Conditionally invariant measures for Anosov maps with small holes

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Abstract

We study Anosov diffeomorphisms on surfaces in which some small 'holes' are cut. The points that are mapped into those holes disappear and never return. We assume that the holes are arbitrary open domains with piecewise smooth boundary, and their sizes are small enough. The set of points whose trajectories stay away from holes in the past is a Cantor-like union of unstable fibers. We establish the existence and uniqueness of a conditionally invariant measure on this set, whose conditional distributions on unstable fibers are smooth. This generalizes previous works by Pianigiani, Yorke, and others.

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1 Introduction

1.1. Let $\hat{T} : \hat{M} \to \hat{M}$ be a topologically transitive Anosov diffeomorphism of class $C^{1+\alpha}$ of a compact Riemannian surface $\hat{M}$. Let $H \subset \hat{M}$ be an open set with a finite number of connected components.

We denote $M = \hat{M} \setminus H$. For any $n \geq 0$ we put

$$M_n = \cap_{i=0}^{n} \hat{T}^i M \quad \text{and} \quad M_{-n} = \cap_{i=0}^{n} \hat{T}^{-i} M, \tag{1.1}$$

and also

$$M_+ = \cap_{n\geq 1} M_n, \quad M_- = \cap_{n\geq 1} M_{-n}, \quad \Omega = M_+ \cap M_- \tag{1.2}$$

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All these sets are closed, $\hat{T}^{-1}M_+ \subset M_+$, $\hat{T}M_- \subset M_-$ and $\hat{T}\Omega = \hat{T}^{-1}\Omega = \Omega$. The set M_+ (resp., M_-) consists of points whose trajectories stay away from H in the past (the future). The set Ω consists of points whose trajectories never enter H.

In this paper, we study the structure of the sets $M_\pm$ and Ω and the dynamics of the map $\hat{T}$ on these sets. We think of the connected components of H as holes (one can also think of H as an absorbing region). The trajectories that fall into H will no longer be considered – they disappear. So we may call Ω the set of nonwandering points in $\hat{M}$.

We denote by T the restriction of $\hat{T}$ on M, which means that for any set $A \subset M$ and $n \geq 1$ we put $T^nA = \hat{T}^n(A \cap M_{-n})$ and $T^{-n}A = \hat{T}^{-n}(A \cap M_n)$.

1.2. The concept of a chaotic dynamical system with holes in its phase space and related problems have been formulated by Pianigiani and Yorke in 1979 [11] by way of the following pictorial example.

Imagine a Sinai billiard table (with dispersing boundary), so that the dynamics of the ball are strongly chaotic. Let one or more holes be cut in the table, so that the ball can fall through. In particular, one can place those holes at the corners of the table and make 'pockets'. Let the initial position of the ball be chosen at random with some smooth probability distribution (which may be the equilibrium distribution for the original system, without holes). Denote by $p(t)$ the probability that the ball stays on the table for at least time t and, if it does, by $\mu(t)$ its (normalized) distribution in the phase space at time t. Natural questions are: does $p(t)$ converge to zero at some exponential rate, as $t \to \infty$? is there a limit probability distribution $\mu_+ = \lim_{t\to\infty} \mu(t)$; is that limit distribution independent of the initial distribution $\mu(0)$?

These questions still remain open. However, since the pioneering work [11], substantial progress has been made in the study of certain classes of chaotic dynamical systems with holes.

Expanding (noninvertible) maps S with holes have been studied in [11] and later by Collet, Martínez and Schmitt [7], where the analogues of the above questions have been answered positively. They called the limit probability distribution μ_+ a *conditionally invariant measure* [11]. The measure μ_+ is not invariant under S; it cannot be because of the holes. Instead, its image under S is proportional to itself: $\mu_+(S^{-1}A) = \lambda_+\mu_+(A)$ for any Borel set A, with some constant $\lambda_+ \in (0,1)$. The constant λ_+ is called the *eigenvalue* of μ_+ [4, 5]. Another constant, $\gamma_+ = -\ln \lambda_+$, is known as the *escape rate*. The paper [7] also constructed a related S-invariant measure η_+ on the set of nonwandering points Ω for S and established the important *escape rate formula*, $\gamma_+ = \chi_+ - h(\eta_+)$, where χ_+ was the sum of positive Lyapunov exponents on Ω and $h(\eta_+)$ the Kolmogorov-Sinai entropy of η_+.

In 1981-86 Čencova [3, 4] studied a class of invertible chaotic transformations with holes, namely smooth Smale's horseshoes. She also answered the analogues of the above questions positively. She constructed the invariant measure η_+ on the set of nonwandering points Ω by pulling the conditionally invariant measure μ_+ backward in time.

Lopes and Markarian in [9] studied an open billiard system – a particle bouncing off several convex scatterers placed sufficiently far apart on a plane, so that they do not cast shadow on each other (this is called the 'no eclipse' condition). Here almost every trajectory eventually escapes through the openings between the scatterers. In [9], a conditionally invariant measure and a related invariant measure were constructed, the latter was shown to be a Gibbs measure, and the escape rate formula was also proved.

Open billiards and other open Hamiltonian systems have become very popular in physics in the past ten years. They have been studied numerically and heuristically, see the survey [8] and the references therein. Physicists call this chaotic scattering theory. Very few results of it, however, are proved mathematically.

Recently, two of us (N.Ch. and R.M.) in [5, 6] studied $C^{1+\alpha}$ transitive Anosov diffeomorphisms $\hat{T} : \hat{M} \to \hat{M}$ with what we called rectangular holes. For an arbitrary finite Markov partition $R_1, \dots, R_{I'}$ of $\hat{M}$ we defined H to be the union of the interiors of some rectangles, say, $H = \text{int}\, R_{I+1} \cup \cdots \cup \text{int}\, R_{I'}$, for some $I < I'$. Even though such holes look quite special, the systems studied in [5, 6] generalize both horseshoes studied in [3, 4] and open billiards of the paper [9].

In [5], we assumed an additional 'mixing condition': there is a $k_0 \geq 1$ such that (in the notations of 1.1) $\text{int} R_i \cap \hat{T}^{k_0}(R_j \cap M_{-k_0}) \neq \emptyset$ for all $i, j \leq I$. We proved the existence and uniqueness of a conditionally invariant measure and a related invariant measure, established the escape rate formula, and generalized other results of [7, 4, 9, 13], including formulas for the fractal dimension of the invariant measure. In the next paper [6], we relaxed the mixing condition and extended the results of [5] to nonmixing and nonergodic cases.

The subject of this paper is the study of Anosov maps with rather arbitrary open holes, not necessarily rectangles. Our key assumptions are that the holes are small enough, and $\dim \hat{M} = 2$. While the latter is assumed only to simplify the arguments, the former is essential – for large holes the conditionally invariant measure is obviously not unique. Our main result is the existence and uniqueness of a conditionally invariant measure μ_+ on M_+. We plan to study the invariant measure η_+ on Ω in a separate paper, since that would require essentially new approach.

1.3. Here we state necessary definitions and conventions.

For any point $x \in \hat{M}$ we denote by W_x^u and W_x^s the local unstable and stable fibers containing x. We denote by J_x^u and J_x^s the Jacobians of the map $\hat{T}$ restricted to W_x^u and W_x^s, respectively, at the point x. We put

$$\Lambda_{\min} = \min_{x \in \hat{M}} \{J_x^u, 1/J_x^s\} > 1 \quad \text{and} \quad \Lambda_{\max} = \max_{x \in \hat{M}} \{J_x^u, 1/J_x^s\} < \infty$$

Let $\phi_0 > 0$ be the minimum angle between stable and unstable fibers in $\hat{M}$. Recall that for any two points $x, y \in W^s$ a holonomy map $h_{x,y} : W_x^u \to W_y^u$ is defined by sliding the points of W_x^u along local stable fibers (symmetrically, $h_{x,y} : W_x^s \to W_y^s$ is defined for $x, y \in W^u$).

A rectangle $R \subset \hat{M}$ is a sufficiently small set such that for any $x, y \in R$ we have $W_x^u \cap W_y^s \in R$. We consider only closed connected rectangles. Those are bounded by two stable and two unstable fibers (called stable and unstable sides of R). Segments of local unstable and stable fibers inside R that terminate, respectively, on the stable and unstable sides of R are called R-fibers. They are 'full-size' local fibers in R stretching completely across it. Any subrectangle $R' \subset R$ whose stable (unstable) sides are on the stable (unstable) sides of R is called a u-subrectangle (s-subrectangle). Any u-subrectangle (s-subrectangle) in R is a union of unstable (resp. stable) R-fibers.

Convention. We say that a measure μ on $\hat{M}$ is smooth if its conditional measures on local unstable fibers $W^u \subset \hat{M}$ are absolutely continuous with respect to the Riemannian length, and their densities are Hölder continuous with Hölder exponent α, which is the same as for the class of smoothness of the map $\hat{T}$.

Recall that every transitive Anosov diffeomorphism has a unique Sinai-Bowen-Ruelle (SBR) measure [14, 2, 12]. It is an invariant measure, whose conditional distributions on local unstable fibers are smooth in the above sense. Motivated by this, we will call these conditional distributions *u-SBR measures* on unstable manifolds. Equivalently, for any local unstable fiber W^u its u-SBR measure is a probability measure ν^u on W^u whose density $\rho(x)$ with respect to the Riemannian length satisfies the equation [1]

$$\frac{\rho(x)}{\rho(y)} = \lim_{n\to\infty} \frac{J^u_{T^{-1}y} \cdots J^u_{T^{-n}y}}{J^u_{T^{-1}x} \cdots J^u_{T^{-n}x}} \tag{1.3}$$

The u-SBR measures are $\hat{T}$-invariant, i.e. the image of ν^u on W^u under $\hat{T}$ is a u-SBR measure on the fiber $\hat{T}W^u$.

We introduce bounds on distorsions as follows. For any $r > 0$ we denote by $D_1(r) \geq 1$ the supremum of all ratios $\rho(x)/\rho(y)$ in (1.3) for all $x, y \in W^u$ on all fibers W^u of length r (length always means the Riemannian length). Next, $D_2(r)$ denotes the supremum of all the Jacobians of holonomy maps $h_{x,y}$ for points $x, y \in W^{u,s}$ at distance $\leq r$ (measured along $W^{u,s}$). We put $D(r) = \max\{D_1(r), D_2(r)\}$. One can think of $D(r)$ as a general upper bound on distorsions within the distance r in $\hat{M}$. Obviously, $D(r) \to 1$ as $r \to 0$. For linear Anosov maps $D(r) \equiv 1$.

For any finite Borel measure μ on M we define its norm by $||\mu|| = \mu(M)$. We denote by T_* the adjoint operator on the class of Borel measures on M defined by $(T_*\mu)(A) = \mu(T^{-1}(A \cap M_1))$ for any $A \subset M$. Due to the holes, the operator T_* does not preserve norm. We also denote by T_+ the (nonlinear) operator on the space of probability measures on M defined by $T_+\mu = T_*\mu/||T_*\mu||$, whenever $||T_*\mu|| \neq 0$.

