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Electrical Current in Sinai Billiards Under General Small Forces

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Abstract The Lorentz gas of $\mathbb{Z}^2$-periodic scatterers (or the so called Sinai billiards) can be used to model motion of electrons on a metal. We investigate the linear response for the system under various external forces (during both the flight and the collision). We give some characterizations under which the forced system is time-reversible, and derive an estimate of the electrical current generated by the forced system. Moreover, applying Pesin entropy formula and Young dimension formula, we get several characterizations of the non-equilibrium steady state of the forced system.

Keywords Sinai billiards · Electrical current · Linear response · External forces · SRB measure

1 Introduction

Lorentz gas is a popular model in mathematical physics introduced in 1905, (see [20]), in studying the motions of a point-particle or a gas of particles (electrons) in a metallic conductor. Here we consider a two-dimensional periodic Lorentz gas, that is, a particle moves on the plane and bounces off a $\mathbb{Z}^2$-periodic ray of scatterers (ions). In this case the dynamic reduces to a dynamical billiard system on the 2-D torus $\mathbb{T}^2$, generated by a billiard moving freely until it bounces off the scatterers. More precisely, let $\mathbf{B}_1, \ldots, \mathbf{B}_s$ be open convex domains on $\mathbb{T}^2$ with mutually disjoint closures, which are occupied by the $\mathbb{Z}^2$-periodic ions, and $Q = \mathbb{T}^2 \setminus \bigcup \mathbf{B}_i$ be the free space where the particle moves. Moreover, we assume that the boundary of each $\mathbf{B}_i$ is C^3 smooth with non-vanishing curvature.

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The study of classical billiard dynamics was originated in 1970 by Sinai. This model is a Hamiltonian system, so it preserves the kinetic energy $E = \frac{1}{2}\|\mathbf{p}\|^2$. Therefore we can restrict the billiard flow to a 3-D submanifold $\mathcal{M}_0 = \{(\mathbf{q}, \mathbf{p}) : E(\mathbf{q}, \mathbf{p}) = c\}$, on which the Liouville measure is also preserved by the flow. Moreover there is a 2-D global cross-section $M_0 \subset \mathcal{M}_0$, which consists of the post-collision vectors: $M_0 = \{(\mathbf{q}, \mathbf{p}) \in \mathcal{M}_0 : \mathbf{q} \in \partial Q, \mathbf{p} \cdot \mathbf{n}(\mathbf{q}) \geq 0\}$. The Poincaré map $T_0 : M_0 \to M_0$ preserves a smooth measure $d\nu_0 = \mathrm{cst} \cdot \cos\varphi dr d\varphi$, where φ is the angle from the normal vector $\mathbf{n}(\mathbf{q})$ and $\mathbf{p}$. The map T_0 and the flow Φ_0 have been shown to be uniformly hyperbolic and Bernoulli [16, 23], and many statistical properties have been well understood and are proved, see [3, 4, 6, 11, 12, 26], and the references therein.

Recently, much attention is shifting to the investigation of statistical properties of nonequilibrium billiards. Nonequilibrium phenomena are characterized by the action of external forces or boundary conditions for transport equations that change the system and generate a steady process that can be measured through mass transfer, energy (heat) transfer, charge transfer (electrical current), entropy production, or others. The laws in equilibrium statistical physics are better understood and proved or almost proved in quite a few cases. However, the apparatus of nonequilibrium statistical mechanics still relies largely on heuristic statements or numerical results, and only very few models have been studied with sufficient mathematical rigor. The main difficulty is that nonequilibrium billiards have singularities and unbounded derivatives, they usually do not preserve smooth measures, and their evolution is described by steady states characterized by singular invariant measures, of which relatively little is known in general. In addition, since Gibbs entropy is invariant under a Hamiltonian time evolution, the study of entropy increase (the second law of thermodynamics) in nonequilibrium systems is far from straightforward [22]. One of the first nonequilibrium physical models that were studied rigorously is the periodic Lorentz gas with a constant electrical field by Chernov, Eyink, Lebowitz and Sinai [8, 9] and the famous Ohm's law was proved for that case. Similar studies on special nonequilibrium dynamics were conducted by Bunimovich and Spohn [2], Gallavotti [15], Ruelle [22], and others.

Here we investigate some physical laws for Sinai billiards (or periodic Lorentz gases) under general external forces. Let $\mathbf{q} = (x, y)$ be the position of a particle in a billiard table $Q := \mathbb{T}^2 \setminus \cup_i \mathbf{B}_i$, and $\mathbf{p} = (\dot{x}, \dot{y})$ be the velocity vector. We add two types of forces to the system in the following steps:

(1) (**during the flight**) Let $\mathbf{F} = \mathbf{F}(\mathbf{q}, \mathbf{p})$ be a stationary external force on Q. The forced billiard flow is governed by the following differential equation between collisions:

$$\begin{cases} \dot{\mathbf{q}} = \mathbf{p}, \\ \dot{\mathbf{p}} = \mathbf{F}, \end{cases} \tag{1.1}$$

where the dot derivative refers to differentiate with respect to the time t.

(2) (**at the moment of collision**) Let $\mathbf{G}$ be an external twisting force spreading on ∂Q that acts on each incoming trajectory right after its elastic collision with ∂Q:

$$\big(\mathbf{q}^+(t_i), \mathbf{p}^+(t_i)\big) = \mathbf{G}\big(\mathbf{q}^-(t_i), \mathcal{R}\mathbf{p}^-(t_i)\big), \tag{1.2}$$

where $\mathcal{R}\mathbf{p}^-(t_i) = \mathbf{p}^-(t_i) + 2(\mathbf{n}(\mathbf{q}^-) \cdot \mathbf{p}^-)\mathbf{n}(\mathbf{q}^-)$ is the usual elastic reflection operator, $\mathbf{n}(\mathbf{q})$ is the unit normal vector to the billiard wall ∂Q at $\mathbf{q}$ pointing inside of the table Q, and $\mathbf{q}^-(t_i)$, $\mathbf{p}^-(t_i)$, $\mathbf{q}^+(t_i)$ and $\mathbf{p}^+(t_i)$ refer to the incoming and outgoing position and velocity vectors, respectively.

Note that the twisting force $\mathbf{G}$ changes not only the outgoing velocity of the billiard, but also the position of the billiard along the boundary ∂Q. The change in velocity can be thought of as a kick, while a change in position can model a slip along the boundary at collision. These forces indicate that the system will experience nonelastic reflections.

The type of forces we consider here are quite common in many physical models. For example we can have a potential function U on Q, or an electromagnetic field (when the moving billiard is an electron), see [5, 7–9]. The twisting force right after the collision is also closely related to real-world models. The first class is the *Goos-Hänchen shift* in the study of microresonators. That is, a finite wavelength of light will experience a lateral shift when it reflects from the interface between two materials of different refractive indices, see [17, 24] for more discussions. The twisting force is also related to the so called *soft scatterers*. Recall that the unforced system can be viewed as a Hamiltonian system under a potential function U given by $U(\mathbf{q}) = 0$ if $\mathbf{q} \in Q$, and $+\infty$ on the scatterers. In this case, the scatterers B_i are said to be rigid (or hard). Now if we replace U by some finite potential function on each B_i, then the resulting B_i is a soft one: a running billiard will climb up B_i and then exit very soon. Different shapes of the scatterer result in different enter-exit relations. If we view what happen on the scatterers as a black box, the effect can be understood as a twisting force $\mathbf{G}$ right after the elastic collision. Clearly this kind of twisting forces not only preserve the tangent collisions, and are also time-reversible. For more details, see [1, 19].

In [5, 7], Chernov considered billiards under small external forces $\mathbf{F}$ between collisions, and proved several ergodic and statistical properties of the SRB measure for the perturbed billiard system. In [14] Dolgoyat and Chernov put a constant electric field on Lorentz gases with infinite horizon and got various characterizations of the steady state electric current generated by the forced system with Gaussian thermostat. The systems with some simple twist forces were considered in [27], assuming that $\mathbf{G}$ depends on and affects only the velocity, not the position. The Green-Kubo type formula was proved, and it was shown that the current generated by the forced flow is closely related to the strength of the force. Very recently, Chernov and Korepanov investigated in [10] the linear response for Sinai billiards under external forces (without twisting). In [13], they consider the dynamics of the Sinai billiard on the table Q, but subject to more general forces $\mathbf{P} = (\mathbf{F}, \mathbf{G})$ both during flight and at collisions. Here we characterized certain properties of the SRB measure for the forced systems by obtaining the Pesin's K-S entropy formula and Young's expression for the fractal dimension. Moreover, we also prove rigorously the Green-Kubo like formula and investigate the linear response formula.

Structure of the paper This paper is organized as follows. In Sect. 2.1 we list the main assumptions on the external forces $\mathbf{P} = (\mathbf{F}, \mathbf{G})$, introduce some basic notations and propositions of our systems. In Sect. 2.2 we state the main theorems about the linear responses and statistical properties of our forced billiard system. We also list quite a few examples of the external forces and study the current generated by these forced systems. Then we divide the analysis of the forced systems into two steps: in Sect. 3 we study the effect of the force $\mathbf{F}$ during the flight, and in Sect. 4 we add the twisting effect and conclude the proof of our main theorems.

2 Main Results

2.1 Assumptions

In this subsection we first state the assumptions on the model, which combine the assumptions in [5, 13, 27]. Let $\mathbf{P} = (\mathbf{F}, \mathbf{G})$, where $\mathbf{F}$ and $\mathbf{G}$ are the two external forces during the flight and right after the reflection, respectively. Let $\Phi_{\mathbf{P}}$ be the induced billiard flow on Q.

Assumption A1 (Invariant space) *The forced flow $\Phi_{\mathbf{P}}$ preserve a smooth function $\mathcal{E}(\mathbf{q},\mathbf{p})$, such that the level surface $\mathcal{M} := \{\mathcal{E}(\mathbf{q},\mathbf{p}) = c\}$ is a compact* 3*-D manifold, for some $c > 0$. Moreover, $\|\mathbf{p}\| > 0$ on $\mathcal{M}$, and for each $\mathbf{q} \in Q$ and $\mathbf{p} \in S^1$, the ray $\{(\mathbf{q}, t\mathbf{p}), t > 0\}$ intersects the manifold $\mathcal{M}$ in exactly one point.*

Assumption (A1) specifies an additional integral of motion, so that we only consider the restricted systems on a compact phase space. For example, we can add a Gaussian thermostat (a heat bath) to the system such that the billiard moves at a constant speed (constant temperature if there are a large number of particles). Then $\mathcal{M} := \{\|\mathbf{p}\| = c\}$ is an invariant compact level set.

