Decay of Correlations for Lorentz Gases and Hard Balls

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Abstract

We discuss rigorous results and open problems on the decay of correlations for dynamical systems characterized by elastic collisions: hard balls, Lorentz gases, Sinai billiards and related models. Recently developed techniques for general dynamical systems with some hyperbolic behavior are discussed. These techniques give exponential decay of correlations for many classes of billiards and open the door to further investigations.

1 Definitions

This is a survey on the statistical properties of Lorentz gases and gases of hard balls. The properties of interest to us include rates of correlation decay, central limit theorems, and invariance principles. We begin with some precise definitions.

Let $\Phi^t : M \to M$ be a dynamical system preserving a probability measure μ. The time t here is either discrete, i.e. $t \in \mathbb{Z}$, or continuous, i.e. $t \in \mathbb{R}$. Let $f : M \to \mathbb{R}$ be a real-valued measurable function which we think of as an observable. Then the family

$$\xi_t = f \circ \Phi^t, \qquad t \in \mathbb{Z} \ \text{ or } \ \mathbb{R} \ ,$$

defines a stationary stochastic process with (M, μ) as the underlying probability space, stationarity following from the invariance of the measure μ.

For $t > 0$, let $S_t : M \to \mathbb{R}$ be the accumulation function of ξ_t. That is to say,

$$S_t = f + f \circ \Phi^1 + f \circ \Phi^2 + \cdots + f \circ \Phi^{t-1}$$

in the case of discrete time and

$$S_t = \int_0^t f \circ \Phi^s \, ds$$

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in the case of continuous time. (We assume in the latter that as a function of t, $f \circ \Phi^t(x)$ is continuous or at least piecewise continuous for a.e. $x \in M$, so that the integral is well defined). The function S_t/t is the time average of the process ξ_t. We denote by $\langle\cdot\rangle$ the expected value of a function with respect to μ.

The Birkhoff Ergodic Theorem asserts that if (M, Φ^t, μ) is ergodic and f is integrable, then S_t/t converges almost surely to $\langle f\rangle$ as $t \to \infty$. In probability theory, this is also called the *strong law of large numbers*.

As we shall see, under suitable assumptions on the system $\Phi^t : M \to M$ and the observable f, many other results in probability theory can be carried over to the stochastic process ξ_t and its accumulation function S_t.

An important object of study is the *time correlation function*

$$C_f(t) = \langle f \cdot (f \circ \Phi^t)\rangle - \langle f\rangle^2. \tag{1.1}$$

This function measures the dependence between the values of f at time 0 and time t. It is also common to study the asymptotics of more general correlation functions than (1.1), namely

$$C_{f,g}(t) = \langle f \cdot (g \circ \Phi^t)\rangle - \langle f\rangle\langle g\rangle \tag{1.2}$$

where $g : M \to \mathbb{R}$ is another measurable function. The function $C_f(t)$ in (1.1) is called an *autocorrelation function*.

In statistical physics, autocorrelation functions are involved in transport theories. Transport coefficients (such as diffusion coefficient, heat or electrical conductivity, shear viscosity) can be expressed through integrals of certain autocorrelation functions. See the survey [Bu].

Next, we say that ξ_t satisfies the *central limit theorem* (CLT) if

$$\lim_{t\to\infty} \mu\left\{\frac{S_t - t\langle f\rangle}{\sqrt{t}} < z\right\} = \frac{1}{\sqrt{2\pi}\sigma}\int_{-\infty}^{z} e^{-\frac{s^2}{2\sigma^2}}\, ds \tag{1.3}$$

for all $-\infty < z < \infty$. Here $\sigma = \sigma_f \geq 0$ is a constant. (In the case $\sigma_f = 0$, the right side of the equation is to be read as 0 for $z < 0$ and 1 for $z > 0$.) Equation (1.3) is equivalent to the convergence of $(S_t - t\langle f\rangle)/\sqrt{t}$ in distribution to the normal random variable $N(0, \sigma_f^2)$. We remark that the central limit theorem is considerably more refined than the Birkhoff Ergodic Theorem; it tells us that the distribution of the deviations of the time average S_t/t from its limit value $\langle f\rangle$, when scaled by $1/\sqrt{t}$, is asymptotically Gaussian.