Definition. A measure μ on M is said to be *conditionally invariant* under T if there is a $\lambda > 0$ such that $T_*\mu = \lambda\mu$. The factor λ is the *eigenvalue* of μ.

Obviously, any conditionally invariant measure μ is supported on M_+, and we have $\mu(T^{-1}A \cap M_+) = \lambda\mu(A \cap M_+)$.

1.4. Here we make preliminary assumptions on the holes.

The holes (the connected components of H) are domains that satisfy the following regularity condition. There is a constant $B_0 > 0$ such that for any local unstable fiber W^u and any local stable fiber W^s that intersect only one hole $H' \subset H$ the sets $W^u \setminus H'$ and

$W^s \setminus H'$ consist of not more than B_0 smooth components. In particular, if the holes are convex and the curvature of their boundary is greater than that of unstable and stable fibers, then $B_0 = 2$. Also, if the holes are rectangles bounded by stable and unstable fibers, then again $B_0 = 2$.

Let N_H be the number of holes. We denote by d_0 the minimum distance between the holes, if there is more than one hole, $N_H > 1$. We also assume that d_0 is smaller than a quarter of the length of the shortest closed geodesic on $\hat{M}$. In the case $N_H = 1$ this will be the definition of d_0.

We fix $D = D(2d_0)$, which will be the only bound on distorsions that we use. Nonlinearity of the map $\hat{T}$ will result in additional factors, all $\le D$, in various estimates.

For certain technical reasons, we will assume that $\Lambda_{\min} > 64D^2$. This is not a restrictive assumption, because it can be always fulfilled by taking higher iterates of $\hat{T}$. (The constant D is determined by the stable and unstable fibers in $\hat{M}$, so it is the same for all iterates of $\hat{T}$).

We denote by h the maximal size of holes defined as follows. For any hole $H' \subset H$ its size is

$$\sup_{x \in H'} \{\operatorname{diam} W_x^u \cap H',\, \operatorname{diam} W_x^s \cap H'\}$$

where the diameter is measured along the fibers $W_x^{u,s}$. We will need h to be small enough compared to d_0, i.e. $h < h_0(\hat{T}, d_0, B_0)$. There will be four specific upper bounds on h assumed in Sections 2 and 3. They are always clearly stated.

1.5. The structure of the paper is as follows. In Section 2 we study the evolution of an arbitrary unstable fiber W^u under T. We estimate the fractions of the images of the u-SBR measure on W^u that are and are not 'eaten' by the holes. In Section 3 we prove the existence of sufficiently long unstable fibers in M_+ and study their properties. In Section 4 we define a sequence of approximations of the holes H by unions of rectangles of increasingly fine Markov partitions of $\hat{M}$. Accordingly, we obtain a sequence of conditionally invariant measures, $\mu_+^{(k)}$, for those rectangular holes based on the results of [5, 6]. In Sections 5 we prove the existence of the conditionally invariant measures μ_+ by taking the weak limit points of the sequence of measures $\mu_+^{(k)}$. In Section 6 we establish the uniqueness of μ_+.

2 Evolution of u-SBR measures

In this section we study the evolution of unstable fibers and u-SBR measures on them under T. This requires the following assumption on h.

Assumption H1. $h < d_0/9$.

Additional cuts. For certain technical reasons, in this section it will be convenient to limit the length of the unstable fibers in M by $2d_0$. This will be done by subdividing all maximal unstable fibers $W^u \subset M$ (terminating on ∂H) whose length is $> 2d_0$ into subfibers of length between d_0 and $2d_0$. This can be accomplished by making a finite number of cuts in M along some local stable fibers. The choice of those stable fibers will not be important, and they will not be actually removed from M. They just determine the way we do the bookkeeping of unstable fibers. Obviously, any long unstable fiber in M will be cut into subfibers that are still longer than d_0, and we never cut the fibers of length $\leq 2d_0$.

We now consider an unstable fiber $W^u \subset M$ and a u-SBR measure ν^u on it. Its images under the iterates of the map T are cut by holes and our additional cuts. Whenever a smooth component of $T^n W^u$ is cut into subcomponents, the further images of those under the iterates of T are treated separately. So, for every $n \geq 1$ the image $T^n W^u = \hat{T}^n(W^u \cap M_{-n})$ will consists of many smooth unstable fibers, which carry the image of the measure ν^u under T^n_*, which we denote by ν^u_n. We do not normalize it. Denote the components of $T^n W^u$ by $W^u_{n,i}$. Let $\varepsilon > 0$. For every component $W^u_{n,i}$ we denote by $W^u_{n,i}(\varepsilon)$ the ε-neighborhood of its endpoints within $W^u_{n,i}$.

Theorem 2.1 *Let W^u be long enough, of length $> \Lambda^{-1}_{\min} d_0$. Then for every $n \geq 0$ and any $\varepsilon > 0$ we have*

$$\frac{\nu^u_n(\cup_i W^u_{n,i}(\varepsilon))}{\nu^u_n(T^n W^u)} \leq C_1 \varepsilon \tag{2.1}$$

where $C_1 = 48D/d_0$.

Proof. We fix an $\varepsilon \in (0, C_1^{-1})$ (the theorem is trivial for $\varepsilon > C_1^{-1}$). The components $W^u_{n,i}$ are divided into three groups – long, medium and short. We say that a component $W^u_{n,i}$ is long if the length of its complete image $\hat{T} W^u_{n,i}$ on the original manifold $\hat{M}$ is $\geq d_0$. The other components $W^u_{n,i}$ are said to be short if their lengths are $< (2D)^{-1}\varepsilon$, otherwise they are said to be medium.

We put

$$s_{n,i} = \frac{\nu^u_n(W^u_{n,i}(\varepsilon))}{\nu^u_n(W^u_{n,i})}$$

and

$$s'_{n,i} = \frac{\sum_j \nu^u_n(T^{-1} W^u_{n+1,j}(\varepsilon))}{\nu^u_n(W^u_{n,i})}$$

where the sum is taken over j such that $W^u_{n+1,j} \subset T W^u_{n,i}$.

If a component $W^u_{n,i}$ is long, then its image $T W^u_{n,i}$ has length $d \geq d_0$ and can be subdivided by holes and our additional cuts into no more than $d/d_0 + 2$ subcomponents. Then we have

$$s'_{n,i} \leq 2D(d/d_0 + 2)d^{-1}\varepsilon \leq C'_1 \varepsilon \tag{2.2}$$

with $C'_1 = 6D/d_0$.

If a component $W^u_{n,i}$ is a medium one of length d', then its image $T W^u_{n,i}$ can intersect just one hole. In this case we have

$$s'_{n,i} \leq \frac{4D}{\Lambda_{\min} d'} \varepsilon$$

and

$$s_{n,i} \geq \min\{1,\, 2(Dd')^{-1}\varepsilon\}$$

Since we have a medium component, $d' > (2D)^{-1}\varepsilon$, and we have

$$s'_{n,i} \leq 8D^2\Lambda_{\min}^{-1} s_{n,i} \tag{2.3}$$

For any short component $W^u_{n,i}$ we have $s_{n,i} = 1$ and $s'_{n,i} \leq 1$.

The proof of the theorem goes by induction on n. Its validity for the current value of n means that

$$\frac{\sum_i s_{n,i}\nu^u_n(W^u_{n,i})}{\sum_i \nu^u_n(W^u_{n,i})} \leq C_1\varepsilon \tag{2.4}$$

It is then sufficient to show that, under this assumption, we have

$$\frac{\sum_i s'_{n,i}\nu^u_n(W^u_{n,i})}{\sum_j \nu^u_{n+1}(W^u_{n+1,j})} \leq C_1\varepsilon \tag{2.5}$$

We split each of the two sums in (2.4) into three parts, corresponding to long, medium and short components. Those parts we denote by x_n, y_n, z_n for the top sum and p_n, q_n, r_n for the bottom one, respectively. The two sums in (2.5) are also split into three groups each, corresponding to the images of long, medium and short components of T^nW^u. (Note the difference in the way we split these sums!) We denote by x'_n, y'_n, z'_n the three parts of the top sum in (2.5) and p'_n, q'_n, r'_n those for the bottom sum, respectively. Note that these quantities depend on ε, but for brevity we suppress this dependence.

It is clear from (2.2) and (2.3) that

$$x'_n + y'_n + z'_n \leq C'_1\varepsilon p_n + 8D^2\Lambda_{\min}^{-1} y_n + r_n$$

We have, from (2.4), that

$$y_n \leq C_1\varepsilon(p_n + q_n + r_n)$$

Note that all the short components $W^u_{n,i}$ lie within $\cup_i W^u_{n,i}(\varepsilon')$, where $\varepsilon' = (4D)^{-1}\varepsilon$. We certainly can assume that (2.4) is true for all $\varepsilon > 0$, in particular for ε'. This implies that

$$\begin{array}{rcl} r_n = r_n(\varepsilon) \leq \nu^u_n(\cup_i W^u_{n,i}(\varepsilon')) & \leq & C_1\varepsilon'\nu^u_n(\cup_i W^u_{n,i}) \\ & = & C_1(4D)^{-1}\varepsilon(p_n + q_n + r_n) \end{array}$$

Combining the above bounds gives

$$\begin{array}{rcl} x'_n + y'_n + z'_n & \leq & (C'_1 + 8D^2\Lambda_{\min}^{-1}C_1)\varepsilon p_n \\ & + & 8D^2\Lambda_{\min}^{-1}C_1\varepsilon(q_n + r_n) + C_1(4D)^{-1}\varepsilon(p_n + q_n + r_n) \end{array}$$

We now recall that $\Lambda_{\min} > 64D^2$ and $C_1 = 8C'_1$. Then we get

$$x'_n + y'_n + z'_n \leq \frac{1}{2}C_1\varepsilon(p_n + q_n + r_n)$$

In order to complete the proof of the theorem, it is enough to show that

$$p'_n + q'_n + r'_n \geq \frac{1}{2}(p_n + q_n + r_n) \tag{2.6}$$

i.e. the holes cannot eat up more than 50% of the images of curves $W^u_{n,i}$ under T, combined. It is clear that if a component $W^u_{n,i}$ is of length $> 4D\Lambda_{\min}^{-1}h$, then no more than 25% of its image under T can be eaten up by holes. Applying the assumption (2.4) with $\varepsilon = 4D\Lambda_{\min}^{-1}h$ to the other components $W^u_{n,i}$ shows that their total ν^u_n-measure does not exceed $4C_1D\Lambda_{\min}^{-1}h(p_n + q_n + r_n)$. Assumption H1 implies that

$$4C_1D\Lambda_{\min}^{-1}h = 192D^2(d_0\Lambda_{\min})^{-1}h < 1/3$$

Then (2.6) follows, and the proof of the theorem is completed.

Remark. A time-symmetric version of Theorem 2.1 also holds for the backward iterations of stable fibers, $T^{-n}W^s$, $n \geq 1$. A precise statement of that is obvious. (Of course, additional cuts of stable fibers in M along a finite number of local unstable fibers can be done in a completely similar way.)