Under the assumption (A1), the speed $p = \|\mathbf{p}\|$ of the billiard along any typical trajectory on $\mathcal{M}$ at time t satisfies

$$0 < p_{\min} \le p(t) \le p_{\max} < \infty,$$

for some constants $p_{\min} \le p_{\max}$. Moreover, $\mathcal{M}$ admits a global coordinate system $\{(x, y, \theta) : (x, y) \in Q, 0 \le \theta < 2\pi\}$, where θ is the angle between $\mathbf{p}$ and the positive x-axis. In particular, the speed $p = \|\mathbf{p}\|$ on $\mathcal{M}$ can be represented as a function $p = p(x, y, \theta)$. Then the velocity $\mathbf{p}$ at $\mathbf{q}$ is given by $\mathbf{p} = p\mathbf{v}$, where $\mathbf{v} = (\cos\theta, \sin\theta)$ is the unit vector in the direction of $\mathbf{p}$. So Eq. (1.1) of the dynamics between collisions can be rewritten as

$$\dot{p}\mathbf{v} + p\dot{\mathbf{v}} = \mathbf{F}. \tag{2.1}$$

Multiplying $\mathbf{v}$ to both sides of (2.1) using dot product and cross product respectively, we then get

$$\dot{p} = \mathbf{v} \cdot \mathbf{F}, \quad \text{and} \quad p\mathbf{v} \times \dot{\mathbf{v}} = \mathbf{v} \times \mathbf{F}. \tag{2.2}$$

Therefore, the equations in (1.1) have the following coordinate representations that, at any $(x, y, \theta) \in \mathcal{M}$,

$$\begin{cases} \dot{x} = p\cos\theta, \\ \dot{y} = p\sin\theta, \\ \dot{\theta} = (-F_1 \sin\theta + F_2 \cos\theta)/p. \end{cases} \tag{2.3}$$

Consider the trajectory $\tilde{\gamma} \subset \mathcal{M}$ of the flow passing through the point $(x, y, \theta) \in \mathcal{M}$, which projects down to a smooth curve $\gamma \subset Q$. Let $h = h(x, y, \theta)$ be the (signed) geometric curvature of γ at the base point $(x, y) \in Q$. Then we have that

$$h(x, y, \theta) = \pm\frac{\|\dot{\mathbf{q}} \times \ddot{\mathbf{q}}\|}{\|\dot{\mathbf{q}}\|^3} = \pm\frac{\|\mathbf{v} \times \mathbf{F}\|}{p^2} = \frac{-F_1 \sin\theta + F_2 \cos\theta}{p^2}, \tag{2.4}$$

where the sign should be chosen accordingly. Then combining with (2.3), we have

$$\dot{\theta} = ph. \tag{2.5}$$

Note that the angle $\theta = \theta(t)$ experiences a discontinuity at the times of reflection. That is, it changes from θ^- to θ^+. In the elastic collision case, all other quantities (x, y and p) stay the same. For example the speed $p(x, y, \theta^+) = p(x, y, \theta^-)$ (here $(x, y) \in \partial Q$). Under the twisting forces, all quantities are subject to change.

For any phase point $(\mathbf{q}, \theta) \in \mathcal{M}$ for the flow, let $\tau(\mathbf{q}, \theta)$ be the time for the trajectory from $(\mathbf{q}, \theta)$ to its next non-tangential collision at ∂Q.

Assumption A2 (Finite horizon) *There exist $\tau_{\max} > \tau_{\min} > 0$ such that free paths between successive reflections are uniformly bounded*: $\tau_{\min} \le \tau(\mathbf{q}, \theta) \le \tau_{\max}$, *for all* $(\mathbf{q}, \theta) \in \mathcal{M}$ *with* $\mathbf{q} \in \partial Q$. *In addition, the curvature $\mathcal{K}(r)$ of the boundary ∂Q is also uniformly bounded for all $r \in \partial Q$.*

Assumption (A2) implies that there exists a 2-D global cross-section, the post-collision space M, of the perturbed billiard flow $(\mathcal{M}, \Phi_{\mathbf{P}})$: $M = \{(\mathbf{q}, \mathbf{p}) \in \mathcal{M} : \mathbf{q} \in \partial Q, \mathbf{p} \cdot \mathbf{n}(\mathbf{q}) \ge 0\}$, which consists all outgoing vectors in $\mathcal{M}$ based at the boundary of the billiard table Q. Denote by $T_{\mathbf{P}} : M \to M$ the Poincaré map induced by the forced flow $\Phi_{\mathbf{P}}$ on $\mathcal{M}$. Moreover, the 2-D space M can be parameterized by $\mathbf{x} = (r, s)$, where r is the arc-length parameter of ∂Q oriented clockwise, and a new parameter $s = \sin\varphi$, where φ is the angle formed by the outgoing vector $\mathbf{p}$ and the normal vector $\mathbf{n}(\mathbf{q})$. This coordinate system has the advantage that the Lebesgue measure $d\mu_0 = \mathrm{Cst}\, dr\, ds$ coincides with the measure $d\nu_0 = \mathrm{Cst} \cos\varphi\, dr\, d\varphi$. Using this new coordinate system, the collision time can be written as $\tau(\mathbf{x}) = \tau(r, s)$, and the twist force $\mathbf{G}$ at the collision can be reformulated as $(\bar{r}, \bar{s}) = \mathbf{G}(r, s) = (r, s) + (g^1(r, s), g^2(r, s))$.

Assumption A3 (Smallness of the external forces) *There exists $\varepsilon > 0$ small enough, such that the forces $\mathbf{P} = (\mathbf{F}, \mathbf{G})$ and $C^{1+\alpha}$, for some $\alpha > 0$, and satisfy*

$$\|\mathbf{F}\|_{C^1} < \varepsilon, \|\mathbf{G} - \mathrm{Id}_M\|_{C^1} < \varepsilon.$$

Moreover, we assume that $\mathbf{G}$ preserves tangential collisions: $\mathbf{G}(r, \pm 1) = (r, \pm 1)$. *In other words, $g^i(r, \pm 1) = 0$ for each $i = 1, 2$.*

In particular, the singularity set of $T_{\mathbf{P}}^{-1}$ is the same as that of untwisted map $T_{\mathbf{F}}^{-1}$. The regularity assumption $C^{1+\alpha}$ enable us to include some common physical models. For example, *Weeks-Chandler-Anderson potential*, given by $\phi(r) = 4\epsilon((\frac{\sigma}{r})^{12} - (\frac{\sigma}{r})^6) + \epsilon$ for $r \le 2^{1/6}\sigma$; and $\phi(r) = 0$ for $r > 2^{1/6}\sigma$, which is $C^{1+\alpha}$, for some $\alpha < 1$. See also [18]

Remark 1 Note that (A2) also put some constrains on the smallness of forces. In fact, the existence of $\tau_{\min}$ not only prevents touching scatterers, but also implies the trajectory can't be bent too much such that the particle falls back to the same scatterer immediately.

Let $\mathcal{I} : \mathcal{M} \to \mathcal{M}$ be the reversal transformation (also called involution), which is defined by $\mathcal{I}(x, y, \theta) = (x, y, \pi + \theta)$. Let $\Phi_t : \mathcal{M} \to \mathcal{M}$ be a general flow. The reversed flow of Φ is defined by $\Phi_t^- = \mathcal{I} \circ \Phi_{-t} \circ \mathcal{I}$. Then the flow Φ is said to be *time-reversible*, if $\Phi_t^- = \Phi_t$. It is well known that the unforced billiard flow is time-reversible.

Assumption A4 (Time-reversibility) *Both forces $\mathbf{F}$ and $\mathbf{G}$ are stationary, and the forced billiard flow $\Phi_{\mathbf{P}}$ is time-reversible.*

Let $\varepsilon > 0$, $\tau_* \in (0, 1)$ and $\mathcal{F}(Q, \tau_*, \varepsilon)$ be the collection of all forced billiard maps defined by the dynamics (1.1) and (1.2) under the external forces $\mathbf{P} = (\mathbf{F}, \mathbf{G})$ that satisfying the assumptions (A1)–(A4) with $\tau_* \le \tau_{\min} \le \tau_{\max} \le \tau_*^{-1}$. The following lemma was proved by Demers and Zhang in [13]:

Lemma 2.1 *Each map $T \in \mathcal{F}(Q, \tau_*, \varepsilon)$ preserves a unique SRB measure (the nonequilibrium steady state) μ_T that is mixing, Bernoulli and positive on open sets in M. Let $\mathcal{H}$ be* *the collection of all piecewise Hölder continuous functions on M where the discontinuities occur on the singularity sets T. Then*

a. (*Equidistribution*) *for any $f \in \mathcal{H}$, $T^n \mu_0(f) \to \mu_T(f)$ at an exponential rate*;
b. (*Decay of Correlations*) *for any $f, g \in \mathcal{H}$, $\mu_0(f \circ T^n \cdot g) \to \mu_T(f) \cdot \mu_0(g)$ at an exponential rate.*
c. (*Central Limit Theorem*) *for any $f \in \mathcal{H}$, $f_n = f + f \circ T + \cdots + f \circ T^{n-1}$, then $\frac{f_n}{\sqrt{n}} \Rightarrow \mathcal{N}(\mu_T(f), \sigma_f^2)$, where $\sigma_f^2 = \sum_{n\in\mathbb{Z}} \mathcal{C}_{f,f}(T^n)$. Here the convergence means that the distributions of $\frac{f_n}{\sqrt{n}}$ converge to the normal distribution $\mathcal{N}(\mu_T(f), \sigma_f^2)$.*

Note that there is one-to-one correspondence between the invariant measures of the billiard map and the invariant measures of the billiard flow (following the general construction of suspension flows). Let $\hat{\mu}_T$ be the corresponding measure of μ_T with respect to the forced flow Φ, which is also a mixing SRB measure.

2.2 Main Results

In this section we state the main results of this paper, the properties of forced billiard systems under the assumptions (A1)–(A4). More precisely, let $\varepsilon > 0$ be small enough, $\tau_* \in (0, 1)$ and $\mathbf{P} = (\mathbf{F}, \mathbf{G})$ be an ε-small external force pair such that $T_{\mathbf{P}} \in \mathcal{F}(Q, \tau_*, \varepsilon)$, $\mu_{\mathbf{P}}$ be the SRB measure on M given in Theorem 2.1. Let $\widetilde{Q}$ be the $\mathbb{Z}^2$-periodic table on the plane, $\widetilde{T}_{\mathbf{P}}$ be the forced collision map on the $\mathbb{Z}^2$-periodic collision space $\widetilde{M}$, and

$$\widetilde{\mathbf{\Delta}}_{\mathbf{P}}(\mathbf{x}) = (\widetilde{\Delta}_{x,\mathbf{P}}, \widetilde{\Delta}_{y,\mathbf{P}}) = \tilde{\pi} \circ \widetilde{T}_{\mathbf{P}}(\mathbf{x}) - \tilde{\pi}(\mathbf{x}) \tag{2.6}$$

be the displacement vector between collisions, where $\tilde{\pi}$ is the projection from $\widetilde{M}$ to the base point in $\widetilde{Q}$. It is easy to see that $\widetilde{\mathbf{\Delta}}_{\mathbf{P}}$ is $\mathbb{Z}^2$-periodic, which induces a displacement function on the collision space M, say $\mathbf{\Delta}_{\mathbf{P}}$. Our first theorem describes estimations on the current for the discrete system.