The variance σ_f^2 in the CLT is related to the correlation function (1.1) by

$$\sigma_f^2 = C_f(0) + 2\sum_{n=1}^{\infty} C_f(n) \tag{1.4}$$

in the case of discrete time and

$$\sigma_f^2 = \int_{-\infty}^{\infty} C_f(t)\, dt \tag{1.5}$$

in the case of continuous time. It follows that a prerequisite for the central limit theorem is the integrability of the correlation function $C_f(t)$. Most existing proofs of the CLT for dynamical systems follow essentially (though not immediately) from slightly stronger estimates on the speed of correlation decay, i.e. the speed with which $C_f(t)$ tends to 0 as $t \to \infty$.

We remark also that under very mild assumptions, $\sigma_f^2 = 0$ if and only if the function f is cohomologous to a constant. This means, in the discrete time case, that $f = K + g - g \circ \Phi$ for some $g \in L^2(M)$ and a constant K, and in the continuous time case, that $f = K + \frac{d}{dt}|_{t=0}(g \circ \Phi^t)$.

A good survey on issues related to the central limit theorem for dynamical systems may be found in [De].

An *invariance principle* often accompanies the central limit theorem in the study of random processes and dynamical systems. For large $T > 0$, $x \in M$ and $0 \leq s \leq 1$, consider the function $W_T(s;x)$ defined by

$$W_T(s;x) = \frac{S_t(x) - t\langle f\rangle}{\sigma\sqrt{T}} \qquad \text{where} \qquad t = sT \ .$$

In the discrete time case, we interpolate linearly between integer values of t, letting

$$W_T(s;x) = (k+1-sT)\, W_T(k/T;x) + (sT-k)\, W_T((k+1)/T;x)$$

for $k/T < s < (k+1)/T$.

For fixed T, the family $\{W_T(s;x), x \in M\}$ induces a probability measure on the space of piecewise continuous functions on $[0,1]$. We say that ξ_t satisfies the *weak invariance principle* (WIP) if this measure converges, as $T \to \infty$, to the Wiener measure. We say that ξ_t satisfies an *almost sure invariance principle* (ASIP) if there is a standard Brownian motion $B(s;x)$ on M with respect to the measure μ so that for some $\lambda > 0$,

$$|W_T(s;x) - B(s;x)| = \mathcal{O}(s^{-\lambda})$$

for μ-almost all $x \in M$. The invariance principle asserts, therefore, that the accumulation function S_t, after a proper rescaling of space and time, converges to the Wiener process (or Brownian motion). The weak invariance principle is sometimes called the functional central limit theorem. More detailed discussions may be found in [C3, DP, PS].

Other refinements of the central limit theorem and related probabilistic limit laws also have their corresponding versions for dynamical systems. For example, one can prove a local central limit theorem, the law of iterated logarithms, renewal theorems, Borel-Cantelli lemmas, Poisson distribution for return times etc. These extensions would take us too far from the main topic and will not be discussed here.

Finally, let us relate the discussion above to some of the standard notions in ergodic theory. Recall that a dynamical system (M, Φ^t, μ) is said to be *mixing* if for any two measurable sets $A, B \subset M$, we have $\mu(A \cap \Phi^{-t}B) \to \mu(A)\mu(B)$ as $t \to \infty$. The following fact is standard in ergodic theory:

Fact $C_{f,g}(t) \to 0$ as $t \to \infty$ for all $f, g \in L^2(M)$ if and only if (M, Φ^t, μ) is mixing.