Remark. We have actually proved more than Theorem 2.1 says. Recall that the proof was done by induction on n: we assumed that the measure ν^u_n on T^nW^u satisfied (2.1) and deduced the same inequality for $\nu^u_{n+1} = T_*\nu^u_n$. The measure ν^u_n was supported on a finite union of unstable fibers in M_n, on each of which it was proportional to the u-SBR measure, but we never used the fact that those fibers were images of one original fiber W^u under T^n. Therefore, we have proved the following:

Let μ be a finite measure supported on a finite union of unstable fibers in M, such that its conditional distributions on those fibers are proportional to the u-SBR measures. Let the relative μ-measure of the union of ε-neigborhoods of the endpoints of those fibers be $\leq C_1\varepsilon$ for all $\varepsilon > 0$. Then the same property holds for $T_*\mu$.

The estimate on the amount of T^nW^u eaten up by holes under the action of T obtained in the proof of the above theorem can be greatly improved.

Theorem 2.2 *Under the conditions of Theorem 2.1, for every $n \geq 0$ we have*

$$\frac{\nu^u_n((T^nW^u) \cap M_{-1})}{\nu^u_n(T^nW^u)} \geq \lambda_h \stackrel{\text{def}}{=} 1 - C_2h \tag{2.7}$$

where

$$C_2 = C_1D(\Lambda_{\max}/\Lambda_{\min} + 1) = 48D^2(\Lambda_{\max}/\Lambda_{\min} + 1)/d_0$$

Consequently,

$$\frac{\nu^u(W^u \cap M_{-n})}{\nu^u(W^u)} \geq \lambda_h^n \tag{2.8}$$

Proof. We prove (2.7), which is equivalent to

$$\frac{\nu_n^u((T^n W^u)\cap \hat{T}^{-1}H)}{\nu_n^u(T^n W^u)} \le C_2 h \tag{2.9}$$

Every component $W_{n,i}^u$ of $T^n W^u$ has length $\le 2d_0$ due to our additional cuts. Its image under $\hat{T}$ has length $\le 2\Lambda_{\max} d_0$. So, it may intersect no more than $2\Lambda_{\max}+1$ holes. The intersection with every hole has length $\le h$ on $\hat{T}W_{n,i}^u$, so that the total length of the subset $W_{n,i}^u \cap \hat{T}^{-1}H$ on the curve $W_{n,i}^u$ is less than $\Lambda_{\min}^{-1}(2\Lambda_{\max}+1)h$. The ν_n^u measure of that subset does not exceed the ν_n^u measure of the $(\Lambda_{\max}/\Lambda_{\min}+1)h$-neighborhood of the endpoints of $W_{n,i}^u$ times the nonlinearity correction factor D. Now the bound (2.9) follows from Theorem 2.1.

From now on, we remove the additional cuts of unstable fibers introduced in this section. In the theorems just proved this will amount to gluing together the components of $T^n W^u$ that have been artificially cut. Clearly, both theorems will hold after we glue together any components of $T^n W^u$.

3 The structure of the sets M_+, M_- and Ω

On every unstable fiber W^u, the set of points whose forward images never fall through holes has ν^u-measure zero, because the original Anosov diffeomorphism $\hat{T}$ is ergodic. However, it follows from Theorem 2.1 that every fiber W^u of length $> d_0/\Lambda_{\min}$ contains points whose forward images never escape (belong in M_-), and the set of those points is a Cantor-like set on W^u. Furthermore, it follows from Theorem 2.1 that for every $n \ge 1$ the set M_n contains unstable fibers of length $\ge 2C_1^{-1} = (24D)^{-1}d_0$. Indeed, the theorem implies that for any $\varepsilon < C^{-1}$ the set $T^n W^u \setminus \cup_i W_{n,i}^u(\varepsilon)$ is not empty, so that there are components $W_{n,i}^u \subset T^n W^u \subset M_n$ of length $\ge 2\varepsilon$. Moreover, since $\Lambda_{\min} > 64D^2$, the images of those components of $T^n W^u$ will contain components of $T^{n+1}W^u \subset M_{n+1}$ of length $\ge d_0$. Since $M_+ = \cap_n M_n$, the closed set M_+ also contains such fibers. Since the forward and backward dynamics are symmetric, we immediately get

Proposition 3.1 *The set M_+ and M_- are not empty. Moreover, the set M_+ contains some unstable fibers of length $\ge d_0$, and the set M_- contains some stable fibers of length $\ge d_0$.*

We say that an unstable fiber $W^u \subset M$ is *eventually long* if for some $n \ge 1$ its image $T^n W^u$ contains a component of length $\ge d_0$. Otherwise the fiber W^u is said to be *forever short.*

Lemma 3.2 *Every fiber $W^u \subset M$ of length $\ge d_1$, where*

$$d_1 = \frac{h}{\Lambda_{\min} - 3}$$

is eventually long.

Proof. The curve $\hat{T}W^u$ has length $\geq \Lambda_{\min} d_1$. If it intersects more than one hole, it has a component of length $\geq d_0$ already. If it intersects one hole, the intersection consists of curves whose union has diameter $\leq h$. Then $TW^u = (\hat{T}W^u) \setminus H$ necessarily contains a component of length $(\Lambda_{\min} d_1 - h)/2 = \frac{3}{2} d_1$. Thus, the maximal length of components of $T^n W^u$ will grow with n until it necessarily exceeds d_0, hence the lemma.

Fibers that are forever short may exist, but their influence on our results will be negligible in view of the following lemma.

Lemma 3.3 *Let W^u be a fiber of length $d > 0$ that is forever short, and let ν^u be the u-SBR measure on W^u. Then for every $n \geq 1$ we have*

$$\frac{\nu^u(W^u \cap M_{-n})}{\nu^u(W^u)} \leq \frac{B_0^n d_1 D}{\Lambda_{\min}^n d}$$

Proof. For every $i \geq 0$ the set $T^i W^u$ consists of short unstable fibers, whose lengths are $< d_1$. Any such fiber can intersect just one hole. Therefore, the number of components in $T^i W^u$ can only increase by a factor of B_0 at every iteration. The set $T^n W^u$ then consists of $\leq B_0^n$ curves of length $< d_1$. Their preimages under T^{-n} have lengths $< d_1/\Lambda_{\min}^n$, so their total length within W^u is less than $B_0^n d_1/\Lambda_{\min}^n$. Hence the lemma.

Assumption H2. $h < (1 - B_0/\Lambda_{\min})/C_2$, so that $\lambda_h > B_0/\Lambda_{\min}$.

Under this assumption, Lemma 3.3 shows that the images of any forever short fiber fall through holes at a higher rate than those of eventually long ones, cf. (2.8). Due to this, only fibers that are eventually long will be essential in our study.

We are now going to prove that all sufficiently long unstable and stable fibers contained in M_+ and M_-, respectively, form a sort of connected 'net' in M. The following lemma is a key argument.

Lemma 3.4 *On any unstable fiber $W^u \subset \hat{M}$ of length $\geq d_2$, where*

$$d_2 = \frac{D \Lambda_{\min} d_1}{\sin \phi_0} = \frac{D \Lambda_{\min} h}{(\Lambda_{\min} - 3) \sin \phi_0}$$

there is a closed uncountable Cantor-like subset $W_-^u \subset W^u$ such that for every $x \in W_-^u$ there is a stable fiber $W_x^s \subset M_-$ containing x whose endpoints are the distance $\geq d_0/3$ away from x.

Proof. For any $\delta \in (0, d_0/3]$ let H_δ^s be the union of all stable fibers of length δ intersecting the set H. In other words, we 'stretch' each hole by the distance δ in the stable directions. We further enlarge these holes so that if x, y belong in one hole (one connected component of H_δ^s) and lie on one local unstable fiber, then the segment of that fiber between x and y is also included in the hole. The union of these stretched and enlarged holes is denoted by $\hat{H}^s_\delta$. Thus, any local unstable fiber intersects any hole in $\hat{H}^s_\delta$ in at most one interval. The length of that interval does not exceed $Dh/\sin\phi_0$.

We set $\delta = d_0/3$. We now consider iterations $\hat{T}^n W^u$, $n \geq 0$, of the given fiber W^u, but at the nth iteration we erase all the points of $\hat{T}^n W^u$ that fall into the larger holes $\hat{H}^s_\delta$ rather than H. If, for a point $x \in W^u$, none of its forward images $\hat{T}^n x$, $n \geq 0$, is erased, then clearly $x \in W^u_-$.

On any fiber W^u of length d_2 we erase at most one curve of length $< Dh/\sin\phi_0$. Then the remaining part of that fiber necessarily contains two disjoint subcurves of length

$$\frac{1}{3}\left(d_2 - \frac{Dh}{\sin\phi_0}\right) = \frac{Dh}{(\Lambda_{\min} - 3)\sin\phi_0}$$

that do not intersect $\hat{H}^s_\delta$. Their images under $\hat{T}$ have lengths $> d_2$, and we can apply the same argument to each of them. Continuing iterating and doubling such subcurves of length $> d_2$ gives, in the limit, an uncountable subset of points $x \in W^u_-$. Lemma 3.4 is proved.

Assumption H3. $h < (10D)^{-1} d_0 \sin\phi_0$, so that $d_2 \leq d_0/9$.

Then Lemma 3.4 says that any unstable fiber of length d_2 is crossed by much longer stable fibers in M_-. Due to the symmetry, the dual statement is true for any stable fiber of length d_2. Since the phase space M is compact and connected, we get the following.

Theorem 3.5 *Denote by $M_\pm(d_2)$ the union of all fibers in $M_\pm$ (unstable or stable, respectively) of length $> d_2$. Then the set $M_+(d_2) \cup M_-(d_2)$ is connected. The set $M_+(d_2) \cap M_-(d_2)$ is a subset of Ω, and it makes a $(2d_2)$-net in the manifold $\hat{M}$.*

So far we have obtained some local properties of the map T on M. The next theorem is the only one in this section requiring a global argument.

Theorem 3.6 *If h is small enough (i.e, it satisfies assumption H4 below), then there is a $k_1 \geq 1$ such that for every two fibers $W^u, W^s \subset M$ of length d_0 there is a smooth component of $T^{k_1} W^u = \hat{T}^{k_1}(W^u \cap M_{-k_1})$ intersecting W^s. Moreover, the endpoints of that smooth component are at least a distance d_0 away from W^s.*

Proof. First, we enlarge our holes so that if x, y belong in one hole and lie on one local stable or unstable fiber, then the segment of that fiber between x and y lies in the hole, too. Thus, any local fiber intersect any hole in at most one interval.