Theorem 2.2 (a) *The discrete-time steady state electrical current is well-defined and given by*:

$$\mathbf{J}_{\mathbf{P}} = \lim_{n\to\infty} \tilde{\mathbf{q}}_n / n = \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}}). \tag{2.7}$$

(b) *The current $\mathbf{J}_{\mathbf{P}}$ satisfies $\mathbf{J}_{\mathbf{P}} = \varepsilon \boldsymbol{\sigma} + o(\varepsilon)$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ is uniformly bounded and satisfies*

$$\sigma_a = \frac{1}{2}\mu_0(\Delta_{a,\mathbf{P}} \cdot H) + \sum_{k=1}^{\infty} \mu_0\big[\big(\Delta_{a,\mathbf{P}} \circ T_0^k\big) \cdot H\big], \quad a \in \{x, y\}, \tag{2.8}$$

where $H(r,s) = (2 - \exp(\int_0^{\tau_{\mathbf{F}}(r,s)} ph_\theta \, dt) - \mathcal{J}_{\mathbf{G}}(r_1, s_1))/\varepsilon$ is a uniformly bounded function, $\mathcal{J}_{\mathbf{G}}$ is the Jacobian of the twisting force $\mathbf{G}$, and (r_1, s_1) is the intermediate position right before the twist.
(c) *As $n \to \infty$,*

$$\frac{\tilde{\mathbf{q}}_n - n\mathbf{J}_{\mathbf{P}}}{\sqrt{n}} \Rightarrow \mathcal{N}(0, \mathbf{D}_{\mathbf{P}}), \tag{2.9}$$

where $\mathbf{D}_{\mathbf{P}}$ is the discrete-time diffusion matrix of the Lorentz particle, which is given by:

$$\mathbf{D}_{\mathbf{P}} = \sum_{n=-\infty}^{\infty} \big[\mu_{\mathbf{P}}\big(\mathbf{\Delta}_{\mathbf{P}} \circ T_{\mathbf{P}}^n \otimes \mathbf{\Delta}_{\mathbf{P}}\big) - \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}}) \otimes \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}})\big] \tag{2.10}$$

Moreover $\mathbf{D}_\mathbf{P}$ *is continuous with respect to the size of the force pair* $\mathbf{P}$ *at* $\mathbf{P}=0$:

$$\mathbf{D}_\mathbf{P} = \mathbf{D}_0 + o(1). \tag{2.11}$$

In particular, the generated current $\mathbf{J}_\mathbf{P}$ is comparable to the size of the external force $\mathbf{P}$, where $\boldsymbol{\sigma}$ resembles the electric conductivity of the forced system. Moreover, it follows from Eq. (2.9) that the drift effect is sub-linear (of order $\sqrt{n}$). One may wonder if we could use the linear approximates $\int_0^{\tau_\mathbf{F}(r,s)} ph_\theta\, dt$ for the function H in (b). Indeed, this may destroy the convergence of the series (2.8) (see Remark 6 for more discussions). However, we do use the linear term for all physical models with a *Gaussian thermostat* (see Proposition 2.6, 2.7 and Corollary 2.8).

Remark 2 It would be natural and elegant to replace $\Delta_{a,\mathbf{P}}$ by $\Delta_{a,0}$ in the definitions Eq. (2.8) of σ_a. However, we don't have a satisfied estimate of the dependence of $\Delta_{a,\mathbf{P}}$ on the force $\mathbf{P}$ (besides Lemma 4.4, which is too rough to use here). We have a similar situation when defining $\hat{\mathbf{J}}_\mathbf{P}$ in Eq. (2.13).

The corresponding results for the continuous-time forced system is provided by the following theorem.

Theorem 2.3 *Suppose a particle move in the domain* $\widetilde{Q}$ *under the external force* $\mathbf{P} = (\mathbf{F}, \mathbf{G})$.

(a) *The steady state current generated by* $\Phi_\mathbf{P}$ *is well-defined and given by*

$$\hat{\mathbf{J}}_\mathbf{P} = \lim_{t\to\infty} \tilde{\mathbf{q}}(t)/t = \mu_\mathbf{P}(\boldsymbol{\Delta}_\mathbf{P})/\mu_\mathbf{P}(\tau_\mathbf{P}). \tag{2.12}$$

(b) *The current* $\hat{\mathbf{J}}_\mathbf{P}$ *satisfies*

$$\hat{\mathbf{J}}_\mathbf{P} = \varepsilon \cdot \frac{\boldsymbol{\sigma}}{\bar{\tau}_\mathbf{P}} + o(\varepsilon). \tag{2.13}$$

(c) *As* $t \to \infty$,

$$\frac{\tilde{\mathbf{q}}(t) - \hat{\mathbf{J}}_\mathbf{P} t}{\sqrt{t}} \Rightarrow \mathcal{N}(0, \hat{\mathbf{D}}_\mathbf{P}), \tag{2.14}$$

where $\hat{\mathbf{D}}_\mathbf{P}$ *is the continuous-time diffusion matrix of the Lorentz particle.*

(d) *The diffusion matrix is continuous with respect to the size of force pair* $\mathbf{P}$ *at* $\mathbf{P}=0$:

$$\hat{\mathbf{D}}_\mathbf{P} = \hat{\mathbf{D}}_0 + o(1) \tag{2.15}$$

We also give some characterizations of the nonequilibrium steady state $\mu_\mathbf{P}$ of the forced system $T_\mathbf{P}$:

Theorem 2.4 (1) *The measure* $\mu_\mathbf{P}$ *satisfies the Pesin entropy formula*:

$$h_{\mu_\mathbf{P}}(T_\mathbf{P}) = \lambda^u_\mathbf{P}, \tag{2.16}$$

where $\lambda^s_\mathbf{P} < 0 < \lambda^u_\mathbf{P}$ *are the Lyapunov exponents for the measure* $\mu_\mathbf{P}$ *and* $h_{\mu_\mathbf{P}}(T_\mathbf{P})$ *is the metric entropy for* $(T_\mathbf{P}, \mu_\mathbf{P})$;

(2) $\mu_{\mathbf{P}}$ *satisfies Young's dimension formula*:

$$\mathrm{HD}(\mu_{\mathbf{P}}) = h_{\mu_{\mathbf{P}}}(T_{\mathbf{P}})\left(\frac{1}{\lambda^u_{\mathbf{P}}} - \frac{1}{\lambda^s_{\mathbf{P}}}\right), \tag{2.17}$$

where $\mathrm{HD}(\mu_{\mathbf{P}})$ *is the Hausdorff dimension of the measure* $\mu_{\mathbf{P}}$.

(3) *Let* $h_0 = h_{\mu_0}(T_0)$ *be the metric entropy of* T_0, *then*

$$\mathrm{HD}(\mu_{\mathbf{P}}) = 2 - \varepsilon^2 \cdot \frac{\sigma_H^2}{2h_0} + o(\varepsilon^2), \quad \textit{where} \quad \sigma_H^2 = \sum_{k=-\infty}^{\infty} \mu_0\big[H \circ T_0^k \cdot H\big]. \tag{2.18}$$

It follows from Eq. (2.18) that $1 < \mathrm{HD}(\mu_{\mathbf{P}}) < 2$ for some small external forces $\mathbf{P}$. So $\mu_{\mathbf{P}}$ is singular with respect to the Lebesgue measure and admits a fractal structure.

Let $\hat{\mu}_{\mathbf{P}}$ be the corresponding SRB measure preserved by the forced flow $\Phi^t_{\mathbf{P}}$ on $\mathcal{M}$. The metric entropy of the measure $\hat{\mu}_{\mathbf{P}}$ is given by $h_{\hat{\mu}_{\mathbf{P}}}(\Phi_{\mathbf{P}}) := h_{\mu_{\mathbf{P}}}(T_{\mathbf{P}})/\mu_{\mathbf{P}}(\tau_{\mathbf{P}})$ and the fractal dimension $\mathrm{HD}(\hat{\mu}_{\mathbf{P}}) = \mathrm{HD}(\mu_{\mathbf{P}}) + 1$. Therefore similar formulas in Theorem 2.4 hold for $\hat{\mu}_{\mathbf{P}}$.

2.3 Applications

Next we provide some example of reversible external forces and give the generated currents by the forced billiard systems.

Example 1 (Conservative forces on Q) We consider a conservative force $\mathbf{F} = -\nabla U(\mathbf{q}) = -(U_x, U_y)$, where $U(\mathbf{q})$ is a (small) potential function on Q. The induced billiard system $\Phi_{\mathbf{F}}$ preserves the total energy $E = \frac{1}{2}p^2 + U(\mathbf{q})$ of the system and hence satisfies Assumption (A1). We restrict the dynamics to a energy level $\mathcal{M} = \{(\mathbf{q}, \mathbf{p}) : E(\mathbf{q}, \mathbf{p}) = 1/2\}$. In particular the speed function $p^2(x, y, \theta) = 1 - 2U(\mathbf{q})$ is independent of θ, say $p = p(x, y)$. It is well known that the billiard flow $\Phi_{\mathbf{F}}$ is time-reversible (see also Lemma 3.1 and Remark 4). Moreover, it is easy to see that the generating vector field $X = \langle p\cos\theta, p\sin\theta, ph\rangle$ is divergence-free:

$$\operatorname{div} X = p_x \cos\theta + p_y \sin\theta + (U_x \cos\theta + U_y \sin\theta)/p = \frac{1}{p}(\dot{p} - \mathbf{v}\cdot\mathbf{F}) = 0. \tag{2.19}$$

In particular, the flow $\Phi_{\mathbf{F}}$ preserves the Lebesgue measure m on the energy level $\mathcal{M}$ and $\hat{\mu}_{\mathbf{F}} = m$. Since there is no slip after the collision, the current is indeed zero (see the related discussion in Remark 7):

$$\mathbf{J}_{\mathbf{F}} = m(\mathbf{p}) = \int_Q p(x, y)\left(\int_{S^1} \mathbf{v}\, d\theta\right) dx\, dy = 0. \tag{2.20}$$

Example 2 (Conservative twists and soft scatterers) Let $\mathbf{B}$ be a small scatterer on Q, U a potential function on $\mathbb{T}^2$ such that $U = 0$ on $Q\backslash\mathbf{B}$ and $U > 0$ on $\mathbf{B}$. We consider the energy surface $E(\mathbf{q}, \mathbf{p}) = \frac{1}{2}\|\mathbf{p}\|^2 + U(\mathbf{q}) = 1$. So a running billiard may climb up the scatterer, and regain its full kinetic energy whenever it exits that scatterer. If we view what happened on the scatterer as a black box, the reduced dynamics is close to the unforced system during its free flight. The only difference is that the exit location and direction are different from the elastic collision (corresponding to $U = +\infty$ on $\mathbf{B}$). Therefore, the effect is equivalent to applying a twist force $\mathbf{G}$ right after the collision. Clearly the perturbed flow Φ is time-reversible. Moreover, by embedding Φ into the real Hamiltonian flow on the ambient torus, we see that $\Phi_{\mathbf{G}}$ preserves the canonical space $\mathcal{M}_0$ and the Liouville measure on $\mathcal{M}_0$. In particular its current $\mathbf{J}_{\mathbf{G}} = 0$.

