the boundary of α. Thus the total measure of points, x, which lie in F_4 and have the property that $C(x) < C$ is less than $\delta^{1/2}$. It follows that the measure of F_4 is less than $c\delta^{1/2}$. Let $\hat{F}_4$ denote the union of all those atoms, A in ω, which have the property $\mu(F_4/A) > \delta^{1/4}$. It follows that $\mu(\hat{F}_4) < c\delta^{1/4}$.

Consider an atom, A in ω, which is in the complement of each of the sets $\hat{F}_1$, $\hat{F}_2$, $\hat{F}_3$, and $\hat{F}_4$. It follows from the above estimates that the union of all such atoms has measure greater than $1 - c\delta^{1/4}$ which is equal to $1 - c\varepsilon$. We shall use Lemma 4.3, substituting $(A, \mu(\cdot/A), (\Phi^{-i}\alpha)/A)$ for (X, μ, α_i), and $(\mathcal{M}, \mu, \Phi^{-i}\alpha)$ for (Y, ν, β_i), to show that A satisfies (11) with ε replaced by $c\varepsilon$.

We define the map $\psi\colon A \to \mathcal{M}$ as follows. First consider those sets $A \cap R_i \cap {F_3}^c$, $i \geq 1$, that have positive measure, and consider a point x_i in each such set. The sets $A \cap R_i \cap {F_3}^c \cap \mathcal{W}^s_{x_i}$ and $\mathcal{W}^s_{x_i} \cap R_i$, equipped with the factor measure μ^s_f (thought of as a measure on $\mathcal{W}^s_{x_i}$) conditioned on them, are non-atomic Lebesgue probability spaces, so there is a bijective measure preserving map from one to the other. Define ψ on $A \cap R_i \cap {F_3}^c \cap \mathcal{W}^s_{x_i}$ to be any such map. Given any other point in the set, y, define its image under ψ by first mapping y along its unstable manifold to the point $W^u_y \cap \mathcal{W}^s_{x_i}$, then mapping this point to its image under ψ, and then mapping back again along the unstable manifold to $\mathcal{W}^s_y$. On the remaining part of A define ψ to be the identity. Notice that ψ has been defined in such a way that for any set $B \subset A \cap R_i \cap {F_3}^c$,

$$\mu^p_{R_i}(B/A \cap {F_3}^c) = \mu^p_{R_i}(\psi B) \tag{18}$$

The set E_1 of Lemma 4.3 which consists of all the points, $x \in A$, for which $h(x, \psi x) > \delta^{1/4} = \varepsilon$ is contained in F_4. It follows that $\mu(E_1/A) < \delta^{1/4} = \varepsilon$ since A lies in the complement of $\hat{F}_4$.

To understand the set E_2 of Lemma 4.3, we must first identify and estimate the measures of some 'bad' subsets of A. E_2 will be the union of these sets.

The first bad set is $A \cap R_0$. Since A lies in the complement of $\hat{F}_1$, and $\mu(R_0) < \delta$, it follows that $\mu(R_0/A) < c\delta$.

The next two bad sets are $A \cap F_2$ and $A \cap F_3$. Since A lies in the complement of both $\hat{F}_2$ and $\hat{F}_3$, it follows that $\mu(F_2/A) < \delta^{1/2}$ and $\mu(F_3/A) < \delta^{1/2}$.

In addition to F_2 and F_3 themselves, we also consider as bad all those rectangles in which A contains a large proportion of F_2 or F_3. These proportions can be measured with respect to product measure or invariant measure. More specifically, let D_2 denote the union of all those rectangles, R_i, for which either

$$\mu_{R_i}(F_2/A) > \delta^{1/4} \quad \text{or} \quad \frac{\mu^p_{R_i}(A \cap F_2)}{\mu(A \cap R_i)} > \delta^{1/4}$$

Since A is in the complement of $\hat{F}_2$, it follows that $\mu(D_2/A) < c\delta^{1/4}$. Similarly, let D_3 denote the union of all those rectangles, R_i, for which $\mu(F_3/A \cap R_i) > \delta^{1/4}$. Since A is in the complement of $\hat{F}_3$, it follows that $\mu(D_3/A) < c\delta^{1/4}$.