Let $P_k(W^u, W^s)$ be the number of points of intersection between $\hat{T}^k W^u$ and W^s. It easily follows from the general theory of Anosov diffeomorphisms that the sequence

$$P_k(d_0) = \inf_{W^u, W^s} P_k(W^u, W^s)$$

(infimum being taken over all fibers of length d_0) grows exponentially in k. More precisely, $P_k(d_0)$ grows asymptotically as $e^{kh_{\rm top}}$, where $h_{\rm top} > 0$ is the topological entropy of $\hat{T}$. We then put

$$k_1 = \min\{k : \; P_k(d_0) > N_H k\}$$

We now have to show that at least one point in $\hat{T}^{k_1}W^u \cap W^s$ is not covered by the set $\cup_{i=1}^{k_1-1}\hat{T}^i H$. The image of every hole under $\hat{T}^i$, $1 \le i < k_1$, covers on the curve $\hat{T}^{k_1}W^u$ one or more intervals. The theorem will follow if we show that

(i) such an interval is unique for every hole and every $i = 1, \dots, k_1 - 1$;
(ii) every interval has length $< d_t/3$, in particular, it intersects the fiber W^s at most once;

Here d_t denotes the minimum length of a stable (unstable) fiber for $\hat{T}$ in $\hat{M}$ whose both endpoints lie on any one unstable (stable) fiber of length d_0. Note that $d_0 + d_t$ is greater than the length of the shortest closed geodesic on $\hat{M}$, and then $d_t/3 > d_0$.

Assumption H4. $h < \frac{1}{3}\, d_t\, \Lambda_{\max}^{-k_1} \sin\phi_0$.

This assumption implies (ii) immediately. To prove (i), assume that some hole in H intersects the curve $\hat{T}^i W^u$ for some $1 \le i \le k_1 - 1$ in two intervals. Then there is a stable fiber W_1^s of length $\le 2h/\sin\phi_0$ whose both endpoints lie on $\hat{T}^i W^u$. Taking its preimage $\hat{T}^{-i}W_1^s$ leads to a contradiction with Assumption H4. The theorem is proved.

Remark. Of course, the above theorem has a time-symmetric counterpart: for any two fibers $W^u, W^s \subset M$ of length d_0 there is a smooth component of $T^{-k_1}W^s$ intersecting W^u, and its endpoints are at least a distance d_0 away from W^u.

4 Markov approximation

To further study the dynamics of T on M we will define a sequence of approximations of the holes H by unions of 'rectangular' holes, $H^{(k)}$, $k \ge 1$. To this end we take a sequence $\hat{\mathcal{R}}^{(k)}$ of increasingly fine Markov partitions of $\hat{M}$, and define $H^{(k)}$ to be the union the interiors of all rectangles in $\hat{\mathcal{R}}^{(k)}$ that intersect H. Our previous results [5, 6] provide us with conditionally invariant measures $\mu_+^{(k)}$ for the map $\hat{T}$ with holes $H^{(k)}$. As $k \to \infty$, the 'rectangular' holes $H^{(k)}$ will be arbitrarily close to the original holes H. This enables us to construct the conditionally invariant measure μ_+ as a weak limit of $\mu_+^{(k)}$ as $k \to \infty$.

Let $\hat{\mathcal{R}} = \{R_1, \dots, R_I\}$ be an arbitrary Markov partition of $\hat{M}$. We assume that the rectangles $R_i \in \hat{\mathcal{R}}$ are of sufficiently small diameter so that symbolic dynamics are defined [2]. Also, we can assume that every rectangle R_i is closed and connected [10], i.e., every R_i is a curvilinear quadrilateral bounded by two unstable and two stable sides.

The union of stable fibers bounding the rectangles $R \in \hat{\mathcal{R}}$ is invariant under $\hat{T}$, while the union of unstable fibers is invariant under $\hat{T}^{-1}$. Therefore, all these fibers lie on the global stable and unstable fibers of a finite number of periodic points, whose union is called the *core* of the given Markov partition.

We call a *generic fiber* any stable or unstable fiber in $\hat{M}$ that is not a part of the global fiber of a core point of $\hat{\mathcal{R}}$. There are then only a countable number of nongeneric R-fibers in the rectangles $R \in \mathcal{R}$.

For any $k \geq 1$ the partition

$$\hat{\mathcal{R}}^{(k)} = \bigvee_{i=-k}^{k} \hat{T}^i \hat{\mathcal{R}}$$

is also Markov and consists of connected rectangles.

We put

$$\mathcal{R}^{(k)} = \{R \in \hat{\mathcal{R}}^{(k)} : R \cap H = \emptyset\}$$

and $M^{(k)} = \cup_{R \in \mathcal{R}^{(k)}} R$, and $H^{(k)} = \hat{M} \setminus M^{(k)}$. Clearly, $H^{(1)} \supset H^{(2)} \supset \cdots$ and $\cap_k H^{(k)} = H$, so that the sets $H^{(k)}$ approximate the holes H 'from outside'. In this section we assume that k is large enough, $k \geq k_0(\mathcal{R}, d_0, h)$, so that the set $H^{(k)}$ consists of small open holes satisfying the assumptions H1-H4. Therefore, all the results of the previous section apply to the map $\hat{T}$ restricted to $M^{(k)}$ for $k \geq k_0$

For every $k \geq k_0$ the set $M^{(k)}$ is a finite union of Markov rectangles. We put, analogously to (1.1) and (1.2)

$$M_n^{(k)} = \cap_{i=0}^n T^i M^{(k)}, \quad M_{-n}^{(k)} = \cap_{i=0}^n T^{-i} M^{(k)}$$

$$M_+^{(k)} = \cap_{n \geq 0} M_n^{(k)}, \quad M_-^{(k)} = \cap_{n \geq 0} M_{-n}^{(k)}, \quad \Omega^{(k)} = M_+^{(k)} \cap M_-^{(k)}$$

The set $M_+^{(k)}$ is a union of some unstable R-fibers in the rectangles $R \in \mathcal{R}^{(k)}$. Likewise, $M_-^{(k)}$ is a union of some stable R-fibers in the rectangles $R \in \mathcal{R}^{(k)}$. The set $\Omega^{(k)}$ is a closed Cantor-like set, which has a direct product structure inside every rectangle $R \in \mathcal{R}^{(k)}$.

We denote by $T^{(k)}$ the map $\hat{T}$ restricted to $M^{(k)}$. A detailed study of Anosov maps with 'rectangular' holes, which are some rectangles of one Markov partition, was performed in [5, 6]. In particular, it was shown that the map $T^{(k)}$ always has a conditionally invariant measure $\mu_+^{(k)}$ with an eigenvalue $\lambda_+^{(k)}$. The conditional distributions of $\mu_+^{(k)}$ on unstable fibers of $M_+^{(k)}$ are u-SBR measures. While the eigenvalue $\lambda_+^{(k)}$ is uniquely determined by $T^{(k)}$, the measure $\mu_+^{(k)}$ may be not unique.

The study in [5, 6] was technically simpler than the one we present here. Yet, the properties of the map $T^{(k)}$ on $M^{(k)}$ greatly depend on the structure of allowed transitions between the rectangles $R \in \mathcal{R}^{(k)}$.

To describe the above structure, we invoke the language of symbolic dynamics. The Markov partition $\hat{\mathcal{R}}^{(k)}$ of $\hat{M}$ defines a symbolic representation of $\hat{T}$ in terms of a subshift of finite type, which is topologically mixing because so is $\hat{T}$. This shift restricted to $\mathcal{R}^{(k)}$ is also a subshift of finite type. That one need not be topologically mixing or even transitive. It generally consists of several ergodic components, on each of which the subshift either is topologically mixing or cyclically permutes several subcomponents, there may be one-way connections between ergodic components and some nonrecurrent rectangles as well, see [6]. The influence of every ergodic component, $E_j^{(k)}$, on invariant and conditionally invariant measures on $\Omega^{(k)}$ is determined by its *eigenvalue*, $\lambda_j^{(k)} \in (0, 1)$ related to the *escape rate* $\gamma_j^{(k)} = -\ln \lambda_j^{(k)}$. In fact, the eigenvalue $\lambda_+^{(k)}$ of $T^{(k)}$ is the largest eigenvalue $\lambda_{\max}^{(k)} = \max_j \lambda_j^{(k)}$ of its ergodic components. Furthermore, only the ergodic components with the largest eigenvalue $\lambda^{(k)}_{\max}$ (i.e., with the smallest escape rate) determine the conditionally invariant measure $\mu^{(k)}_+$ [6]. We call them *dominating ergodic components*. If there is more than one dominating component in the system, their influences on $\mu^{(k)}_+$ are determined by the structure of one-way connections between them, see [6] for more detail.

The next few lemmas show that if h is small enough, then there exists a unique dominating ergodic component, on which the subshift is topologically mixing.

Lemma 4.1 *There is an ergodic component $E^{(k)}_j$ with eigenvalue $\lambda^{(k)}_j \geq \lambda_h$ (the latter was introduced by (2.7)).*

The lemma immediately follows from Theorem 2.2.

Lemma 4.2 *Let a rectangle $R \in \mathcal{R}^{(k)}$ belong to an ergodic component $E^{(k)}_j$ with $\lambda^{(k)}_j \geq \lambda_h$. Then its interior contains $T^{(k)}$-eventually long unstable fibers and $T^{(k)}$-eventually long stable fibers.*

Proof. If every unstable R-fiber $W^u \subset \operatorname{int} R$ were forever short, then according to Lemma 3.3 the eigenvalue $\lambda^{(k)}_j$ could not exceed $B_0/\Lambda_{\min}$, which would contradict to Assumption H2.

Now let every stable R-fiber $W^s \subset \operatorname{int} R$ be forever short. As it was also shown in Lemma 3.3, the preimage $[T^{(k)}]^{-n}W^s$ consists of not more than B_0^n short fibers. So, $[T^{(k)}]^{-n}R \cap R$ consists of not more than B_0^n connected subrectangles. Hence, the topological entropy of the subshift on the ergodic component $E^{(k)}_j$ is $\leq \ln B_0$. Then the invariant measure $\eta^{(k)}_j$ on the component $E^{(k)}_j$ has measure-theoretic entropy $h(\eta^{(k)}_j) \leq \ln B_0$. Its positive Lyapunov exponent $\chi^{(k)}_j$ is definitely $\geq \ln \Lambda_{\min}$. Now, recall [5, 6] that these quantities satisfy the escape rate formula

$$\chi^{(k)}_j = h(\eta^{(k)}_j) - \ln \lambda^{(k)}_j$$

This immediately leads to a contradiction with Assumption H2. The lemma is proved.

Lemma 4.3 *Let a rectangle $R_1 \in \mathcal{R}^{(k)}$ contain an eventually long unstable fiber $W^u \subset \operatorname{int} R_1$ and a rectangle $R_2 \in \mathcal{R}^{(k)}$ contain an eventually long stable fiber $W^s \subset \operatorname{int} R_2$. Then there is an $m_0(R_1, R_2) \geq 1$ such that transitions from R_1 to R_2 are allowed in any number of steps $m \geq m_0(R_1, R_2)$.*

Proof. Once the image $[T^{(k)}]^{m_1}W^u$ contains a component of length $\geq d_0$, every further image $[T^{(k)}]^m W^u$, $m \geq m_1$, will contain such components. The same is true for all $[T^{(k)}]^{-m}W^s$, $m \geq m_2$, if only $[T^{(k)}]^{-m_2}W^s$ contains a long component. Then the lemma follows from Theorem 3.6, and $m_0(R_1, R_2) = m_1 + m_2 + k_1$.