One might surmise that stronger versions of mixing (for example, multiple mixing, K-mixing, or Bernoulli) imply fast decay of correlations. This is not true. Even the Bernoulli property cannot guarantee any speed of convergence of $C_{f,g}(t)$ to zero for arbitrary functions $f, g \in L^2(M)$, not even for bounded or continuous functions! Moreover, let Φ^t be the "most chaotic" of all known dynamical systems, such as an expanding interval map or a hyperbolic toral automorphism. Even then, for typical integrable or continuous functions f, g on M, the convergence of $C_{f,g}(t)$ to zero is arbitrarily slow, and the central limit theorem (1.3) fails (cf. [C3, CC, JR, V]). In order to have a reasonable speed of correlation decay common to a family of observables, or to guarantee the CLT, it is necessary to restrict ourselves to observables with some regularity. In general, Hölder continuity is sufficient. Fortunately, all interesting functions in physics (such as temperature, energy, velocity) are smooth or at least piecewise smooth. This makes it possible to obtain strong statistical properties for many physically relevant observables.

2 Historical overview

Among the dynamical systems first rigorously studied are geodesic flows on manifolds of constant negative curvature [Ha, He, Ho]. E. Hopf [Ho] proved ergodicity for these flows as early as 1940. His argument relies on the existence of a pair of transversal foliations whose leaves are made up of stable and unstable manifolds (called horocycles in the case of geodesic flows). Hopf's argument is simple yet far reaching. While it alone is not adequate in more complicated situations, it lies at the heart of many proofs of ergodicity, including those for the dynamical systems considered in this survey.

Following Hopf's work, the dynamics of geodesic flows on manifolds of constant negative curvature were investigated by Ya. Sinai. In the late fifties, Sinai noticed a striking resemblance between these flows and stationary random processes. He proved the central limit theorem in 1960 [Si1] and the K-mixing property in 1961 [Si2]. In 1967, D. Anosov [An] completed a study of geodesic flows on manifolds of variable negative curvature, proving ergodicity, in particular, along the lines of Hopf. The Bernoulli property was later proved by D. Ornstein and B. Weiss [OW] and by Ratner [Ra2].

Generalizing a key property of geodesic flows on manifolds of negative curvature, Anosov [An] introduced a class of flows with the property that Φ^t leaves invariant a pair of foliations transversal to the vector field, uniformly expanding distances in one of them and uniformly contracting distances in the other. He called these flows and their discrete-time versions C-systems; they are now known as *Anosov diffeomorphisms* and *flows*, and the dynamical property described above is called *uniform hyperbolicity.* At roughly the same time, S. Smale [Sm] introduced his famous horseshoe model and the notion of *Axiom A*. The Axiom A condition requires only that the system be uniformly hyperbolic on certain recurrent sets and not necessarily on the entire manifold. Axiom A systems, therefore, are more general than Anosov systems. Unlike geodesic flows, however, Anosov and Axiom A systems do not always admit invariant probability measures that are compatible with volume. This perhaps motivated the study of more general invariant measures.

An important class of invariant measures for dynamical systems has its origins in statistical physics. This came about in a curious way. In 1967 Adler and Weiss [AW] constructed Markov partitions for linear toral automorphisms. In 1968 Sinai [Si3, Si4] constructed Markov partitions for all Anosov diffeomorphisms. Via a Markov partition, the trajectory of each point is coded by an infinite sequence of symbols from a finite alphabet, and the dynamics of the map is represented by a topological Markov chain. Spaces of symbol sequences are naturally reminiscent of one-dimensional lattice models in statistical mechanics. For lattice models, Gibbs measures were constructed in 1968-1969 by R. Dobrushin [Db1, Db2, Db3] and by O. Lanford and D. Ruelle [LR], who also showed that translationally invariant Gibbs measures are equilibrium states, i.e. they are characterized by a variational principle. Using Markov partitions, Sinai developed a theory of Gibbs measures for Anosov diffeomorphisms in 1972 [Si6] in analogy with that for lattice models. In the meantime, Bowen constructed Markov partitions for all Axiom A diffeomorphisms [B1], making it possible to extend the theory of Gibbs measures to this larger context. See [B3] for an exposition.