To get an intuition of the vanishing currents, we consider a special case explicitly.

Proposition 2.5 *Suppose all the scatterers $\mathbf{B}_i$ are all round disks centered at $\mathbf{q}_i = (x_i, y_i)$ of radius δ_i, the potential is given by $U(\mathbf{q}) = \varepsilon^{-1} \cdot (\delta_i - \|\mathbf{q} - \mathbf{q}_i\|)$ on $\mathbf{B}_i$, and $U = 0$ outside the scatterers. Then the forced flow has vanishing current.*

Proof In the case it is easy to see that the outgoing direction is the same as the hard scatterer case, the slip does not depend on where the collision happens: $\mathbf{G}(r, s) = (r + g_i(s), s)$. So the Jacobian $\mathcal{J}_{\mathbf{G}} \equiv 1$. Therefore the current must be zero. □

Example 3 (Isokinetic forces) Next we consider forces $\mathbf{F}$ that are always perpendicular to the momentum. In this case, the forced billiard flow preserves the kinetic energy. Without loss of generality, we assume $\|\mathbf{p}\| = 1$ (as in the classical billiards), and reformulate the force as

$$\mathbf{F} = F\mathbf{v}^{\perp} = F(x, y, \theta)(-\sin\theta, \cos\theta),$$

where $F = F(x, y, \theta)$ is a scalar function (may be negative) with $\|F\|_{C^1} < \varepsilon$. By Eq. (2.4), the geometric curvature $h(x, y, \theta)$ satisfies

$$h(x, y, \theta) = F(x, y, \theta)\big(\sin^2\theta + \cos^2\theta\big)/p^2 = F(x, y, \theta). \tag{2.21}$$

Not all isokinetic forced systems are time-reversible. For example, the system of an electron moving under a constant magnetic field perpendicular to Q is not time-reversible.

In the following we assume that F satisfies $F(x, y, \theta + \pi) = -F(x, y, \theta)$. Then the forced billiard flow $\Phi_{\mathbf{F}}$ is time-reversible (see Lemma 3.1). A special feature of these isokinetic forces is that we can use the linear term of H to estimate the generated current:

Proposition 2.6 *The discrete-time steady state current under a general isokinetic force $\mathbf{F}$ is given by $\mathbf{J}_{\mathbf{F}} = \varepsilon\boldsymbol{\sigma} + o(\varepsilon)$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ is given by*

$$\sigma_a = -\frac{1}{2}\mu_0\left[\Delta_{a,\mathbf{F}} \cdot \int_0^{\tau_{\mathbf{F}}(\mathbf{x})} F_\theta\, dt/\varepsilon\right] - \sum_{k=1}^{\infty}\mu_0\left[\big(\Delta_{a,\mathbf{F}} \circ T_0^k\big) \cdot \int_0^{\tau_{\mathbf{F}}(\mathbf{x})} F_\theta\, dt/\varepsilon\right], \quad a \in \{x, y\}.$$

Example 4 (Electric field with thermostat) We consider an electric field $\mathbf{E}(\mathbf{q}) = (\varepsilon e_1(\mathbf{q}), \varepsilon e_2(\mathbf{q}))$ on $\mathbf{q} \in Q$. It may generate a net velocity in the force direction and keep accelerating the electron. We can modify the system by adding a constraining force to maintain the system at a constant temperature (a compact level set, say $\mathcal{E}_1 = \{\|\mathbf{p}\| \equiv 1\}$), and to preserve a steady state on that level. More precisely, the system of the forced equations on $\mathcal{E}_1$ is given by: $\dot{\mathbf{q}} = \mathbf{p}$, $\dot{\mathbf{p}} = \mathbf{E} - \alpha\mathbf{p}$, where $\alpha = \mathbf{E} \cdot \mathbf{p}$ is a thermostat. For such systems, we have the following results.

Proposition 2.7 *Let $\mathbf{E}(\mathbf{q}) = (\varepsilon e_1(\mathbf{q}), \varepsilon e_2(\mathbf{q}))$ be an electric field on $\mathbf{q} \in Q$. The discrete-time steady state electrical current under $\mathbf{E}$ with thermostat is given by $\mathbf{J}_{\mathbf{E}} = \varepsilon\boldsymbol{\sigma} + o(\varepsilon)$, where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ is given by*

$$\sigma_a = \frac{1}{2}\mu_0\left(\Delta_{a,\mathbf{E}} \cdot \int_0^{\tau_{\mathbf{E}}} (e_1, e_2) \cdot \mathbf{p}\, dt\right) + \sum_{k=1}^{\infty}\mu_0\left[\big(\Delta_{a,\mathbf{E}} \circ T_0^k\big) \cdot \int_0^{\tau_{\mathbf{E}}} (e_1, e_2) \cdot \mathbf{p}\, dt\right].$$

Now we consider a constant electric field $\mathbf{E} = (\varepsilon e_1^0, \varepsilon e_2^0)$ on the table Q. In this case it is easy to see that $\int_0^{\tau_{\mathbf{E}}(r,s)} (e_1^0, e_2^0) \cdot \mathbf{p} dt = (e_1^0, e_2^0) \cdot \boldsymbol{\Delta}_{\mathbf{E}}(r,s)$. So we have

Corollary 2.8 *Let* $\mathbf{E}(\mathbf{q}) = (\varepsilon e_1^0, \varepsilon e_2^0)$ *be a constant electric field on* $\mathbf{q} \in Q$. *The discrete-time steady state electrical current under* $\mathbf{E}$ *with thermostat is given by* $\mathbf{J}_{\mathbf{E}} = \varepsilon \boldsymbol{\sigma} + o(\varepsilon)$, *where* $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ *is given by*

$$\sigma_a = \frac{1}{2}\mu_0(\Delta_{a,\mathbf{E}} \cdot H) + \sum_{k=1}^{\infty} \mu_0\big[\big(\Delta_{a,\mathbf{E}} \circ T_0^k\big) \cdot H\big], \quad a \in \{x, y\},$$

where $H(r,s) = (e_1^0, e_2^0) \cdot \boldsymbol{\Delta}_{\mathbf{E}}(r,s)$.

This is the current formula obtained in [8].

3 Preliminary Properties of $T_{\mathbf{F}}$ Under the Force F

We divide our study of the forced system $T_{\mathbf{P}}$ into two steps according to the nature of the force pair **P**: the step of a elastic reflection under the force **F**, and the twisting by **G** right after the collision. In this section we consider the pre-twist step, that is, the effect of the force **F** between one collision. First we state the time-reversibility of the forced system:

Lemma 3.1 *Let* **F** *be an external force on the table* Q, p *and* h *be the speed and the curvature functions of the forced billiard flow* $\Phi_{\mathbf{F}}$. *Then* $\Phi_{\mathbf{F}}$ *is time-reversible if and only if the following conditions hold for any* $(x,y,\theta) \in \mathcal{M}$:

$$p(x,y,\pi+\theta) = p(x,y,\theta), \tag{3.1a}$$

$$h(x,y,\pi+\theta) = -h(x,y,\theta). \tag{3.1b}$$

The proof is straightforward and omitted here. See also [5, pp. 209–210] for detailed discussions.

Remark 3 There is a canonical involution I on the post-collision space M, which is given by $I : M \to M$, $(r,s) \mapsto (r,-s)$. Let $T_{\mathbf{F}}^-$ be the Poincaré map of the reversed flow $\Phi_{\mathbf{F}}^-$. It is easy to see that $T_{\mathbf{F}}^- = I \circ T_{\mathbf{F}}^{-1} \circ I$. So time-reversibility of the forced flow $\Phi_{\mathbf{F}}^- = \Phi_{\mathbf{F}}$ implies the time-reversibility of the forced map: $T_{\mathbf{F}}^- = T_{\mathbf{F}}$.

Remark 4 A special case is that the force $\mathbf{F} = \mathbf{F}(\mathbf{q})$ depends only on the position **q**. For example, for a particle moving in the gravity field and an electron moving in an electric field, the forces do not depend on the velocity **p**. According to Eq. (2.4), we see that (3.1b) follows from (3.1a). So these forced systems are time-reversible if and only if $p(x,y,\pi+\theta) = p(x,y,\theta)$ for any $(x,y,\theta) \in \mathcal{M}$.

Lemma 3.2 *Let* m *be the Lebesgue measure on* M, $\mathbf{x} = (r,s)$ *and* $T_{\mathbf{F}}\mathbf{x} = (r_1, s_1)$. *Then the Jacobian of* $DT_{\mathbf{F}}$ *with respect to the Lebesgue measure* μ_0 *is given by*

$$\det D_{\mathbf{x}} T_{\mathbf{F}} = \exp\left(\int_0^{\tau_{\mathbf{F}}(\mathbf{x})} p h_\theta dt\right). \tag{3.2}$$

Proof Let $X(x,y,\theta)=\langle p\cos\theta, p\sin\theta, ph\rangle$ be the vector field on $\mathcal{M}$ that generates the flow $\Phi_{\mathbf{F}}^t$. Consider the Lebesgue measure $dm=dx\,dy\,d\theta$ on $\mathcal{M}$. Note that m is not necessarily invariant under the forced flow $\Phi_{\mathbf{F}}$, and its rate of change is given by the divergence of the generating vector field X:

$$\begin{aligned}\operatorname{div} X(x,y,\theta) &= p_x\cos\theta+p_y\sin\theta+p_\theta h+ph_\theta=(p_x\dot{x}+p_y\dot{y}+p_\theta\dot{\theta})/p+ph_\theta\\ &=\frac{d\ln p}{dt}+ph_\theta.\end{aligned}$$

Note that the force billiard flow can also be represented as a suspension of the forced map $(M, T_{\mathbf{F}}, \mu_0)$ with respect to the roof function $\tau_{\mathbf{F}}$. Along the suspension direction, the arc length differential $d\ell=pdt$, where p is the speed of the flow. In particular, the volume $dm=dx\,dy\,d\theta$ on $\mathcal{M}$ has the suspension form: $dm=\mathrm{Cst}\cdot d\mu_0\,d\ell=\mathrm{Cst}\cdot p\cdot d\mu_0\,dt$. Clearly the t-direction is invariant. So we have

$$\frac{dT_{\mathbf{F}}^{-1}\mu}{d\mu}(\mathbf{x})=\frac{p(\mathbf{x})}{p(T_{\mathbf{F}}\mathbf{x})}\exp\left(\int_0^{\tau_{\mathbf{F}}(\mathbf{x})}\operatorname{div} X\left(\Phi_{\mathbf{F}}^t\mathbf{x}\right)dt\right)=\exp\left(\int_0^{\tau_{\mathbf{F}}(\mathbf{x})}ph_\theta dt\right). \tag{3.3}$$

This finishes the proof of the lemma. □

4 Properties of $T_{\mathbf{P}}=T_{(\mathbf{F},\mathbf{G})}$

We consider the twisting step of a force pair $\mathbf{P}=(\mathbf{F},\mathbf{G})$ satisfying the assumptions (A1)–(A4) with $\|\mathbf{F}\|_{C^1}<\varepsilon$ and $\|\mathbf{G}-\mathrm{Id}_M\|_{C^1}<\varepsilon$. Our main approach is to compare the combined effect $T_{\mathbf{P}}\mathbf{x}=\mathbf{G}\circ T_{\mathbf{F}}\mathbf{x}$ with the pre-twist map $T_{\mathbf{F}}$.