Corollary 4.4 *There is just one ergodic component, $E^{(k)}_+$ with eigenvalue $> \lambda_h$. On this component the subshift is topologically mixing.*

Let $T_*^{(k)}$ be the adjoint operator on Borel measures on $M^{(k)}$, i.e. $(T_*^{(k)}\mu)(A) = \mu([T^{(k)}]^{-1}(A \cap M_1^{(k)}))$ for any Borel $A \subset \hat{M}$. Let $T_+^{(k)}$ be the operator on probability measures on $M^{(k)}$ such that $T_+^{(k)}\mu = T_*^{(k)}\mu/||T_*^{(k)}\mu||$, where the norm of a measure μ on $M^{(k)}$ is set to be $||\mu|| = \mu(M^{(k)})$.

The next theorem follows from the results of [5, 6].

Theorem 4.5 *For all sufficiently large k ($k \geq k_0(\mathcal{R}, d_0, h)$) we have the following:*
(i) *the map $T^{(k)}$ on $M^{(k)}$ has a conditionally invariant probability measure $\mu_+^{(k)}$ supported on $M_+^{(k)}$, i.e. $T_*^{(k)}\mu_+^{(k)} = \lambda_+^{(k)}\mu_+^{(k)}$ for some constant $\lambda_+^{(k)} \in (0,1)$;*
(ii) *the measure $\mu_+^{(k)}$ conditioned on any unstable R-fiber $W_R^u(x) \subset M_+^{(k)}$, $R \in \mathcal{R}^{(k)}$ is the u-SBR measure on that fiber;*
(iii) *for any smooth*[1] *measure μ on $M^{(k)}$ the measures $[T_+^{(k)}]^n\mu$ weakly converge, as $n \to \infty$, to $\mu_+^{(k)}$;*
(iv) *for any smooth measure μ on $M^{(k)}$ there is a constant $c_\mu > 0$ such that the measures $c_\mu[\lambda_+^{(k)}]^{-n}[T_*^{(k)}]^n\mu$ weakly converge to $\mu_+^{(k)}$. If μ is a probability measure, then c_μ is bounded away from 0 and ∞.*

5 Approximation of the conditionally invariant measure

Here we define the conditionally invariant measure μ_+ for the map T as a weak limit of the measures $\mu_+^{(k)}$.

Since $M^{(1)} \subset M^{(2)} \subset \cdots$, we have $\lambda_+^{(1)} \leq \lambda_+^{(2)} \leq \cdots$, so that there is a limit

$$\lim_{k\to\infty} \lambda_+^{(k)} = \lambda_+ \in (\lambda_h, 1) \tag{5.1}$$

The measures $\mu_+^{(k)}$ are all supported on the compact set M_+, hence the sequence of these measures has at least one weak limit point. Let μ_+ be any weak limit point of this sequence, it will be a probability measure on M_+. We will first investigate the properties of any such μ_+. In the next section we will show that there cannot be two distinct limit points, so that the sequence $\mu_+^{(k)}$ weakly converges to μ_+. In this section, we denote by μ_+ an arbitrary weak limit point of the sequence $\{\mu_+^{(k)}\}$.

Proposition 5.1 *The measure μ_+ is conditionally invariant under T with eigenvalue λ_+, i.e. $T_*\mu_+ = \lambda_+\mu_+$.*

This follows directly from Theorem 4.5 and (5.1).

Proposition 5.2 *The measure μ_+ conditioned on any generic unstable fiber $W^u \subset M_+$ is a u-SBR measure on W^u.*

[1] We remind that our convention on smoothness, cf. Introduction, is still valid.

Proof. For any $k \geq k_0$ we put $W_+^{u,(k)} = W^u \cap M_+^{(k)}$. The diameter of the set $T^{-n}W_+^{u,(k)}$ converges exponentially to zero as $n \to \infty$. Either this set lies in one rectangle $R \in \mathcal{R}^{(k)}$ for every sufficiently large n, or it intersects more than one rectangle for every n. In the latter case, $T^{-n}W_+^{u,(k)}$ converges, as $n \to \infty$, to the periodic orbit of a core point of the Markov partition $\hat{\mathcal{R}}$, so it is not generic. In the former case, the measure $\mu_+^{(k)}$ conditioned on $T^{-n}W_+^{u,(k)}$ is the u-SBR measure on it. Since $\mu_+^{(k)}$ is conditionally invariant, it is also a u-SBR measure on $W_+^{u,(k)}$. Taking the limit as $k \to \infty$ proves the proposition.

It also follows from the above proof that if a finite number of generic unstable fibers in M_+ lie on one global unstable fiber of the original Anosov diffeomorphism $\hat{T}$, then the measure μ_+ conditioned on their union is also a u-SBR measure on that union.

Proposition 5.3 *The μ_+-measure of the union of nongeneric fibers is zero.*

Proof. Since there are countably many nongeneric fibers, it is enough to show that the μ_+-measure of every one is zero. If a fiber $W^u \subset M_+$ is nongeneric, then according to the proof of Proposition 5.2 it can be divided into two subsegments on each of which the conditional μ_+-measure is u-SBR. The fiber W^u belongs to a global unstable fiber, Γ^u, of a core periodic point, which is invariant under some iterate of $\hat{T}$, say, under $\hat{T}^k$. Now, if the set $\Gamma^u \cap M_+$ has positive μ_+-measure, the map $\hat{T}^k$ stretches Γ^u by a factor $\geq \Lambda_{\min}^k$ decreasing μ_+ on $\Gamma^u \cap M_+$ by at least $\Lambda_{\min}^{-k}$. On the other hand, Proposition 5.1 says that the measure μ_+ under the action of T^k decreases by the factor of λ_+^k. Since $\Lambda_{\min}^{-1} < \lambda_+$ due to Assumption H2, we get a contradiction proving Proposition 5.3.

Combining the last two propositions shows that the measure μ_+ conditioned on *any* unstable fiber is a u-SBR measure. Also, we can extend the last proposition showing that any unstable fiber in M_+ has zero μ_+-measure.

Our next step is to extend Theorem 2.1 to the measures $\mu_+^{(k)}$ and μ_+. For any $x \in M_+^{(k)}$ let $W_{x,+}^{u,(k)}$ be the largest segment of the unstable fiber in $M_+^{(k)}$ containing x. Likewise, $W_{x,+}^u$ is the largest segment of the unstable fiber in M_+ containing $x \in M_+$. For any $\varepsilon > 0$ we put

$$U_\varepsilon^{(k)} = \{x \in M_+^{(k)} : \operatorname{dist}(x, \partial W_{x,+}^{u,(k)}) < \varepsilon\}$$

where the distance is measured along the unstable fiber $W_{x,+}^{u,(k)}$. Removing the superscript (k) in the above formula will define U_ε.

Lemma 5.4 *For any $k \geq k_0$ and $\varepsilon > 0$ the set $M_+^{(k)} \setminus U_\varepsilon^{(k)}$ is compact (it may be empty for large ε). The same holds for the set $M_+ \setminus U_\varepsilon$.*

Proof. Let $x_l \in M_+^{(k)} \setminus U_\varepsilon^{(k)}$ be a sequence of points converging to a point $x \in \hat{M}$. Since the map $\hat{T}$ is smooth and the holes $H^{(k)}$ are open, it is easy to verify that the segment of the unstable fiber through x of length 2ε centered at x belongs in $M_+^{(k)}$ (in other words, its preimages under $\hat{T}^{-n}$, $n \geq 0$, never cross holes). This proves the lemma.

Corollary 5.5 *Suppose that a sequence of probability measures* $\{\nu_l\}$ *on* $M_+^{(k)}$ *converges, as* $l \to \infty$*, to a probability measure* ν*. Suppose that* $\nu_l(U_\varepsilon^{(k)}) \le p$ *for some* $p \ge 0$ *and for all* $l \ge l_0$*. Then* $\nu(U_\varepsilon^{(k)}) \le p$*. The same holds for* U_ε*.*

Theorem 5.6 *For any* $k \ge k_0$ *and* $\varepsilon > 0$ *we have* $\mu_+^{(k)}(U_\varepsilon^{(k)}) \le C_1\varepsilon$*, where* $C_1 = 48D/d_0$*.*

Proof. Let $W^u \in M_+^{(k)}$ be an unstable fiber whose endpoints are on two stable fibers bounding some rectangles in $\mathcal{R}^{(k)}$. Let ν^u be a u-SBR probability measure supported on W^u. It follows from [5] that the measure $\nu_n^u = [T_+^{(k)}]^n \nu^u$ weakly converges, as $n \to \infty$, to the measure $\mu_+^{(k)}$.

Proposition 3.1 applies to the map $T^{(k)}$ on $M^{(k)}$ for all $k \ge k_0$. Based on it, we can find a fiber $W^u \subset M_+^{(k)}$ longer than d_0. Theorem 2.1 then implies that $\nu_n^u(U_\varepsilon^{(k)}) \le C_1\varepsilon$. Using Corollary 5.5 now completes the proof of Theorem 5.6.

Theorem 5.7 *For any* $\varepsilon > 0$ *we have* $\mu_+(U_\varepsilon) \le C_1\varepsilon$*.*

Proof. Obviously, $U_\varepsilon \cap M_+^{(k)} \subset U_\varepsilon^{(k)}$ for any k. Therefore, $\mu_+^{(k)}(U_\varepsilon) \le C_1\varepsilon$. Applying Corollary 5.5 to U_ε and the subsequence of measures $\mu_+^{(k)}$ that converges to μ_+ proves the theorem.

Corollary 5.8 *For any* $\varepsilon > 0$ *let* V_ε *be the union of all maximal unstable fibers in* M_+ *of length* $< \varepsilon$*. Then* $\mu_+(V_\varepsilon) = o(\varepsilon)$ *as* $\varepsilon \to 0$*.*

Our last step in this section is to estimate measures of rectangles that are sufficiently long in the stable direction. Denote by $\mathcal{B}$ the set of closed rectangles $R \subset \hat{M}$ such that

(i) every stable R-fiber has length between d_0 and $2d_0$;
(ii) every unstable R-fiber has length $\le d_0$.

We denote by $\mu_{\rm SBR}$ the SBR measure on $\hat{M}$. For any rectangle R let $d^u_{\min}(R)$ and $d^u_{\max}(R)$ be the minimal and maximal length of unstable R-fibers. General properties of SBR measures imply that

$$C'\, d^u_{\max}(R) \le \mu_{\rm SBR}(R) \le C''\, d^u_{\min}(R) \qquad (5.2)$$

for some constants C', C'' depending only on $\hat{T}$.

Lemma 5.9 *There is a constant* $C_3' > 0$ *such that for every* $R \in \mathcal{B}$*, every unstable fiber* $W^u \subset \hat{M}$ *of length* d_0 *and every* $k \ge k_0$ *we have*

$$[T_*^{(k)}]^{k_1} \nu^u(R) \ge C_3'\, d^u_{\min}(R)$$

Here ν^u *is the u-SBR measure on* W^u*, and the constant* k_1 *appears in Theorem 3.6.*

The lemma readily follows from Theorem 3.6 with $(C_3')^{-1} = d_0 D \Lambda_{\max}^{k_1}$.