For dynamical systems, certain invariant measures are more important than others from the standpoint of physics. Here we take the view that observable events correspond to sets of positive Lebesgue measure in the phase space. It follows that the invariant measures of interest in physics are those that reflect the distributions of orbits starting from sets of positive Lebesgue measure. This is all very natural for conservative systems, i.e. for systems that preserve smooth measures. In the presence of dissipation, the situation is more subtle: there is no reason *a priori* why any invariant measure with the desired property should exist. For Anosov diffeomorphisms and Axiom A attractors, it turns out these measures can be found among Gibbs measures [Si6]. Equivalent characterizations emphasizing their connection to Lebesgue-almost every initial condition are given in [Ru2, BR]; see also [B3]. Today these special invariant measures are known as *Sinai-Ruelle-Bowen measures* or *SRB measures*.

In the discrete time case, strong statistical properties for Gibbs measures were obtained (mostly) by Sinai [Si6] and Ruelle [Ru1, Ru2]; a version of it is given in [B3]. In these papers, exponential decay of correlations for Hölder continuous functions is proved for Anosov and Axiom A diffeomorphisms. From this the central limit theorem is easily deduced. For Axiom A flows, the Bernoulli property and central limit theorem were proved by Ratner [Ra2], and the ASIP was obtained by M. Denker and W. Philipp [DP]. Asymptotic bounds on correlations in continuous time have turned out to be considerably more delicate. First, Ruelle [Ru4] and M. Pollicott [Po] obtained negative results: they found Axiom A flows with arbitrarily slow rates of correlation decay. Recently D. Dolgopyat [Do1] proved the exponential decay of correlations for Anosov flows under certain additional assumptions and for all geodesic flows on surfaces of negative curvature. He also proved [Do2] that there is an open and dense set of Axiom A flows that enjoy rapid mixing in the sense of Schwarz. Whether "most" Anosov flows have exponential decay of correlations is unknown at this time.

Today, the theory of Anosov and Axiom A systems can be regarded as fairly complete. In statistical physics, these systems have become a reference model, a paradigm for many heuristic studies of chaotic multiparticle systems – gases and fluids (both in and out of equilibrium). G. Gallavotti and E. Cohen [GC] spelled this out in their Axiom C in 1995. They stated that chaotic multiparticle systems could be regarded, for the purpose of averaging of phase observables, as Anosov systems.

Since the seventies, mathematicians have tried to extend the theory of Axiom A systems to dynamical systems having some hyperbolic behavior but satisfying less stringent conditions. Particularly important to physicists are billiard models, including hard-ball gases and Lorentz gases. Analogies between collisions of hard balls and geodesic flows on manifolds of negative curvature had been noticed by N. Krylov [K] decades earlier. Krylov pointed out that the convex surface of hard balls produces the same scattering effect on phase trajectories as the negative curvature of the manifold on geodesic curves.

In 1970 Sinai [Si5] undertook a systematic study of planar periodic Lorentz gases and, more generally, of planar billiards in tables with concave boundaries (now called dispersing billiards or Sinai billiards). He investigated the hyperbolic properties of these billiards and proved ergodicity and the K-mixing property. A major difference between billiards and Anosov systems lies in the fact that billiard flows are not continuous. Singularity sets break up stable and unstable curves into arbitrarily small pieces, making the proof of ergodicity considerably more involved and ruling out the existence of finite Markov partitions, which, as we recall, were the main tool in understanding the ergodic theory of Anosov systems. Sinai's seminal work paved the way for many subsequent developments in this direction.

In 1980, L. Bunimovich and Sinai considered billiard maps associated with planar periodic Lorentz gases and constructed for them countable Markov partitions [BS1]; see also [BSC1]. (Billiard maps are return maps on the Poincare sections of billiard flows corresponding to collisions.) In 1981, Bunimovich and Sinai [BS2] established the CLT and WIP with the help of these Markov partitions. Note that from the point of view of physics, the WIP for Lorentz gases has the following important interpretation: it says that typical particle trajectories (x_t, y_t) on the covering plane converge, after a suitable rescaling of time and space, to Brownian motion. See also [BSC2]. Bunimovich and Sinai [BS2] obtained, in fact, an upper bound on the time correlation function. They showed that

$$|C_{f,g}(t)| \leq \text{const} \cdot \exp(-at^{\gamma}) \tag{2.1}$$

for some $a > 0$ and $\gamma \in (0, 1)$. (As usual, f and g are Hölder continuous functions and Φ^t here is the billiard *map*). This mode of decay, later termed *stretched exponential decay* of correlations, is slower than exponential but fast enough to allow them to derive their results on the CLT and WIP.