In Lemma 3.1 we gave a characterization for the forced flow $\Phi_{\mathbf{F}}$ to be time-reversible. To ensure that $\Phi_{\mathbf{P}}$ is time-reversible, it suffices to know if the twisting process is also time-reversible. More precisely, let $S=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$, and $G=\begin{pmatrix}\mathbf{0}&S\\S&\mathbf{0}\end{pmatrix}$, where $\mathbf{0}$ is the 2×2 zero matrix. Then we have the following result.

Lemma 4.1 *Assume the condition* (3.1a), (3.1b) *hold. Then $\Phi_{\mathbf{P}}$ is time-reversible if and only if the graph of* $\mathbf{G}$ *is G-invariant.*

Proof Let $T_{\mathbf{P}}\mathbf{x}=\mathbf{G}\circ T_{\mathbf{F}}\mathbf{x}$ be decomposition of the forced collision map, $\Phi_{\mathbf{P}}^-$ be the time-reversal flow of $\Phi_{\mathbf{P}}$, and $T_{\mathbf{P}}^-$ be the induced Poincaré map of $\Phi_{\mathbf{P}}^-$. Let $\mathcal{I}:\mathcal{M}\to\mathcal{M}$ be the involution on $\mathcal{M}$ and $I:M\to M$ be the induced involution on M.

Under the assumption (3.1a), (3.1b), we have that $T_{\mathbf{F}}^-=T_{\mathbf{F}}$ and

$$T_{\mathbf{P}}^-\mathbf{x}=I\circ\mathbf{G}^{-1}\circ T_{\mathbf{F}}^{-1}\circ I(\mathbf{x})=I\circ\mathbf{G}^{-1}\circ I\circ I\circ T_{\mathbf{F}}^{-1}\circ I(\mathbf{x})=\mathbf{G}^-\circ T_{\mathbf{F}}^-(\mathbf{x})=\mathbf{G}^-(T_{\mathbf{F}}\mathbf{x}),$$

where $\mathbf{G}^-=I\circ\mathbf{G}^{-1}\circ I$. In order to have $T_{\mathbf{P}}^-=T_{\mathbf{P}}$, a necessary and sufficient condition is $\mathbf{G}^-=\mathbf{G}$.

Let $(\bar{r},\bar{s})=\mathbf{G}(r,s)$. Then $\mathbf{G}^-(\bar{r},-\bar{s})=I\circ\mathbf{G}^{-1}(\bar{r},\bar{s})=I(r,s)=(r,-s)$. So $\mathbf{G}^-=\mathbf{G}$ is equivalent to $\mathbf{G}(\bar{r},-\bar{s})=(r,-s)$. Combining with the assumption $(\bar{r},\bar{s})=\mathbf{G}(r,s)$, we see that $\mathbf{G}^-=\mathbf{G}$ is equivalent to the graph of $\mathbf{G}$ being G-invariant. □

Remark 5 In above lemma we take the Poincaré map $T_{\mathbf{P}}^-$ of the time-reversed flow, which can be viewed as the *physically* time-reversal of the map $T_{\mathbf{P}}$. We can also define a *formally* time-reversed map of the forced map $T_{\mathbf{P}}$ by $\widehat{T}_{\mathbf{P}}^-:=I\circ T_{\mathbf{P}}^{-1}\circ I$. Generally speaking, this formally reversed map $\widehat{T}_{\mathbf{P}}^{-}$ may not coincide with the Poincaré map $T_{\mathbf{P}}^{-}$ of the time-reversed flow $\Phi_{\mathbf{P}}^{-}$ (after we apply a twist force right after the elastic collision). In this paper we always use the physically time-reversal definition.

By Assumption (A3), the twist force $\mathbf{G}$ preserves tangential collisions. Therefore, the discontinuity set of $T_{\mathbf{P}}$ is the same as that of $T_{\mathbf{F}}$, which comprises the preimage of $\mathcal{S}_0 := \{s = \pm\pi/2\}$. Similarly, the singularity sets of $T_{\mathbf{P}}^{-1}$ and $T_{\mathbf{F}}^{-1}$ are the same due to (A3). But the singular sets for higher iterates are no longer the same. Let $\mathcal{S}_{\pm n}^{\mathbf{P}} = \cup_{i=0}^{n} T_{\mathbf{P}}^{\mp i}\mathcal{S}_{0,H}$ with $n \in \mathbb{N}$. Then $T_{\mathbf{P}}^{\pm n}$ is smooth on all the cells of $M \setminus \mathcal{S}_{\pm n}^{\mathbf{P}}$.

For any phase point $\mathbf{x} = (r, s) \in M$, let $T_{\mathbf{F}}\mathbf{x} = (r_1, s_1)$ and $T_{\mathbf{P}}\mathbf{x} = (\bar{r}_1, \bar{s}_1)$. According to the discussion between (A2) and (A3), we express the twist force $\mathbf{G}$ in local coordinates via two smooth functions g^1 and g^2 such that

$$(\bar{r}_1, \bar{s}_1) = \mathbf{G}(r_1, s_1) = (r_1, s_1) + \big(g^1(r_1, s_1), g^2(r_1, s_1)\big). \tag{4.1}$$

Note that g^i is a $C^{1+\alpha}$ function whose C^1 norm is uniformly bounded from above by $c \cdot \varepsilon$, for some uniform constant $c > 0$. Moreover $g^i(r, \pm 1) = 0$, $i = 1, 2$.

According to (4.1), the differential of the twisting force $\mathbf{G}$ satisfies

$$\begin{cases} d\bar{r}_1 = (1 + g_1^1(r_1, s_1))dr_1 + g_2^1(r_1, s_1)ds_1, \\ d\bar{s}_1 = g_1^2(r_1, s_1)dr_1 + (1 + g_2^2(r_1, s_1))ds_1, \end{cases} \tag{4.2}$$

where $g_1^i = \partial g^i/\partial r$ and $g_2^i = \partial g^i/\partial s$. So the differential of the map $T_{\mathbf{P}}$ is given by

$$D_{\mathbf{x}}T_{\mathbf{P}} = D_{T_{\mathbf{F}}\mathbf{x}}\mathbf{G} \circ D_{\mathbf{x}}T_{\mathbf{F}}. \tag{4.3}$$

Note that $T_{\mathbf{P}}$ may not be a C^1-perturbation of $T_{\mathbf{F}}$, since $T_{\mathbf{F}}$ is unbounded around the boundary of M. However, it follows from Eq. (4.3) that

$$\det D_{\mathbf{x}}T_{\mathbf{P}} = \mathcal{J}_{\mathbf{G}}(r_1, s_1) \cdot \det D_{\mathbf{x}}T_{\mathbf{F}} = \big(1 + \hat{g}(r_1, s_1)\big) \cdot \det D_{\mathbf{x}}T_{\mathbf{F}}, \tag{4.4}$$

where

$$\begin{aligned} \hat{g}(r_1, s_1) &= \mathcal{J}_{\mathbf{G}}(r_1, s_1) - 1 \\ &= g_1^1(r_1, s_1) + g_2^2(r_1, s_1) + g_1^1(r_1, s_1)g_2^2(r_1, s_1) - g_2^1(r_1, s_1)g_1^2(r_1, s_1). \end{aligned} \tag{4.5}$$

Note that $\mathcal{J}_{\mathbf{G}}$ is a C^1 function with $\hat{g}(r_1, s_1) = \mathcal{O}(\varepsilon)$. So we have that

$$\det D_{\mathbf{x}}T_{\mathbf{P}} = \big(1 + \hat{g}(r_1, s_1)\big) \cdot \exp\left(\int_0^{\tau_{\mathbf{F}}(\mathbf{x})} ph_\theta dt\right) = 1 + \mathcal{O}(\varepsilon). \tag{4.6}$$

Once again, let $\mu_0 = \mathrm{Cst}\, dr\, ds$ be the normalized Lebesgue measure on M, $\mathcal{J}_{\mathbf{P}}(\mathbf{x}) := dT_{\mathbf{P}}^{-1}\mu_0/d\mu_0(\mathbf{x})$ be the density function defined by $T_{\mathbf{P}}$. Note that $\mathcal{J}_{\mathbf{P}}(\mathbf{x}) = \det D_{\mathbf{x}}T_{\mathbf{P}}$. So we have:

Lemma 4.2 *Let $\mathbf{P} = (\mathbf{F}, \mathbf{G})$ be an ε-small force pair, and $T_{\mathbf{P}}$ the forced collision map. Let $\mathbf{x} = (r, s) \in M$, $T_{\mathbf{F}}\mathbf{x} = (r_1, s_1)$ and $T_{\mathbf{P}}\mathbf{x} = (\bar{r}_1, \bar{s}_1)$. Then the Jacobian $\mathcal{J}_{\mathbf{P}}(\mathbf{x})$ satisfies*

$$1 - \mathcal{J}_{\mathbf{P}}(\mathbf{x}) = \varepsilon H(\mathbf{x}) + \varepsilon^2 R_{\mathbf{P}}, \tag{4.7}$$

where

$$H(r,s)=\frac{1}{\varepsilon}\left(2-\exp\left(\int_0^{\tau_{\mathbf{F}}(r,s)} ph_\theta dt\right)-\mathcal{J}_{\mathbf{G}}(T_{\mathbf{F}}\mathbf{x})\right)$$
$$=\frac{1}{\varepsilon}\left(1-\exp\left(\int_0^{\tau_{\mathbf{F}}(r,s)} ph_\theta dt\right)-\hat{g}(r_1,s_1)\right).$$

Moreover, $\mu_0(H)=0$, *and both* H *and* $R_{\mathbf{P}}$ *are uniformly bounded and* C^α *on each component of* $M\backslash\mathcal{S}_1^T$.