Combining this lemma with Theorems 5.6 and 5.7 gives the following:

Corollary 5.10 *There is a constant $C_3 > 0$ such that for every $R \in \mathcal{B}$ and every $k \geq k_0$ we have*

$$\mu_+^{(k)}(R) \geq C_3\, \mu_{\mathrm{SBR}}(R) \quad \text{and} \quad \mu_+(R) \geq C_3\, \mu_{\mathrm{SBR}}(R)$$

Lemma 5.11 *There is a constant $C_4 > 0$ such that for every $R \in \mathcal{B}$ and every $k \geq k_0$ we have*

$$\mu_+^{(k)}(R) \leq C_4\, \mu_{\mathrm{SBR}}(R) \quad \text{and} \quad \mu_+(R) \leq C_4\, \mu_{\mathrm{SBR}}(R)$$

Proof. Put $\varepsilon = d_{\max}^u(R)/2$. Let W^u be an unstable fiber in $M_+^{(k)}$ (respectively, M_+), and ν^u the u-SBR measure on W^u. Then $\nu^u(W^u \cap R) \leq D\nu^u(W^u \cap U_\varepsilon)$. Applying Theorems 5.6 and 5.7 proves the lemma with $C_4 = DC_1/(2C')$.

6 Uniqueness of the conditionally invariant measure

This section is devoted to the following theorem:

Theorem 6.1 *There is a unique measure μ_+ on M_+ with the following properties:*
(i) *μ_+ is a conditionally invariant probability measure with the eigenvalue λ_+;*
(ii) *the conditional distributions of μ_+ on unstable fibers in M_+ are u-SBR measures;*
(iii) *for any $\varepsilon > 0$ we have $\mu_+(U_\varepsilon) \leq C_1\varepsilon$.*
(iv) *for any $R \in \mathcal{B}$ we have $C_3\, \mu_{\mathrm{SBR}}(R) \leq \mu_+(R) \leq C_4\, \mu_{\mathrm{SBR}}(R)$.*

Before proving it, we will make some additional constructions.

Let μ_+ be an arbitrary measure satisfying the assumptions of this theorem. Let $W^s \subset \hat{M}$ be a stable fiber and R a rectangle whose one side is W^s. Denote by R_+ the union of all unstable R-fibers that entirely belong in M_+, i. e. $R_+ = \{x \in R: \; W_x^u \cap R \subset M_+\}$.

There are two finite limits then,

$$\mathcal{D}_{1,2}(\mu_+, W^s) = \lim{}^* \frac{\mu_+(R_+)}{\mu_{\mathrm{SBR}}(R)} \tag{6.1}$$

Here $\lim^*$ is taken as the rectangle R shrinks in the unstable direction, so that its stable side opposite to W^s approaches W^s. The two values, $\mathcal{D}_1$ and $\mathcal{D}_2$ correspond to two possible locations of R, which may be placed on either side of the curve W^s (the subscripts, 1 and 2, are assigned arbitrarily). The existence of the finite limits in (6.1) follows from Propositions 5.2 and 5.3 and Corollary 5.8. Note also that Corollary 5.8 implies that

$$\mu_+(R_+) = \mu_+(R) + o(d_{\min}^u(R)) = \mu_+(R) + o(\mu_{\mathrm{SBR}}(R)) \tag{6.2}$$

Convention. In this section, for any rectangle R we will denote by R_+ the union of unstable R-fibers that entirely belong in M_+. Then, for a rectangle denoted by R_n or $R_{k,l}$, for example, we denote by $R_{n,+}$ and $R_{k,l,+}$, respectively, the union of unstable R_n-fibers or $R_{k,l}$-fibers belonging entirely in M_+.

The values of $\mathcal{D}_{1,2}(\mu_+, W^s)$ for the given measure μ_+ and all the stable fibers $W^s \subset \hat{M}$ characterize the distribution of μ_+ in the stable direction.

Note that, although R is a rectangle, the preimages of R under T^{-n}, after the removal of holes, are no longer rectangles. They are some domains adjacent to the smooth components of $T^{-n}W^s$, where W^s is a stable side of the original rectangle R. Since the connected components of $T^{-n}R$ are not rectangles, it would be difficult to compute with them. Instead, we consider R_+, which has the following invariance property. For any $n \geq 1$ let $R_{n,i}$, $i = 1, 2, \ldots$, be u-subrectangles in $\hat{T}^{-n}R$ whose stable sides are the smooth components of $T^{-n}W^s$. Then

$$T^{-n}R_+ = \cup_i R_{n,i,+} \tag{6.3}$$

This property is easy to verify by induction on n. Note that, according to (6.2), the loss of measure incurred by the replacement of R by R_+ is relatively small as the rectangle R shrinks in the unstable direction.

We call a fiber W^s *essential* for the measure μ_+ if $\mathcal{D}_{1,2}(\mu_+, W^s) > 0$. We will see later, cf. Corollary 6.5, that this is equivalent to $\mu_+(R_+) > 0$ for at least one (and then for every sufficiently narrow) rectangle R adjacent to W^s. As a result, if a fiber W^s contains a piece that is essential, then W^s itself is an essential fiber.

It follows immediately from Corollary 5.10 and Lemma 5.11 that if a fiber W^s has length $\geq d_0$, then it is essential and

$$0 < C_3 \leq \mathcal{D}_{1,2}(\mu_+, W^s) \leq C_4 < \infty \tag{6.4}$$

Assume that there are two distinct measures, μ_1 and μ_2, on M_+ satisfying the assumptions of this theorem. We want to compare μ_1 and μ_2 quantitatively. Since their distributions along unstable fibers coincide by the assumption (ii), it is enough to compare their distributions along stable directions. For any stable fiber $W^s \subset \hat{M}$ that is essential for both μ_1 and μ_2 (see also Corollary 6.5 below) and $j = 1, 2$ there exists a finite positive limit

$$\mathcal{D}_j(\mu_1, \mu_2, W^s) = \lim{}^* \frac{\mu_1(R_+)}{\mu_2(R_+)} \tag{6.5}$$

Note that $\mathcal{D}_j(\mu_1, \mu_2, W^s) \cdot \mathcal{D}_j(\mu_2, \mu_1, W^s) = 1$. It is easy to see that if $\mathcal{D}_j(\mu_1, \mu_2, W^s) = 1$ for all essential fibers W^s and $j = 1, 2$, then $\mu_1 = \mu_2$.

We then put

$$\mathcal{D}(\mu_1, \mu_2) = \sup\{\mathcal{D}_j(\mu_1, \mu_2, W^s) : \ W^s \ \text{ is essential and } \ j = 1, 2\}$$

and

$$\mathcal{D}^*(\mu_1, \mu_2) = \sup\{\mathcal{D}_j(\mu_1, \mu_2, W^s) : \ W^s \ \text{ has length } \ \geq d_0 \ \text{ and } \ j = 1, 2\}$$

We now sketch the proof of Theorem 6.1. It is enough to prove that $\mathcal{D}(\mu_1, \mu_2) = 1$.

Our first goal is to prove that $\mathcal{D}(\mu_1, \mu_2)$ and $\mathcal{D}^*(\mu_1, \mu_2)$ are equal (Lemma 6.3). This means that it is enough to compare μ_1 and μ_2 along stable fibers of length $\geq d_0$.

Our second -and more difficult- goal is to prove that this common value is **one** (Lemma 6.6). Since W^s in the definition of $\mathcal{D}^*(\mu_1, \mu_2)$ is longer than d_0 we can effectively use the global Theorem 3.6.

First we show that if $\mathcal{D}^*(\mu_1, \mu_2) > 1$, then there are two fibers W_1^s, W_2^s crossing a fixed unstable fiber $\tilde{W}^u \subset \hat{M}$ of length d_0 such that the $\mathcal{D}_j$-values defined by (6.5) for W_1^s and W_2^s approximate the values of $\mathcal{D}^*(\mu_1, \mu_2) > 1$ and $\mathcal{D}^*(\mu_2, \mu_1) < 1$ arbitrarily well (Sublemma 6.7).

Next we consider the rectangles whose stable sides belong in long components of $T^{-n}W_i^s$, $i = 1, 2$, respectively, and which are related by the holonomy map in M (projections along unstable directions). These sides are getting closer and closer as n grows, which allows us to show that the $\mathcal{D}_j$-values for W_1^s and W_2^s must be arbitrarily close. This contradiction will prove Lemma 6.6.

We now begin the proof of Theorem 6.1.

Lemma 6.2 *We have*

$$1 \leq \mathcal{D}^*(\mu_1, \mu_2) \leq C_4/C_3$$

The lower bound is obvious since μ_1 and μ_2 are probability measures. The upper bound immediately follows from (6.4).

Lemma 6.3 *We have*

$$\mathcal{D}(\mu_1, \mu_2) = \mathcal{D}^*(\mu_1, \mu_2)$$

Proof. Let W^s be a stable fiber of length $< d_0$, and R a rectangle whose one side is W^s. Since both measures μ_1 and μ_2 are conditionally invariant, we have $\mu_i(R_+) = \lambda_+^{-n} T_*^n \mu_i(R_+) = \lambda_+^{-n} \mu_i(\hat{T}^{-n} R_+)$ for $i = 1, 2$ and all $n \geq 1$. Therefore,

$$\frac{\mu_1(R_+)}{\mu_2(R_+)} = \frac{\mu_1(T^{-n}R_+)}{\mu_2(T^{-n}R_+)}$$

Let $W_{1,i}^s$, $i = 1, 2, \ldots$, be the smooth components of $T^{-1}W^s$. For every i let $R_{1,i}$ be the rectangle whose one side is $W_{1,i}^s$ and the opposite side belongs to $\hat{T}^{-1}(\partial R)$. Note that $T^{-1}R_+ = \cup_i R_{1,i,+}$ according to (6.3).

Some of the curves $W_{1,i}^s$ may have length already larger than d_0, then the adjacent rectangles $R_{1,i}$ are not iterated backward any further. If not, we pull them back under $\hat{T}^{-1}$ and construct rectangles $R_{2,i}$ adjacent to the smooth components of $T^{-2}W^s$ in the same way, etc. At every iteration $n \geq 1$, we hold the obtained rectangles $R_{n,p}$, $p = 1, 2, \ldots$, which are adjacent to a smooth component of $T^{-n}W^s$ of length $\geq d_0$, and map the other (shorter) rectangles further under $\hat{T}^{-1}$. Note that if we map all the eventually held (larger) rectangles $R_{n,p}$ (for all n, p) back on R under $\hat{T}^n$, then we get a collection of disjoint u-subrectangles in R, which we denote by R'_j, $j \geq 1$. In this way we obtain a finite or countable collection of disjoint u-subrectangles $R'_j \subset R$, such that for every j there is a $n_j \geq 1$ for which $R'_j \subset M_{n_j}$ and the length of the stable side of the rectangle $R''_j = T^{-n_j}R'_j$ that belongs in $T^{-n_j}W^s$ is $\geq d_0$.