Our final bad set is the inverse image under ψ of F_2. It will be sufficient to estimate the measure of that part of this set which lies in the complement of R_0, F_2, F_3, and D_2 since the measures of these sets have already been estimated. Consider a rectangle, R_i, which lies in the complement of $R_0 \cup D_2$.

$$\begin{aligned}
&\mu(\psi^{-1}(F_2) \cap (F_2 \cup F_3)^c \cap R_i \,/\, A) \\
&\quad = \mu(\psi^{-1}(F_2 \cap R_i) \cap {F_2}^c \cap {F_3}^c \,/\, A) \\
&\quad = \left(\frac{\mu(\psi^{-1}(F_2 \cap R_i) \cap {F_2}^c \cap {F_3}^c)}{\mu^p(\psi^{-1}(F_2 \cap R_i) \cap {F_2}^c \cap {F_3}^c)}\right) \left(\frac{\mu^p(\psi^{-1}(F_2 \cap R_i) \cap {F_2}^c \cap {F_3}^c)}{\mu^p(A \cap R_i \cap {F_3}^c)}\right) \times \\
&\qquad \left(\frac{\mu^p(A \cap R_i \cap {F_3}^c)}{\mu(A \cap R_i)}\right) \left(\frac{\mu(A \cap R_i)}{\mu(A)}\right)
\end{aligned} \tag{19}$$

The first term in the above expression is the ratio of the product measure of a set to its invariant measure. Since the set lies in the complement of F_2, it follows that this term is less than $1 + c\delta$. Using (18) we can bound the second term as follows:

$$\begin{aligned}
\frac{\mu^p(\psi^{-1}(F_2 \cap R_i) \cap {F_2}^c \cap {F_3}^c)}{\mu^p(A \cap R_i \cap {F_3}^c)} &= \frac{\mu^p(\psi(\psi^{-1}(F_2 \cap R_i) \cap {F_2}^c \cap {F_3}^c))}{\mu^p(R_i)} \\
&\leq \frac{\mu^p(F_2 \cap R_i)}{\mu^p(R_i)} \\
&= 1 - \frac{\mu^p(G_i)}{\mu^p(R_i)} \\
&\leq 1 - \frac{(1-\delta)\mu(G_i)}{(1+\delta)\mu(R_i)} \\
&\leq 1 - \frac{(1-\delta)^2}{(1+\delta)} \\
&\leq c\delta
\end{aligned}$$

To bound the size of the third term we notice that

$$\begin{aligned}
\frac{\mu^p(A \cap R_i \cap {F_3}^c)}{\mu(A \cap R_i)} &\leq \frac{\mu^p(A \cap R_i)}{\mu(A \cap R_i)} \\
&\leq \frac{\mu^p(A \cap R_i \cap {F_2}^c)}{\mu(A \cap R_i \cap {F_2}^c)} + \frac{\mu^p(A \cap R_i \cap F_2)}{\mu(A \cap R_i)} \\
&\leq 1 + c\delta^{1/4}
\end{aligned}$$

Substituting into (19) and summing over all rectangles in the complement of $R_0 \cup D_2$ we get

$$\mu(\psi^{-1}(F_2) \cap (R_0 \cup F_2 \cup F_3 \cup D_2)^c \,/\, A) \quad \leq \quad c\delta$$

The set E_2 of Lemma 4.3 is $A \cap (R_0 \cup F_2 \cup F_3 \cup D_2 \cup D_3 \cup \psi^{-1} F_2)$. Combining all the estimates above we see that $\mu(E_2/A) < c\delta^{1/4} = c\varepsilon$. To show that this set has the desired property we will need the following claim.

Claim: If A is an atom which lies in the complement of both $\hat{F}_2$ and $\hat{F}_3$, and $B \subset A \cap R_i \cap {E_2}^c$, for some rectangle, R_i, then

$$\left| \frac{\mu(B/A \cap R_i)}{\mu(\psi B/R_i)} - 1 \right| < c\delta^{1/4}$$

Proof: Since B lies in the complement of F_3 we have by (18) that $\mu^p_{R_i}(B/A \cap F_3^c) = \mu^p_{R_i}(\psi B)$. It follows that

$$\begin{aligned} \frac{\mu(B/A \cap R_i)}{\mu(\psi B/R_i)} &= \left(\frac{\mu(B)}{\mu^p(B)}\right) \left(\frac{\mu^p(A \cap R_i \cap {F_3}^c)}{\mu(A \cap R_i \cap {F_3}^c)}\right) \left(\frac{\mu(A \cap R_i \cap {F_3}^c)}{\mu(A \cap R_i)}\right) \times \\ & \left(\frac{\mu(R_i)}{\mu^p(R_i)}\right) \left(\frac{\mu^p(\psi B)}{\mu(\psi B)}\right) \end{aligned} \tag{20}$$