The true asymptotics of the time correlation function $C_{f,g}(t)$ for dispersing billiards were not known for some time. Numerical results, including estimates on the constant γ, were produced by various people (e.g. [BlD, CCG]). See also [FM1, MR, FM2, GG].

There was disagreement in the mathematical physics community on whether this decay rate is in fact exponential or if it is substantially slowed down by the cutting and folding action of the singularity set. Analytic evidence in favor of exponential estimates came in the early 1990s: Chernov [C1] and later C. Liverani [L] proved exponential decay of correlations for certain piecewise hyperbolic maps (with singularities) in 2-dimensions.

For Lorentz gases, this question was resolved in the late nineties. In 1998, L.-S. Young [Y1] developed a general method for determining if a map has exponential decay of correlations with respect to its SRB measure. She [Y1] applied her method to planar Lorentz gases with finite horizon and obtained exponential decay of correlations. Shortly thereafter, Chernov applied the scheme in [Y1] to other classes of planar dispersing billiards [C6] and to Lorentz gases under external forces and their SRB measures [C7], obtaining in both cases exponential decay rates.

Young [Y2] has since expanded her results to deal with arbitrary decay rates. With tools for establishing polynomial decay now available, there is hope that rigorous correlation decay results for certain billiard models with very nonuniform hyperbolic behavior, such as the stadium, may be forthcoming.

The main ingredient of Young's approach is a tower construction that captures the renewal properties of a dynamical system. She focuses on return times to a reference set rather than the partitioning up of the phase space. In particular, Markov partitions are not used. It is becoming increasingly clear that this approach is quite generic, in the sense that it has given a unified way of understanding correlation decay and related statistical properties for many (different) dynamical systems that have some degree of expansion or hyperbolicity. In Section 3 we describe in a fairly general context her tower construction and the statistical information it carries. In Section 4 we explain what this tells us when applied to Lorentz gases. Further applications of this method are discussed in Section 5.

Finally, we turn to gases of hard balls, whose ergodic and statistical properties are among the less tractable problems in dynamical systems. In the general setting consisting of an arbitrary number of balls on a torus, full hyperbolicity (i.e. the absence of zero Lyapunov exponents) has been proved only recently by N. Simanyi and D. Szasz [SS1]. A proof of ergodicity is not yet available except in certain special cases (see the survey [SS2]). Nothing is known for gases of $n \geq 3$ balls in a rectangular box (see [Sim] for the case $n = 2$). What is clear is that hyperbolicity is very nonuniform in hard ball systems: there are "traps" of various kind in the phase space. Trajectories temporarily lose hyperbolicity as they get caught in these traps, and they may remain there for arbitrarily long periods of time. A careful quantitative analysis of these traps is necessary for rigorous results on correlation bounds and the CLT. This appears to be hopelessly out of reach at the present time. Numerical and heuristic studies have, however, been carried out by physicists. Some of their results and conjectures are discussed in Section 5.

3 Statistical properties of chaotic dynamical systems

In this section we discuss a general method for capturing statistical information for chaotic dynamical systems. This approach is introduced in [Y1], expanded in [Y2], and has been used successfully to obtain results for a number of the systems considered in this survey.

Some of the material in this section is valid only for discrete time systems, i.e. for Φ^t with $t \in \mathbb{Z}$ or $\mathbb{Z}^+$. It can be applied, in principle, to all discrete time systems that are predominantly hyperbolic, that is to say, Φ^t does not have to be uniformly hyperbolic or Anosov; singularities and other nonhyperbolic behaviors are allowed as long as there is "enough" hyperbolicity.

The key idea of this approach is to extract statistical information from certain distributions of return times. This is motivated by similar considerations in probability theory, in, for example, the theory of countable state Markov chains, the statistical properties of which are known to be closely related to the recurrence properties of the "tail states" to a fixed block of states.