Proof According to Chain Rule and Lemma 3.2, we have

$$\mathcal{J}_{\mathbf{P}}(\mathbf{x})=\frac{dT_{\mathbf{P}}^{-1}\mu_0}{d\mu_0}(\mathbf{x})=\frac{dT_{\mathbf{P}}^{-1}\mu_0}{dT_{\mathbf{F}}^{-1}\mu_0}(\mathbf{x})\cdot\frac{dT_{\mathbf{F}}^{-1}\mu_0}{d\mu_0}(\mathbf{x})$$
$$=\mathcal{J}_{\mathbf{G}}(T_{\mathbf{F}}\mathbf{x})\cdot\det D_{\mathbf{x}}T_{\mathbf{F}}=\left(1+\hat{g}(r_1,s_1)\right)\cdot\exp\left(\int_0^{\tau_{\mathbf{F}}(\mathbf{x})} ph_\theta dt\right). \tag{4.8}$$

Let $H(\mathbf{x})=(1-\exp(\int_0^{\tau_{\mathbf{F}}(r,s)} ph_\theta dt)-\hat{g}(r_1,s_1))/\varepsilon$. Then we get the following expansion for $\mathcal{J}_{\mathbf{P}}$ at $\mathbf{P}=0$:

$$\mathcal{J}_{\mathbf{P}}(\mathbf{x})=\mathcal{J}_0(\mathbf{x})-\varepsilon H(\mathbf{x})-\varepsilon^2 R_{\mathbf{P}}, \tag{4.9}$$

where $R_{\mathbf{P}}$ is the residual term (up to a factor ϵ^2).

Now it is easy to see that $\mu_0(H)=0$. Firstly, $1+\mu_0(\hat{g})=\mu_0(1+\hat{g})=\mu_0(\mathcal{J}_{\mathbf{G}})=\mu_0(M)=1$. So $\mu_0(\hat{g})=0$. Secondly, we note that $\mu_0(e^{\int_0^{\tau_{\mathbf{F}}(r,s)} ph_\theta dt})=\mu_0(DT_{\mathbf{F}})=\mu_0(T_{\mathbf{F}}M)=\mu_0(M)=1$. Therefore, we have $\mu_0(H)=0$.

By assumptions (A1)–(A3), both H and $R_{\mathbf{P}}$ are uniformly bounded on M, and are C^α continuous functions on each component of $M\backslash\mathcal{S}_1^{\mathbf{P}}$. This completes the proof. □

Let $\mu_{\mathbf{P}}$ be the SRB measure of $T_{\mathbf{P}}$ on M given by Theorem 2.1. This measure represents the natural non-equilibrium steady state (NESS) for the system (see [22]), which might be singular with respect to the Liouville measure on M. There are several physically interested quantities associated to the NESS $\mu_{\mathbf{P}}$. For example, the current of the billiard flow on the $\mathbb{Z}^2$-periodic table $\widetilde{Q}$ is given by

$$\mathbf{J}_{\mathbf{P}}=\mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}})=\mu_{\mathbf{P}}^o(\widetilde{\mathbf{\Delta}}_{\mathbf{P}}),$$

where $\mathbf{\Delta}_{\mathbf{P}}$ is the induced displacement vector function on M, $\mu_{\mathbf{P}}^o$ is a copy of $\mu_{\mathbf{P}}$ on a fundamental domain (say $\widetilde{M}_o$) of $\widetilde{M}$. This current was derived in [8, 9] when the billiard is moving in an electric field with a Gaussian thermostat. The study of the current as a function of a general electric field was carried out in [5, 7]. In the other direction, these results were generalized in [27] to systems where the collision rule is perturbed. Here we study the current for billiards under more general external forces. Firstly we will prove the linear response property for general observable with respect to the forced billiard system.

Lemma 4.3 *Let* $\mathbf{P}=(\mathbf{F},\mathbf{G})$ *be an* ε*-small force pair,* $T_{\mathbf{P}}$ *be the induced billiard map and* $\mu_{\mathbf{P}}$ *be the SRB measure of* $T_{\mathbf{P}}$ *on* M, $\mathcal{H}$ *be the set of piecewise Holder continuous functions* *on M whose discontinuities occur at the singularities $S_1^{\mathbf{P}}$. Then for any $f \in \mathcal{H}$, we have*

$$\mu_{\mathbf{P}}(f) = \mu_0(f) + \varepsilon \cdot \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_0^k\big) \cdot H\big] + o(\varepsilon) \tag{4.10}$$

Proof Let $f \in \mathcal{H}$. Then for all $n \geq 1$, we have the following identity:

$$\begin{aligned} T_{\mathbf{P}}^n \mu_0(f) - \mu_0(f) &= \sum_{k=1}^{n} \big(T_{\mathbf{P}}^k \mu_0(f) - T_{\mathbf{P}}^{k-1} \mu_0(f)\big) = \sum_{k=1}^{n} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k\big)(1 - \mathcal{J}_{\mathbf{P}})\big] \\ &= \sum_{k=1}^{n} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k\big)\big(\varepsilon H(\mathbf{x}) + \varepsilon^2 R_{\mathbf{P}}\big)\big]. \end{aligned}$$

Note that $T_{\mathbf{P}}^n \mu_0(f) \to \mu_{\mathbf{P}}(f)$ exponentially (by Theorem 2.1-a). Passing $n \to \infty$, we get that

$$\mu_{\mathbf{P}}(f) = \lim_{n \to \infty} T_{\mathbf{P}}^n \mu_0(f) = \mu_0(f) + \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k\big)\big(\varepsilon H(\mathbf{x}) + \varepsilon^2 R_{\mathbf{P}}\big)\big]. \tag{4.11}$$

It follows from the fact that $\mathcal{J}_{\mathbf{P}}$ is the density function of a probability measure, that $\mu_0(1 - \mathcal{J}_{\mathbf{P}}) = 0$. Combining with the fact that $\mu_0(H) = 0$, we get that $\mu_0(R_{\mathbf{P}}) = 0$, too. In addition H and $R_{\mathbf{P}}$ are piecewise C^{α} functions whose discontinuities occur only at the singularities of $T_{\mathbf{P}}$. Thus H and $R_{\mathbf{P}}$ belong to $\mathcal{H}$. Now (4.11) implies that

$$\begin{aligned} &\sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k\big)\big(\varepsilon H + \varepsilon^2 R_{\mathbf{P}}\big)\big] = \varepsilon \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k\big) H\big] + \varepsilon^2 \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k\big) R_{\mathbf{P}}\big] \\ &\quad = \varepsilon \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_0^k\big) H\big] + \varepsilon \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k - f \circ T_0^k\big) H\big] + \varepsilon^2 \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_{\mathbf{P}}^k\big) R_{\mathbf{P}}\big] \\ &\quad = \varepsilon \sum_{k=1}^{\infty} \mu_0\big[\big(f \circ T_0^k\big) H\big] + o(\varepsilon), \end{aligned} \tag{4.12}$$

where we have used the Lebesgue Dominated Convergence Theorem in the last step, since both $\sum_{k=1}^{\infty} \mu_0[(f \circ T_{\mathbf{P}}^k) H]$ and $\sum_{k=1}^{\infty} \mu_0[(f \circ T_0^k) H]$ converge exponentially fast (by Theorem 2.1-b) and $T_{\mathbf{P}}^k \to T_0^k$ as $\mathbf{P} \to 0$. Therefore the series are dominated by $o(1)$. This completes the proof. □

Let (x, y) be the coordinates of the position of the particle $\tilde{\mathbf{q}} \in \widetilde{Q}$, and

$$\widetilde{\boldsymbol{\Delta}}_{\mathbf{P}} := (\Delta_{x,\mathbf{P}}, \Delta_{y,\mathbf{P}}) = \tilde{\pi} \circ \widetilde{T}_{\mathbf{P}} - \tilde{\pi}, \tag{4.13}$$

be the displacement vector map defined on $\widetilde{M}$. As pointed out before, $\widetilde{\boldsymbol{\Delta}}_{\mathbf{P}}$ is $\mathbb{Z}^2$-periodic and hence induces a well-defined, $\mathbb{R}^2$-valued function $\boldsymbol{\Delta}_{\mathbf{P}}$ on M. Using the fact that $\mu_0(\boldsymbol{\Delta}_0) = 0$ and $\mu_0(\tau_0) = \bar{\tau}$, we get the following results.

Lemma 4.4 *Both functions $\boldsymbol{\Delta}_{\mathbf{P}}$ and $\tau_{\mathbf{P}}$ belong to $\mathcal{H}$ with Holder exponent $\frac{1}{2}$ and uniformly bounded. Moreover,*

$$\mu_0(\Delta_{a,\mathbf{P}}) = \mathcal{O}(\varepsilon) \quad \text{and} \quad \mu_0(\tau_{\mathbf{P}}) = \bar{\tau} + \mathcal{O}(\varepsilon)$$

The proof is omitted here, since proof of Hölder continuity follows closely from [14, Lemma 8.2] and the estimations are almost identical with the proof of Lemma 8.1 in [27].

Next we calculate the current for the moving billiards under small force depending on ε. We only consider the case when the new system $T_{\mathbf{P}}$ is time-reversible.

Theorem 4.5 *Suppose the system is time-reversible under the external forces* $\mathbf{P} = (\mathbf{F}, \mathbf{G})$. *Then*

$$\mathbf{J}_{\mathbf{P}} := \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}}) = \varepsilon\boldsymbol{\sigma} + o(\varepsilon), \tag{4.14}$$

where $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$ *is given by*

$$\sigma_a = \frac{1}{2}\mu_0(\Delta_{a,\mathbf{P}} \cdot H) + \sum_{k=1}^{\infty} \mu_0\big[\big(\Delta_{a,\mathbf{P}} \circ T_0^k\big) \cdot H\big], \quad a \in \{x, y\}. \tag{4.15}$$

Moreover, the current for the flow satisfies $\hat{\mathbf{J}}_{\mathbf{P}} = \varepsilon\boldsymbol{\sigma}/\bar{\tau}_{\mathbf{P}} + o(\varepsilon)$, *where* $\bar{\tau} = \mu_0(\tau_0)$.