Since $B \subset (F_2 \cup R_0)^c$,

$$\left| \frac{\mu(B)}{\mu^p(B)} - 1 \right| < c\delta \tag{21}$$

To estimate the second term notice that

$$\begin{aligned} & \left| \frac{\mu^p(A \cap R_i \cap {F_3}^c)}{\mu(A \cap R_i \cap {F_3}^c)} - 1 \right| \leq \\ & \quad \left| \frac{\mu^p(A \cap R_i \cap {F_2}^c \cap {F_3}^c)}{\mu(A \cap R_i \cap {F_2}^c \cap {F_3}^c)} \frac{\mu(A \cap R_i \cap {F_2}^c \cap {F_3}^c)}{\mu(A \cap R_i)} \frac{\mu(A \cap R_i)}{\mu(A \cap R_i \cap {F_3}^c)} - 1 \right| + \\ & \quad \frac{\mu^p(A \cap R_i \cap F_2 \cap {F_3}^c)}{\mu(A \cap R_i)} \frac{\mu(A \cap R_i)}{\mu(A \cap R_i \cap {F_3}^c)} \end{aligned}$$

By our choice of F_2, the first term inside the absolute value signs above is within δ of 1. Since $B \subset (D_2 \cup D_3)^c$, the second and third terms are within $c\delta^{1/4}$ of 1. Also, since $B \subset {D_2}^c$

$$\frac{\mu^p(A \cap R_i \cap F_2 \cap {F_3}^c)}{\mu(A \cap R_i)} \leq \frac{\mu^p(A \cap R_i \cap F_2)}{\mu(A \cap R_i)} \leq \delta^{1/4}$$

Thus,

$$\left| \frac{\mu^p(A \cap R_i \cap {F_3}^c)}{\mu(A \cap R_i \cap {F_3}^c)} - 1 \right| \leq c\delta^{1/4} \tag{22}$$

Since $B \subset {D_3}^c$, we have for the third term in (20) that

$$\left| \frac{\mu(A \cap R_i \cap {F_3}^c)}{\mu(A \cap R_i)} - 1 \right| < c\delta^{1/4}$$

By the characteristic property (c) of regular coverings

$$\left|\frac{\mu(R_i)}{\mu^p(R_i)} - 1\right| < c\delta$$

and, since $B \subset (\psi^{-1}F_2)^c$,

$$\left|\frac{\mu^p(\psi B)}{\mu(\psi B)} - 1\right| < c\delta \tag{23}$$

Combining (20), (21), and (22) to (23), we get

$$\left|\frac{\mu(B/A \cap R_i)}{\mu_(\psi B/R_i)} - 1\right| < c\delta^{1/4}$$

as required. Q.E.D.

Now, consider a set $B \subset A \backslash E_2$.

$$\begin{aligned}\left|\frac{\mu(B/A)}{\mu(\psi B)} - 1\right| &= \frac{1}{\mu(\psi B)} \left|\mu(B/A) - \mu(\psi B)\right| \\ &\leq \sum_{R_i \subset (R_0 \cup D_2 \cup D_3)^c} \mu(R_i/\psi B) \left|\frac{\mu(B/A \cap R_i)}{\mu(\psi B/R_i)} \frac{\mu(R_i/A)}{\mu(R_i)} - 1\right| \end{aligned} \tag{24}$$

Since A is in the complement of $\hat{F}_1$,

$$\left|\frac{\mu(R_i/A)}{\mu(R_i)} - 1\right| < \delta \tag{25}$$

and it follows from the claim that

$$\left|\frac{\mu(B/A \cap R_i)}{\mu(\psi B/R_i)} - 1\right| < c\delta^{1/4} \tag{26}$$

for each rectangle $R_i \subset (R_0 \cup D_2 \cup D_3)^c$. Thus, combining (24), (25) and (26) we get that

$$\left|\frac{\mu(B/A)}{\mu(\psi B)} - 1\right| \leq c\delta^{1/4} = c\varepsilon$$

Having shown that the hypotheses of Lemma 4.3 are satisfied we conclude that

$$\bar{d}(\{\Phi^{-i}\alpha\}_{i=1+m}^{n+m}, \{\Phi^{-i}\alpha/A\}_{i=1+m}^{n+m}) < c\varepsilon$$

Since ε was chosen arbitrarily and the atom A came from a set of atoms whose union has measure greater than $1 - c\varepsilon$, it follows that α is vwB. It then follows from Theorems 4.1 and 4.2 that the flow Φ^t enjoys the Bernoulli property.

Acknowledgements. We would like to thank L. Bunimovich and D. Szasz for helpful discussions on Bernoulli billiards.

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