Leaving more precise discussions for later, we give an indication here of what the proposed scheme entails: Pick a suitable reference set Λ in the phase space, and regard a subset of Λ as having "renewed" itself when it makes a "full" or "Markov" return to Λ, meaning its image covers all of Λ or at least stretches across all of Λ in the unstable direction. When successfully carried out, this construction gives rise to a representation, or a model, of the system in question, described in terms of a reference set and return times. As a dynamical system, this model is often much simpler than the original one. Consequently, its statistical properties are more easily accessible, and, as we will see, they can be expressed explicitly in terms of the tail distribution of the return times.

This, in summary, is the approach proposed by Young: *Set aside the individual characteristics of the original dynamical system, focus only on return times to a reference set, study the statistical properties of the resulting (abstract) model, and pass the findings back to the original dynamical system.*

In the rest of this section, we will limit ourselves to discrete time systems. Writing $\Phi = \Phi^1$, we let $\Phi : M \to M$ denote the given dynamical system (which we do not need to assume *a priori* to have an invariant density or to admit an SRB measure). All test functions $f : M \to \mathbb{R}$ are assumed to be at least Hölder continuous. In Sect. 3.1 we discuss the construction of reference sets and Markov return maps, using a very simple example, namely the "cat map" or "2-1-1-1 map" of the 2-torus, to illustrate how precisely this is done. The abstract models that result from these constructions will be denoted by $F : \Delta \to \Delta$. In Sect. 3.2 we describe $F : \Delta \to \Delta$, and discuss its statistical properties and their implications for Φ. Sect. 3.3 contains a general discussion.

3.1 Reference set and distribution of return times (for the cat map)

We begin with a discussion of the "cat map". This example will be used to illustrate (1) how to pick a reference set Λ, (2) how to define legitimate return maps, and (3) how to estimate the tail of the distribution of return times. Once this example is understood, we will comment on how it differs from the general situation.

Let $\Phi : \mathbb{T}^2 \to \mathbb{T}^2$ be the "cat map" or any invertible map induced from a hyperbolic linear map of $\mathbb{R}^2$. We require that Λ be a rectangle, with two of its sides aligned with the stable direction and two with the unstable direction; other than that, its choice is entirely arbitrary. Next we describe a procedure for defining a return map to Λ. This map will be denoted by $\Phi^R : \Lambda \to \Lambda$, where R here is to be thought of as an integer-valued function or random variable and not a fixed integer. That is to say, $R : \Lambda \to \mathbb{Z}^+$ is a function, and Φ^R evaluated at $x \in \Lambda$ is equal to $\Phi^n(x)$ with $n = R(x)$.