Sublemma 6.4 *The union $\cup_j R'_j$ essentially covers the set $R \cap M_+$, i.e.*

$$\begin{aligned} \mu_i(R_+) = \sum_j \mu_i(R'_j \cap R_+) &= \sum_j \lambda_+^{-n_j} \mu_i(T^{-n_j}(R'_j \cap R_+)) \\ &= \sum_j \lambda_+^{-n_j} \mu_i(R''_{j,+}) \end{aligned} \tag{6.6}$$

for $i = 1, 2$. In other words, the measure μ_i on R_+ is supported on the u-subrectangles R'_j that are 'eventually long' in the past.

Proof. In the iterative process of construction of rectangles R'_j, at every step $n \geq 1$ we may have some 'unfortunate' rectangles $R'''_{n,p} \subset R$, $p = 1, 2, \ldots$, whose preimages $T^{-i}R'''_{n,p}$ for all $i = 1, \ldots, n$ belong in shorter rectangles, i.e., those adjacent to smooth components of $T^{-i}W^s$ of length $< d_0$. Now, by repeating the main argument of the proof of Lemma 3.3 one can see that for every $n \geq 1$ the number of those short rectangles $T^{-n}R'''_{n,p}$ does not exceed B_0^n. The μ_i-measure of every short rectangle $T^{-n}R'''_{n,p}$ is smaller than const$\cdot\Lambda_{\min}^{-n}$, because its width in the unstable direction is less than const$\cdot\Lambda_{\min}^{-n}$. Therefore,

$$\mu_i(\cup_p R'''_{n,p}) \leq \text{const} \cdot \left(\frac{B_0}{\lambda_+ \Lambda_{\min}} \right)^n \tag{6.7}$$

which exponentially approaches zero as $n \to \infty$ due to our Assumption H2. The sublemma is proved.

Now, since all the rectangles $T^{-n_j}R'_j$ are long, the decomposition (6.6) completes the proof of Lemma 6.3. Here we made use of a simple fact that

$$(c_1 + c_2 + \cdots)/(d_1 + d_2 + \cdots) \leq \max_i \{c_i/d_i\} \tag{6.8}$$

for positive c_i, d_i.

The proof of Sublemma 6.4 justifies the following corollary:

Corollary 6.5 *A stable fiber W^s is essential to μ_+ if and only if it is eventually long in the past, i.e. for some $n \geq 1$ a smooth component of $T^{-n}W^s$ is of length $\geq d_0$. It also follows from (6.7) that for every nonessential fiber W^s we have $\mu_i(R_+) = 0$ for all adjacent rectangles R and $i = 1, 2$.*

This corollary is an analogue of Lemma 3.3. Since the property of being eventually long in the past does not refer to any measure, a fiber W^s in our definition (6.5) is essential either for both μ_1 and μ_2 or for neither.

Lemma 6.6 *We have*

$$\mathcal{D}(\mu_1, \mu_2) = \mathcal{D}^*(\mu_1, \mu_2) = 1$$

Proof. Assume that $\mathcal{D}(\mu_1, \mu_2) = \mathcal{D}^*(\mu_1, \mu_2) > 1$. Then for any $\delta > 0$ we can find a stable fiber W^s of length between d_0 and $2d_0$ such that

$$\mathcal{D}_j(\mu_1, \mu_2, W^s) > (1 - \delta)\, \mathcal{D}^*(\mu_1, \mu_2)$$

for $j = 1$ or $j = 2$.

Next, we introduce additional cuts of stable fibers in the same way as we cut unstable fibers in Sect. 2. That is, we fix a finite number of local unstable fibers so that every maximal stable fiber in M (terminating on ∂H) with length $> 2d_0$ is cut into segments of length between d_0 and $2d_0$. None of the maximal stable fibers of length $\leq 2d_0$ should be cut.

Now, for any $n \geq 1$ the set $T^{-n}W^s$ consists of a finite number of stable fibers, say, $W^s_{n,q}$, $q = 1, 2, \ldots$. (These are defined just like the components $W^u_{n,i}$ in Sect. 2). Denote by Q_n the number of these fibers. It follows from our construction and the definition of B_0 that

$$Q_n \leq 4B_0 \Lambda^n_{\max} \tag{6.9}$$

Let R be a sufficiently narrow rectangle whose one side is W^s. For every q let $R_{n,q}$ be a narrow rectangle such that (i) one side of it is $W^s_{n,q}$ and (ii) its opposite side belongs in $\hat{T}^{-n}(\partial R)$. Then $T^{-n}R_+ = \cup_q R_{n,q,+}$ (here $R_{n,q,+}$ is the union of the unstable $R_{n,q}$-fibers that belong in M_+). Since the measures μ_1 and μ_2 are conditionally invariant with the same eigenvalue, we have

$$\frac{\mu_1(R_+)}{\mu_2(R_+)} = \frac{\sum_q \mu_1(R_{n,q,+})}{\sum_q \mu_2(R_{n,q,+})} > (1 - 2\delta)\mathcal{D}^*(\mu_1, \mu_2) \tag{6.10}$$

The last inequality is ensured if the rectangle R is thin enough in the unstable direction for the given n, which will be our standing assumption in the proof of this lemma. Then we also have

$$\frac{\mu_1(R_{n,q,+})}{\mu_2(R_{n,q,+})} \leq (1 + \delta)\, \mathcal{D}_j(\mu_1, \mu_2, W^s_{n,q}) \leq (1 + \delta)\, \mathcal{D}(\mu_1, \mu_2) = (1 + \delta)\, \mathcal{D}^*(\mu_1, \mu_2) \tag{6.11}$$

for every q such that the fiber $W^s_{n,q}$ is essential. For nonessential fibers, both measures in (6.11) are zero, cf. Corollary 6.5.

Note that, according to (6.2), $\mu_i(R_{n,q,+}) \geq \mu_i(R_{n,q})/2$ for $i = 1, 2$. If for some q_0 the curve W^s_{n,q_0} has length $\geq d_0$, then the assumption (iv) of Theorem 6.1 and (5.2) imply that

$$\begin{aligned}
\mu_i(R_{n,q_0,+}) &\geq \frac{1}{2}\mu_i(R_{n,q_0}) \geq \frac{C'C_3}{2} d^u_{\max}(R_{n,q_0}) \\
&\geq \frac{C'C_3}{2}\Lambda_{\max}^{-n}\, d^u_{\min}(R) \geq \frac{C'C_3}{2D}\Lambda_{\max}^{-n}\, d^u_{\max}(R) \\
&\geq \frac{C'C_3}{2D}\left(\frac{\Lambda_{\min}}{\Lambda_{\max}}\right)^n \max_q\{d^u_{\max}(R_{n,q})\} \\
&\geq \frac{C'C_3}{2DC''C_4}\left(\frac{\Lambda_{\min}}{\Lambda_{\max}}\right)^n \max_q\{\mu_i(R_{n,q})\} \\
&\geq \frac{C'C_3}{8DC''C_4B_0}\left(\frac{\Lambda_{\min}}{\Lambda^2_{\max}}\right)^n \sum_{q\neq q_0}\mu_i(R_{n,q,+}) \qquad (6.12)
\end{aligned}$$

for $i = 1, 2$, where in the last line we have used the bound (6.9). Denote

$$G_n = \frac{C'C_3}{8DC''C_4B_0}\left(\frac{\Lambda_{\min}}{\Lambda^2_{\max}}\right)^n$$

It is a simple calculation to combine the above estimates (6.10) and (6.11) that gives

$$\mu_1(R_{n,q_0,+}) \geq (1-2\delta)\,\mathcal{D}^*(\mu_1,\mu_2)\,\mu_2(R_{n,q_0,+}) - 3\delta\,\mathcal{D}^*(\mu_1,\mu_2)\sum_{q\neq q_0}\mu_2(R_{n,q,+})$$

Here we used again the simple fact (6.8). Now, applying the bound (6.12) with $i = 2$ gives

$$\frac{\mu_1(R_{n,q_0,+})}{\mu_2(R_{n,q_0,+})} \geq (1-2\delta-3G_n^{-1}\delta)\,\mathcal{D}^*(\mu_1,\mu_2)$$

Taking a limit as R shrinks in the unstable direction gives

$$\mathcal{D}_j(\mu_1,\mu_2,W^s_{n,q_0}) \geq (1-2\delta-3G_n^{-1}\delta)\,\mathcal{D}^*(\mu_1,\mu_2) \qquad (6.13)$$

We now fix an unstable fiber, $\tilde{W}^u \subset \hat{M}$ of length $\geq d_0$. According to Theorem 3.6 (see also the remark after it), there is a curve $W^s_{k_1,q_0}$ intersecting $\tilde{W}^u$ whose endpoints are at least a distance d_0 away from $\tilde{W}^u$. By choosing δ sufficiently small we can make the right hand side of (6.13) with $n = k_1$ arbitrary close to $\mathcal{D}^*(\mu_1,\mu_2)$. We can also switch μ_1 and μ_2 and repeat the above construction. As a result, we get the following sublemma.

Sublemma 6.7 *Let $\tilde{W}^u \subset \hat{M}$ be an unstable fiber of length d_0. For any $\delta > 0$ there are two stable fibers, W^s_1 and W^s_2 crossing $\tilde{W}^u$, whose endpoints are at least a distance d_0 away from $\tilde{W}^u$, such that*

$$(1-\delta)\,\mathcal{D}^*(\mu_1,\mu_2) \leq \mathcal{D}_{j_1}(\mu_1,\mu_2,W^s_1) \leq \mathcal{D}^*(\mu_1,\mu_2) \qquad (6.14)$$

and

$$(1-\delta)\,\mathcal{D}^*(\mu_2,\mu_1) \le \mathcal{D}_{j_2}(\mu_2,\mu_1,W_2^s) \le \mathcal{D}^*(\mu_2,\mu_1) \tag{6.15}$$

for some $j_1, j_2 \in \{1,2\}$.

Recall that $\mathcal{D}_{j_2}(\mu_1,\mu_2,W_2^s) = 1/\mathcal{D}_{j_2}(\mu_2,\mu_1,W_2^s)$. Thus, by choosing δ sufficiently small we can make the difference

$$\mathcal{D}_{j_1}(\mu_1,\mu_2,W_1^s) - \mathcal{D}_{j_2}(\mu_1,\mu_2,W_2^s)$$

arbitrary close to the fixed positive value

$$\Delta = \mathcal{D}^*(\mu_1,\mu_2) - 1/\mathcal{D}^*(\mu_2,\mu_1) > 0$$

We now have two stable fibers, W_1^s and W_2^s, which are long (longer than $2d_0$), and by sliding one of them along unstable fibers (applying a holonomy map) a distance less than Dd_0 we can cover at least $2D^{-1}d_0$ of the other fiber. We now iterate both fibers backward, and erase the parts of their preimages that fall through holes. Also, whenever a part of the preimage of one fiber is erased, its image under the holonomy map on the preimage of the other fiber is erased, too. It is also convenient to make additional cuts, like in the proof of Lemma 6.6, to ensure that the preimages consist of smooth components of length $\le 2d_0$. Then, for any $n \ge 1$ on the curves $T^{-n}W_1^s$ and $T^{-n}W_2^s$ we obtain a finite number of subsegments, $W_{1,n,l}^s$ and $W_{2,n,l}^s$ $(l = 1,2,\ldots)$, respectively, such that $W_{1,n,l}^s$ is the image of $W_{2,n,l}^s$ under the holonomy map for every l.