Proof We first use the invariance of $\mu_{\mathbf{P}}$ and the Kawasaki formula (4.11) to write $\mathbf{J}_{\mathbf{P}}$ as

$$\begin{aligned}
\mathbf{J}_{\mathbf{P}} &= \mu_{\mathbf{P}}(\Delta_{\mathbf{P}}) = \frac{1}{2}\big(\mu_{\mathbf{P}}(\Delta_{\mathbf{P}}) + \mu_{\mathbf{P}}\big(\Delta_{\mathbf{P}} \circ T_{\mathbf{P}}^{-1}\big)\big) \\
&= \frac{1}{2}\big(\mu_0(\Delta_{\mathbf{P}}) + \mu_0\big(\Delta_{\mathbf{P}} \circ T_{\mathbf{P}}^{-1}\big) + \mu_0\big[\Delta_{\mathbf{P}}(1 - \mathcal{J}_{\mathbf{P}})\big]\big) + \sum_{k=1}^{\infty} \mu_0\big[\big(\Delta_{\mathbf{P}} \circ T_{\mathbf{P}}^k\big)(1 - \mathcal{J}_{\mathbf{P}})\big].
\end{aligned} \tag{4.16}$$

Recall that $\widetilde{M}$ is the collision space of the $\mathbb{Z}^2$-periodic table $\widetilde{Q} \subset \mathbb{R}^2$. Let $\widetilde{M}_o$ be a fundamental domain of $\widetilde{M}$, say $(\partial\widetilde{Q} \cap [0,1]^2) \times [-1,1]$, and $\mu_0^o = \mathrm{Cst}\, dr\, ds$ be a copy of μ_0 on $\widetilde{M}_o$. So for any function f on M, $\mu_0(f) = \mu_0^o(\tilde{f})$, where $\tilde{f}$ is a $\mathbb{Z}^2$-periodic lift of f on $\widetilde{M}$.

Since μ_0^o is evenly distributed on $\widetilde{M}_o$, it is invariant under the involution I. In particular, we have

$$\mu_0^o(\tilde{\pi} \circ \widetilde{T}_{\mathbf{P}}) = \mu_0^o\big(\tilde{\pi} \circ \widetilde{T}_{\mathbf{P}}^-\big) = \mu_0^o\big(\tilde{\pi} \circ I \circ \widetilde{T}_{\mathbf{P}}^{-1} \circ I\big) = \mu_0^o\big(\tilde{\pi} \circ \widetilde{T}_{\mathbf{P}}^{-1}\big). \tag{4.17}$$

This implies that

$$\begin{aligned}
\mu_0(\mathbf{\Delta}_{\mathbf{P}}) + \mu_0\big(\mathbf{\Delta}_{\mathbf{P}} \circ T_{\mathbf{P}}^{-1}\big) &= \mu_0^o(\widetilde{\mathbf{\Delta}}_{\mathbf{P}}) + \mu_0^o\big(\widetilde{\mathbf{\Delta}}_{\mathbf{P}} \circ \widetilde{T}_{\mathbf{P}}^{-1}\big) \\
&= \mu_0^o(\tilde{\pi} \circ \widetilde{T}_{\mathbf{P}} - \tilde{\pi}) + \mu_0^o\big(\tilde{\pi} - \tilde{\pi} \circ \widetilde{T}_{\mathbf{P}}^{-1}\big) \\
&= \mu_0^o(\tilde{\pi} \circ \widetilde{T}_{\mathbf{P}}) - \mu_0^o\big(\tilde{\pi} \circ \widetilde{T}_{\mathbf{P}}^{-1}\big) = 0.
\end{aligned}$$

See [8, 14] for related discussions. Then according to (4.7), the two components of the current are given by

$$\mu_{\mathbf{P}}(\Delta_{a,\mathbf{P}}) = \frac{1}{2}\mu_0\big[\Delta_{a,\mathbf{P}}\big(\varepsilon H + \varepsilon^2\mathcal{R}\big)\big] + \sum_{k=1}^{\infty} \mu_0\big[\big(\Delta_{a,\mathbf{P}} \circ T_{\mathbf{P}}^k\big) \cdot \big(\varepsilon H + \varepsilon^2\mathcal{R}\big)\big] = \varepsilon\sigma_a + o(\varepsilon),$$

where

$$\sigma_a = \frac{1}{2}\mu_0(\Delta_{a,\mathbf{P}} \cdot H) + \sum_{k=1}^{\infty} \mu_0\big[\big(\Delta_{a,\mathbf{P}} \circ T_0^k\big) \cdot H\big]. \tag{4.18}$$

Here we use a similar argument in the last step as in (4.12), since $T_{\mathbf{P}} \to T_0$ as $\mathbf{P} \to 0$.

Denote $\boldsymbol{\sigma} = (\sigma_x, \sigma_y)$. So we have shown that $\mathbf{J_P} = \varepsilon\boldsymbol{\sigma} + o(\varepsilon)$. The current of the forced flow $\Phi_{\mathbf{P}}$ is given by $\hat{\mathbf{J}}_{\mathbf{P}} = \mu_{\mathbf{P}}(\boldsymbol{\Delta}_{\mathbf{P}})/\bar{\tau}_{\mathbf{P}} = \varepsilon\boldsymbol{\sigma}/\bar{\tau}_{\mathbf{P}} + o(\varepsilon)$, where $\bar{\tau}_{\mathbf{P}} = \mu_{\mathbf{P}}(\tau_{\mathbf{P}})$. This completes the proof. □

Remark 6 It is worth to point out that $\mu_0(H) = 0$ is crucial to define σ_a, $a \in \{x, y\}$. Otherwise the series in (4.18) used to define σ_a don't even converge if we take the linear term $-\int_0^{\tau_{\mathbf{F}}(r,s)} ph_\theta dt$. This is the main reason why we keep all nonlinear terms in Lemma 4.2. However, we can use the linear term $\int_0^{\tau_{\mathbf{F}}(r,s)} ph_\theta dt$ in all physical models with a *Gaussian thermostat* (see [5, 8, 9, 21]). This is exactly the content in Proposition 2.6, 2.7 and Corollary 2.8.

Proof of Proposition 2.6 When restricted to the level set $\mathcal{E}_1 = \{\|\mathbf{p}\| \equiv 1\}$, we have $h = F$. Then we only need to prove that the series (4.18) do converge with the linear choice $H = \int_0^{\tau_{\mathbf{F}}(\mathbf{x})} F_\theta\, dt$. By Theorem 2.1-b and Lemma 4.4, it is sufficient to show that $\mu_0(\int_0^{\tau_{\mathbf{F}}(\mathbf{x})} F_\theta\, dt) = 0$. Then according to the fact that $dm = \mathrm{Cst} \cdot p_0 \cdot d\mu_0\, dt = dx\, dy\, d\theta$, we get

$$\begin{aligned}\mu_0\left(\int_0^{\tau_{\mathbf{F}}(\mathbf{x})} F_\theta\, dt\right) &= \mathrm{Cst} \cdot \int_M \int_0^{\tau_{\mathbf{F}}(\mathbf{x})} F_\theta\, dt d\mu_0 = \int_{\mathcal{M}} F_\theta dm \\ &= \int_Q \left(\int_{S^1} F_\theta\, d\theta\right) dx\, dy = \int_Q (F|_{\partial S^1}) dx\, dy = 0. \end{aligned} \tag{4.19}$$

This finishes the proof of Proposition 2.6. □

Proof of Proposition 2.7 Let $\mathbf{E} = (\varepsilon e_1(\mathbf{q}), \varepsilon e_2(\mathbf{q}))$ be the electric field on Q, $\dot{\mathbf{q}} = \mathbf{p}$, $\dot{\mathbf{p}} = \mathbf{E} - \alpha\mathbf{p}$ be the thermostatted system. When restricted to the level set $\mathcal{E}_1 = \{\|\mathbf{p}\| \equiv 1\}$, the function $h = -e_1 \sin\theta + e_2 \cos\theta$, and $h_\theta = -e_1 \cos\theta - e_2 \sin\theta = -(e_1, e_2) \cdot \mathbf{p}$. So $H = -\int_0^{\tau_{\mathbf{F}}(\mathbf{x})} h_\theta\, dt = (e_1, e_2) \cdot \int_0^{\tau_{\mathbf{E}}(\mathbf{x})} \mathbf{p}\, dt = (e_1, e_2) \cdot \boldsymbol{\Delta}_{\mathbf{E}}(\mathbf{x})$. This finishes the proof of Proposition 2.7. □

Remark 7 There is another classical representation of the current of the billiard flow $\hat{\mathbf{J}} = \hat{\mu}(\mathbf{p})$, which may not hold for general forces. More precisely, let $\hat{\mu}_{\mathbf{P}}$ be the SRB measure of the flow $\Phi_{\mathbf{P}}$, which can also be obtained as the suspension of $\mu_{\mathbf{P}}$ on $\mathcal{M}$. Then $\hat{\mathbf{J}}_{\mathbf{P}} = \hat{\mu}_{\mathbf{P}}(\mathbf{p})$ holds if and only if $\boldsymbol{\Delta}_{\mathbf{P}}(\mathbf{x}) = \boldsymbol{\Delta}_{\mathbf{F}}(\mathbf{x}) := \int_0^{\tau_{\mathbf{F}}(\mathbf{x})} \mathbf{p} dt$, that is, there is no slip on ∂Q after each collision. For a general twist force with slip, we would have an additional term $\boldsymbol{\Delta}_{\mathbf{G}}(T_{\mathbf{F}}\mathbf{x}) = \boldsymbol{\Delta}_{\mathbf{P}}(\mathbf{x}) - \boldsymbol{\Delta}_{\mathbf{F}}(\mathbf{x})$. So we have $\mu_{\mathbf{P}}(\boldsymbol{\Delta}_{\mathbf{P}}) = \int_{M_o} (\tilde{\pi}(\widetilde{T}_{\mathbf{P}}\mathbf{x}) - \tilde{\pi}(\widetilde{T}_{\mathbf{F}}\mathbf{x})) d\mu_{\mathbf{P}}^o + \mu_{\mathbf{P}}(\boldsymbol{\Delta}_{\mathbf{F}})$. Since $\mu_{\mathbf{P}}$ is $T_{\mathbf{P}}$-invariant, we get an modified formula of the current for the forced flow

$$\hat{\mathbf{J}}_{\mathbf{P}} = \hat{\mu}_{\mathbf{P}}(\mathbf{p}) + \frac{1}{\mu_{\mathbf{P}}(\tau_{\mathbf{P}})} \int_{M_o} \left(\tilde{\pi}(\mathbf{x}) - \tilde{\pi}\left(\mathbf{G}^{-1}\mathbf{x}\right)\right) d\mu_{\mathbf{P}}^o. \tag{4.20}$$

Now we turn to the proof of (2.9) and (2.11).