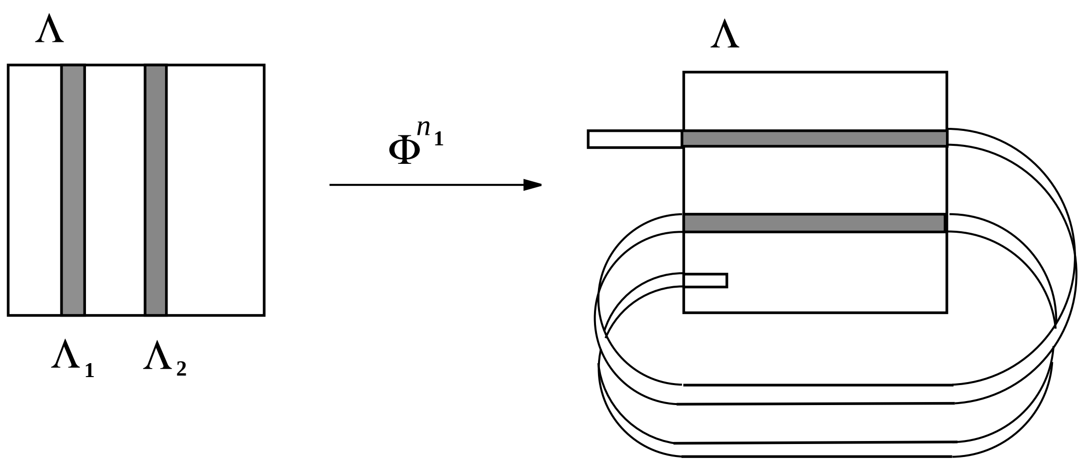

Figure 1: Markov returns

To describe the Markov property of the return map, we introduce the following language: $\Gamma \subset \Lambda$ is called an *s-subrectangle* if it spans Λ in the stable direction, a *u-subrectangle* if it spans Λ in the unstable direction. As Φ is iterated, Λ is transformed into a long and thin ribbon running parallel to the unstable direction. Let $n_1 > 0$ be the first time when part of $\Phi^{n_1}\Lambda$ contains a u-subrectangle of Λ as shown, and let $\Lambda_1, \Lambda_2, \cdots, \Lambda_{k_1}$ be the s-subrectangles of Λ that are mapped under Φ^{n_1} onto u-subrectangles of Λ. We declare that these Λ_i have "returned" with return time $R = n_1$ and stop considering them. Focusing on the part that has not returned, we continue to iterate until its image contains a u-subrectangle of Λ, say at time $n_2 > n_1$. Label the s-subrectangles that return at this time $\Lambda_{k_1+1}, \cdots, \Lambda_{k_2}$ and set their return time to be $R = n_2$. The process is continued *ad infinitum*. We will show momentarily that almost all points in Λ eventually return to Λ under this procedure, defining, modulo a set of Lebesgue measure zero, a return map Φ^R on Λ.

Since we are interested in the process of dynamical renewal and not just recurrence alone, partial crossings are not counted as returns in the procedure above. The fact that the Φ^R-image of each Λ_i is a u-subrectangle of Λ means that $\Phi^R(\Lambda_i)$ contains a sample of all possible future trajectories starting from Λ; it is as though after Φ^R steps, the system is starting anew. This is what we mean by "Markov returns".

Note that we do not insist that $\Phi^R(x)$ be the first time the orbit starting at x returns to Λ. The choice of Φ^R is, in general, quite flexible in the sense that many reasonable choices will not significantly affect the outcome of this discussion. To facilitate the estimation of the distribution of R, we will, in fact, use the following rule: fix a number $\delta > 0$, comparable, say, to the size of Λ, and count an s-subrectangle $\Gamma \subset (\Lambda \cap \{R > n-1\})$ as returning at time n if and only if the component of $\Phi^n(\Lambda \cap \{R > n-1\})$ containing $\Phi^n(\Gamma)$ not only crosses Λ but extends beyond Λ on both sides by lengths at least δ.

Next we show how to estimate the tail distribution of R. Let μ denote Lebesgue measure on the torus. We claim that

$$\mu\{R > n\} < C\theta^n \quad \text{for some } \ \theta < 1.$$

We will see in the next subsection that this tail estimate implies exponential decay of correlations.

To prove the claim, observe that at step n, the set $\Phi^{n-1}(\Lambda \cap \{R > n-1\})$ is the union of a finite number of very thin "ribbons", disconneted due to the fact that the parts that have returned have been removed. Our rule of what constitutes a return in the last paragraph ensures that each component of this "ribbon" has length at least δ. We divide $\Phi^{n-1}(\Lambda \cap \{R > n-1\})$ into segments of length $\sim \delta$, consider them one at a time and argue as follows: By the topological transitivity of Φ, there exists $N = N(\delta)$ such that if B_1 and B_2 are any two balls of radius $\frac{\delta}{3}$, we have $\Phi^k B_1 \cap B_2 \neq \emptyset$ for some $k \leq N$. Now let B_1 be centered at the midpoint of one of the δ-segments Γ of $\Phi^{n-1}(\Lambda \cap \{R > n-1\})$, and let B_2 be centered at the midpoint of Λ. Suppose $\Phi^k B_1 \cap B_2 \neq \emptyset$. Then by the geometry of hyperbolic maps, $\Phi^k(\Gamma)$ crosses Λ with two pieces sticking out on both sides as required (see Figure 2). We have thus shown that for every n, $\mu\{R \leq n + N \mid R > n-1\}$ is greater than some $c > 0$, proving the claim.