It follows from Theorem 2.1 (see the first remark after it) that the larger n the more of the segments $W_{1,n,l}^s$ and $W_{2,n,l}^s$ are long, i.e. have length $\ge d_0$. More precisely, there is a sequence N_n (independent of the choice of the fibers W_1^s and W_2^s) such that $N_n \to \infty$ as $n \to \infty$ and the number of long pairs of unstable fibers $W_{1,n,l}^s$, $W_{2,n,l}^s$ is larger than N_n. We denote by L_n^* the set of indices $\{l\}$ for which both fibers $W_{1,n,l}^s$ and $W_{2,n,l}^s$ are long. Then $\operatorname{card} L_n^* \ge N_n$.

The distance between each $W_{1,n,l}^s$ and its counterpart $W_{2,n,l}^s$ in the unstable direction is less than $d_0 D \Lambda_{\min}^{-n}$. For any given n we can argue like in the proof of Sublemma 6.7 and choose δ small enough to make the value of $\mathcal{D}_{j_1}(\mu_1,\mu_2,W_{1,n,l}^s)$ arbitrarily close to $\mathcal{D}^*(\mu_1,\mu_2)$ and the value of $\mathcal{D}_{j_2}(\mu_1,\mu_2,W_{2,n,l}^s)$ arbitrarily close to $1/\mathcal{D}^*(\mu_2,\mu_1)$ for all $l \in L_n^*$. Therefore, for every n there is a $\delta_n > 0$ such that for any $\delta \le \delta_n$ we have

$$\mathcal{D}_{j_1}(\mu_1,\mu_2,W_{1,n,l}^s) - \mathcal{D}_{j_2}(\mu_1,\mu_2,W_{2,n,l}^s) \ge \Delta/2$$

for all $l \in L_n^*$. Equivalently,

$$\frac{\mathcal{D}_{j_1}(\mu_1,W_{1,n,l}^s)}{\mathcal{D}_{j_1}(\mu_2,W_{1,n,l}^s)} - \frac{\mathcal{D}_{j_2}(\mu_1,W_{2,n,l}^s)}{\mathcal{D}_{j_2}(\mu_2,W_{2,n,l}^s)} \ge \frac{\Delta}{2} \tag{6.16}$$

for all $l \in L_n^*$.

We represent the left hand side of (6.16) by $a/b - c/d$. Then, according to (6.4) we have

$$\frac{ad-bc}{bd} = \frac{(a-c)d+(d-b)c}{bd} \le \frac{|a-c|C_4+|d-b|C_4}{C_3^2}$$

This gives the following alternative: For every $l \in L_n^*$ either

$$|\mathcal{D}_{j_1}(\mu_1, W^s_{1,n,l}) - \mathcal{D}_{j_2}(\mu_1, W^s_{2,n,l})| \ge \frac{1}{4}\Delta\, C_3^2/C_4 \tag{6.17}$$

or

$$|\mathcal{D}_{j_1}(\mu_2, W^s_{1,n,l}) - \mathcal{D}_{j_2}(\mu_2, W^s_{2,n,l})| \ge \frac{1}{4}\Delta\, C_3^2/C_4 \tag{6.18}$$

We put $\Delta_1 = \frac{1}{4}\Delta\, C_3^2/C_4 > 0$. Denote by $L_n^{(1)}$ and $L_n^{(2)}$ the sets of indices in L_n^* for which (6.17) and (6.18) hold, respectively. Note that $L_n^{(1)} \cup L_n^{(2)} = L_n^*$. Without loss of generality, we assume that for some infinite sequence $\{n_m\}$ of indices we have $\operatorname{card} L_{n_m}^{(1)} \ge N_{n_m}/2$.

Let $l \in L_n^{(1)}$. Denote the rectangle whose two stable sides are $W^s_{1,n,l}$ and $W^s_{2,n,l}$ by $\hat{R}_{n,l}$. Its width in the unstable direction is smaller than $d_0 D \Lambda_{\min}^{-n}$. Let $R_{n,l}$ be a rectangle containing $\hat{R}_{n,l}$ as an s-subrectangle, such that its unstable sides are at distance $d^u_{\max}(\hat{R}_{n,l})/100$ from those of $\hat{R}_{n,l}$. Now let R_1 and R_2 be two very thin rectangles adjacent to $W^s_{1,n,l}$ and $W^s_{2,n,l}$ as prescribed by the indices j_1 and j_2, respectively. In particular, $R_1, R_2 \subset R_{n,l}$. For $i = 1, 2$ we put $R_{i,++} = R_i \cap R_{n,l,+}$ and

$$\mathcal{D}^+_{j_i}(\mu_1, W^s_{i,n,l}) = \lim{}^* \frac{\mu_1(R_{i,++})}{\mu_{\mathrm{SBR}}(R_i)}$$

similarly to (6.1). Note that the sets $R_{1,++}$ and $R_{2,++}$ consist of pieces unstable fibers that lie on the same $R_{n,l}$-fibers. It is then a direct calculation with the help of the uniform continuity of the u-SBR density (1.3) and the jacobian of the holonomy map that

$$|\mathcal{D}^+_{j_1}(\mu_1, W^s_{1,n,l}) - \mathcal{D}^+_{j_2}(\mu_1, W^s_{2,n,l})| \le g(d^u_{\max}(R_{n,l})) \le g(2d_0 D\Lambda_{\min}^{-n}) \tag{6.19}$$

where the function $g(x) \to 0$ as $x \to 0$ depends on $\hat{T}$ only. Let $\gamma_n = g(2d_0 D\Lambda_{\min}^{-n})$.

For large $n = n_m$ such that $\gamma_n < \Delta_1$ and for any $l \in L_{n_m}^{(1)}$, the inequality (6.17) can only hold due to the contribution from the sets $R_{i,+} \setminus R_{i,++}$, $i = 1, 2$, which lie on unstable fibers in M_+ terminating inside $R_{n_m,l}$.

Sublemma 6.8 *Let n_m be so large that $\gamma_{n_m} < \Delta_1/2$. Then for any $l \in L_{n_m}^{(1)}$ and small $\varepsilon > 0$ (i.e., $\varepsilon < \varepsilon_0(n_m, l)$) we have*

$$\mu_1(R_{n_m,l} \cap U_\varepsilon) \ge \frac{1}{3} C' D^{-1} \Delta_1 \varepsilon \tag{6.20}$$

where $C' > 0$ appears in (5.2).

Proof. Let $\delta > 0$. If the rectangles R_1 and R_2 are thin enough (in the unstable direction), it follows from (6.17) and (6.19) that

$$\left| \frac{\mu_1(R_{1,+} \setminus R_{1,++})}{\mu_{SBR}(R_1)} - \frac{\mu_1(R_{2,+} \setminus R_{2,++})}{\mu_{SBR}(R_2)} \right| \geq \Delta_1(1-\delta)/2$$

Now, without loss of generality we can assume that

$$\frac{\mu_1(R_{1,+} \setminus R_{1,++})}{\mu_{SBR}(R_1)} \geq \Delta_1(1-\delta)/2$$

Due to (5.2), we have

$$\mu_1(R_{1,+} \setminus R_{1,++}) \geq C' d^u_{\max}(R_1)\Delta_1(1-\delta)/2$$

Let $\varepsilon = d^u_{\max}(R_1)$ (this is not a restrictive choice, since R_1 is an arbitrary sufficiently narrow rectangle adjacent to $W^s_{1,n,l}$). The set $R_{1,+} \setminus R_{1,++}$ is a union of unstable R_1-fibers that belong in M_+. They all have length $\geq d^u_{\min}(R_1) \geq D^{-1}\varepsilon$. Moreover, the continuations of these fibers in M_+ terminate within our rectangle $R_{n_m,l}$ (otherwise those fibers would be in $R_{1,++}$). We now slide every fiber in this set along the fiber in M_+ that contains it, down to the endpoint of the latter lying in $R_{n_m,l}$. We then get another fiber, of the same length, which is between ε and $D^{-1}\varepsilon$, so that the new fiber lies in U_ε. By sliding the fiber we may change its measure, but at most by a factor of D (actually, very little since the rectangle $R_{n_m,l}$ is also thin in the unstable direction). Therefore, the union of the new fibers has measure $\geq \varepsilon D^{-1} C' \Delta_1(1-\delta)/2$. Sublemma 6.8 is proved.

Now note that the number of rectangles $R_{n_m,l}$ satisfying (6.20) is $\geq N_{n_m}/2$.

Claim. The number of *disjoint* rectangles satisfying (6.20) can be made arbitrarily large by increasing n_m.

Proof. Assume that the number of disjoint rectangles satisfying (6.20) is uniformly bounded, say $\leq K_0$. For large n_m we have card $L^{(1)}_{n_m} \geq N_n/2 \gg K_0$, so that the rectangles $R_{n_m,l}$, $l \in L^{(1)}_{n_m}$, have to form $\leq K_0$ clusters, and they necessarily converge to a finite union of some stable curves $\bar{W}^s_k \subset M$, $1 \leq k \leq K_0$. For any $r \geq 1$ denote by $\bar{N}_r$ the number of smooth components of the set $T^{-r}(\cup_k \bar{W}^s_k)$. It follows from the time-symmetric version of Theorem 2.1, cf. a remark after it, that $\bar{N}_r \to \infty$ as $r \to \infty$. Under our assumptions on the sequence $\{n_m\}$ there is an infinite sequence $\{r_l\}$ such that at least $\bar{N}_{r_l}/2$ smooth components of $T^{-r_l}(\cup_k \bar{W}^s_k)$ must be also curves to which some clusters of rectangles $R_{n_m,l}$, $l \in L^{(1)}_{n_m}$, converge as $n_m \to \infty$. This proves the claim.

The existence of arbitrarily many disjoint rectangles $R_{n_m,l}$ satisfying (6.20) clearly contradicts Theorem 5.7. The proof of Lemma 6.6 is completed. Theorem 6.1 now readily follows.

Corollary 6.9 *The sequence of measures $\mu^{(k)}_+$ converge weakly, as $k \to \infty$, to a unique measure μ_+ satisfying all the assumptions of Theorem 6.1.*

Lastly, note that the conditionally invariant measure μ_+ for the map T is also such for any iteration T^k, $k \geq 2$, of the map T, with eigenvalue λ_+^k. Moreover, it is fairly straighforward that all our arguments apply to any iteration of T. Therefore, the measure μ_+ is the only conditionally invariant measure for the map T^k for every $k \geq 2$.

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