Proof of Theorem 2.2 and Theorem 2.3 The convergence of the distribution $\frac{\tilde{\mathbf{q}}_n - n\mathbf{J_P}}{\sqrt{n}}$ to a normal law $\mathcal{N}(\mathbf{0}, \mathbf{D_P})$ follows directly from the central limit theorem (see Theorem 2.1-c), since $\tilde{\mathbf{q}}_n = \sum_{k=0}^{n-1} \mathbf{\Delta}_{\mathbf{P}} \circ T_{\mathbf{P}}^k$, where $\mathbf{\Delta}_{\mathbf{P}}$ belongs to $\mathcal{H}$ by Lemma 4.4. Thus it is enough to estimate the covariance matrix $\mathbf{D}_{\mathbf{P}}$, which is given by the following sum of correlations

$$\mathbf{D}_{\mathbf{P}} = \sum_{n=-\infty}^{\infty} \left[\mu_{\mathbf{P}}\left(\mathbf{\Delta}_{\mathbf{P}} \circ T_{\mathbf{P}}^n \otimes \mathbf{\Delta}_{\mathbf{P}}\right) - \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}}) \otimes \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}})\right]. \tag{4.21}$$

Note that for any $n \geq 1$,

$$\lim_{\mathbf{P}\to 0} \mathbf{\Delta}_{\mathbf{P}} \circ T_{\mathbf{P}}^n = \mathbf{\Delta}_0 \circ T_0^n.$$

Furthermore by Theorem 2.1, there exist $C > 0$ and $\theta \in (0, 1)$, such that for any ε-small force pair $\mathbf{P}$ and for all $a, b \in \{x, y\}$,

$$\left|\mathcal{C}_{a,b}(n)\right| = \left|\mu_{\mathbf{P}}\left(\Delta_{a,\mathbf{P}} \circ T_{\mathbf{P}}^n \cdot \Delta_{b,\mathbf{P}}\right) - \mu_{\mathbf{P}}(\Delta_{a,\mathbf{P}}) \cdot \mu_{\mathbf{P}}(\Delta_{b,\mathbf{P}})\right| \leq C\theta^{|n|}.$$

Therefore the series $\sum_{n=-\infty}^{\infty} \mathcal{C}_{a,b}(n)$ converges uniformly (and exponentially) for all $a, b \in \{x, y\}$. So the diffusion matrix varies continuously at $\mathbf{P} = 0$:

$$\begin{aligned}\mathbf{D}_{\mathbf{P}} &= \sum_{n=-\infty}^{\infty} \left[\mu_{\mathbf{P}}\left(\mathbf{\Delta}_{\mathbf{P}} \circ T_{\mathbf{P}}^n \otimes \mathbf{\Delta}_{\mathbf{P}}\right) - \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}}) \otimes \mu_{\mathbf{P}}(\mathbf{\Delta}_{\mathbf{P}})\right] \\ &= \sum_{n=-\infty}^{\infty} \mu_0\left(\mathbf{\Delta}_0 \circ T_0^n \otimes \mathbf{\Delta}_0\right) + o(1) = \mathbf{D}_0 + o(1),\end{aligned} \tag{4.22}$$

since $\mu_0(\mathbf{\Delta}_0) = 0$. This finishes the proof of Theorem 2.2. Theorem 2.3 follows directly from Theorem 2.2. □

Finally we prove Theorem 2.4.

Proof of Theorem 2.4 Let $\lambda_{\mathbf{P}}^s < 0 < \lambda_{\mathbf{P}}^u$ denote the Lyapunov exponents of the ergodic system $(T_{\mathbf{P}}, \mu_{\mathbf{P}})$. The sum $\xi_{\mathbf{P}} := -(\lambda_{\mathbf{P}}^s + \lambda_{\mathbf{P}}^u)$ represents the *physical entropy production rate* of the perturbed system $(T_{\mathbf{P}}, \mu_{\mathbf{P}})$. Then by Oseledets Multiplicity Ergodic Theorem, we have

$$\xi_{\mathbf{P}} = -\left(\lambda_{\mathbf{P}}^s + \lambda_{\mathbf{P}}^u\right) = -\mu_{\mathbf{P}}(\log \mathcal{J}_{\mathbf{P}}).$$

Note that $1 - \mathcal{J}_{\mathbf{P}} = \mathcal{O}(\epsilon)$. So we have

$$\xi_{\mathbf{P}} = -\mu_{\mathbf{P}}\left(\log\left(1 - (1 - \mathcal{J}_{\mathbf{P}})\right)\right) = \mu_{\mathbf{P}}(1 - \mathcal{J}_{\mathbf{P}}) + \frac{1}{2}\mu_{\mathbf{P}}\left((1 - \mathcal{J}_{\mathbf{P}})^2\right) + \mathcal{O}\left(\varepsilon^3\right).$$

Using the similar analysis as in (4.12) and $\mu_0(\mathcal{J}_{\mathbf{P}}) = 1$, one can check that

$$\begin{aligned}\mu_{\mathbf{P}}(1 - \mathcal{J}_{\mathbf{P}}) &= \mu_0(1 - \mathcal{J}_{\mathbf{P}}) + \sum_{k=1}^{\infty} \mu_0\left[(1 - \mathcal{J}_{\mathbf{P}}) \circ T_{\mathbf{P}}^k \cdot (1 - \mathcal{J}_{\mathbf{P}})\right] \\ &= \varepsilon^2 \sum_{k=1}^{\infty} \mu_0\left(H \circ T_{\mathbf{P}}^k \cdot H\right) + \mathcal{O}\left(\varepsilon^3\right) = \varepsilon^2 \sum_{k=1}^{\infty} \mu_0\left(H \circ T_0^k \cdot H\right) + o\left(\varepsilon^2\right).\end{aligned} \tag{4.23}$$

In addition, we have

$$\mu_{\mathbf{P}}\left((1 - \mathcal{J}_{\mathbf{P}})^2\right) = \mu_0\left((1 - \mathcal{J}_{\mathbf{P}})^2\right) + \sum_{k=1}^{\infty} \mu_0\left[\left((1 - \mathcal{J}_{\mathbf{P}})^2\right) \circ T_{\mathbf{P}}^k \cdot (1 - \mathcal{J}_{\mathbf{P}})\right]$$

$$= \varepsilon^2 \mu_0\big(H^2\big) + \frac{\varepsilon^3}{2} \sum_{k=1}^{\infty} \mu_0\big(H^2 \circ T_0^k \cdot H\big) + o\big(\varepsilon^3\big), \tag{4.24}$$

where we used (4.7) and (4.11) in the above estimates. Combining these facts, we get

$$\xi_{\mathbf{P}} = \frac{\varepsilon^2}{2} \mu_0\big(H^2\big) + \varepsilon^2 \sum_{k=1}^{\infty} \mu_0\big(H \circ T_0^k \cdot H\big) + o\big(\varepsilon^2\big) = \varepsilon^2 \cdot \frac{\sigma_0^2(H)}{2} + o\big(\varepsilon^2\big),$$

where $\sigma_0^2(H) = \sum_{k=-\infty}^{\infty} \mu_0[H \circ T_0^k \cdot H]$. Being an SRB measure, the metric entropy of $\mu_{\mathbf{P}}$ satisfies $h_{\mu_{\mathbf{P}}}(T_{\mathbf{P}}) = \lambda_{\mathbf{P}}^u = \mu_{\mathbf{P}}(\Lambda_{\mathbf{P}}^u)$, by Pesin's Entropy Formula and by Birkhoff Ergodic Theorem, respectively, where $\Lambda_{\mathbf{P}}^u(\mathbf{x})$ is the local expansion rate of $\mathbf{x} \in M$ along the unstable direction under map $T_{\mathbf{P}}$. Then according to (4.11) and the exponential decay of correlations, we have

$$\begin{aligned} h_{\mu_{\mathbf{P}}}(T_{\mathbf{P}}) &= \mu_0\big(\Lambda_{\mathbf{P}}^u\big) + \sum_{k=1}^{\infty} \mu_0\big(\Lambda_{\mathbf{P}}^u \circ T_{\mathbf{P}}^k \cdot (1 - \mathcal{J}_{\mathbf{P}})\big) \\ &= \mu_0\big(\Lambda_0^u\big) + \sum_{k=1}^{\infty} \mu_0\big(\Lambda_0^u \circ T_{\mathbf{P}}^k \cdot \big(\varepsilon H + \varepsilon^2 R_{\mathbf{P}}\big)\big) + o(1) \\ &= h_0 + o(1), \end{aligned} \tag{4.25}$$

since $\Lambda_{\mathbf{P}}^u \to \Lambda_0^u$ as $\mathbf{P} \to 0$. Here $h_0 := h_{\mu_0}(T_0) > 0$ is the metric entropy of the unforced billiard map T_0. Combining the above facts and Young's Dimension Formula (2.17) in [25], we get

$$\begin{aligned} \mathrm{HD}(\mu_{\mathbf{P}}) &= h_{\mu_{\mathbf{P}}}(T_{\mathbf{P}})\left(\frac{1}{\lambda_{\mathbf{P}}^u} - \frac{1}{\lambda_{\mathbf{P}}^s}\right) = 1 - \frac{\lambda_{\mathbf{P}}^u}{\lambda_{\mathbf{P}}^s} = 2 - \frac{\xi_{\mathbf{P}}}{h_{\mu_{\mathbf{P}}}(T_{\mathbf{P}}) + \xi_{\mathbf{P}}} \\ &= 2 - \frac{\varepsilon^2 \cdot \sigma_0^2(H)/2 + o(\varepsilon^2)}{h_0 + \varepsilon^2 \cdot \sigma_0^2(H)/2 + o(1)} \\ &= 2 - \varepsilon^2 \cdot \frac{\sigma_0^2(H)}{2h_0} + o\big(\varepsilon^2\big). \end{aligned}$$

This completes the proof of Theorem 2.4. □

Then by the suspension property, we see that the dimension of the measure $\hat{\mu}_{\mathbf{P}}$ is given by

$$\mathrm{HD}(\hat{\mu}_{\mathbf{P}}) = \mathrm{HD}(\mu_{\mathbf{P}}) + 1 = 3 - \varepsilon^2 \cdot \frac{\sigma_H^2}{2h_0} + o\big(\varepsilon^2\big).$$

So if ε is small and $\sigma_H^2 > 0$, then $1 < \mathrm{HD}(\mu_{\mathbf{P}}) < 2$ and $2 < \mathrm{HD}(\hat{\mu}_{\mathbf{P}}) < 3$. Therefore, both $\mu_{\mathbf{P}}$ and $\hat{\mu}_{\mathbf{P}}$ are singular with respect to the Lebesgue measures and admit the fractal structures if $\sigma_H^2 \neq 0$ under a small force pair $\mathbf{P}$. Recall that $\mu_{\mathbf{P}}$ is positive on every open set. The first intuitive impression of $\mu_{\mathbf{P}}$ is that it might be a smooth measure. Clearly this is not the case under generic small forces. In fact, it is believed [9] (based on numerical evidences) that $\mu_{\mathbf{P}}$ should be multifractal with a continuous spectrum of fractal dimensions.

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