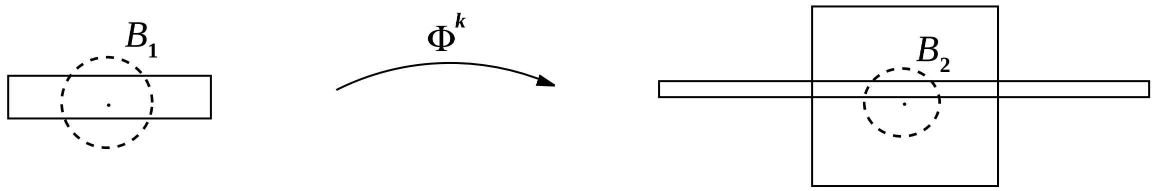

Figure 2: The geometry of hyperbolic maps

We discuss next some of the similarities and differences between the example above and what one may expect to find in general.

With regard to the choice of Λ and Φ^R, the construction above is quite indicative of what is done in general. To make sense of the "Markov" property, we need to be able to talk about s- and u-subrectangles, and Λ should be chosen so that it has a product structure of stable and unstable manifolds. When Φ is not Anosov, there may not be "solid" rectangles made up of stable and unstable leaves; when that is the case, take Λ to be the product of two Cantor sets. Also, since our main focus is on smooth invariant measures or SRB measures, Λ must be chosen so that each unstable leaf meets Λ in a set of positive (1-dimensional) Lebesgue measure. These differences aside, the general construction is quite similar to that for the cat map.

As to the estimation of the measures of $\{R > n\}$, the argument above suggests that pure and uniform hyperbolicity leads to return time distributions with exponentially decaying tails. Indeed, essentially the same argument gives an alternate proof of the exponential decay of correlations with respect to SRB measures for all Anosov diffeormophisms, a result proved earlier in [Ru1, Ru2, Si6].

For systems that are not Anosov, other aspects of the map, such as discontinuities or other forms of nonhyperbolic behavior, may affect the distribution of return times, the nature of which depends entirely on the dynamical system in question.

3.2 The abstract model $F : \Delta \to \Delta$

Let $\Phi : M \to M$ be the dynamical system of interest. As indicated at the beginning of this section, our strategy is to construct from Φ another dynamical system $F : \Delta \to \Delta$ which we think of as an abstract model of Φ. The reason for passing from Φ to F is that F will be a much simpler object, with the relevant information that determines the statistical properties of Φ conveniently displayed. The goal of this subsection is to take a closer look at F and the statistical information it contains.

The maps $F : \Delta \to \Delta$ that may potentially arise as abstract models for some Φ have the structure of a tower or skyscraper, in which every point moves upward until it reaches its highest level, i.e. there is nothing above it, before it returns to the base of the tower which we denote by Δ_0 (see Figure 3). For the cat map, F is related to the construction in Sect. 3.1 as follows. The set Δ_0 is obtained from Λ by collapsing stable manifolds into points, and the return map of F from Δ_0 to itself is the quotient map of $\Phi^R : \Lambda \to \Lambda$.

We now give a more precise description of $F : \Delta \to \Delta$. The bottom level of this tower, denoted by Δ_0, is partitioned into a countable number of sets $\Delta_{0,i}$, $i = 1, 2, \cdots$. The first level of this tower is denoted by Λ_1, the next level up Λ_2 etc. Under the action of F, each $\Delta_{0,i}$ moves upward one level per iterate, to $\Delta_{1,i}$, then $\Delta_{2,i}$, etc. Think of this upward movement as rigid translations. Nothing interesting happens until a point reaches the highest level above $\Delta_{0,i}$, which we call $\Delta_{R_i,i}$. At the next iteration, F maps $\Delta_{R_i,i}$ bijectively onto Δ_0. We assume that the partition $\{\Delta_{\ell,i}\}$ separates points, meaning for every $x, y \in \Delta$, there exists $n \geq 0$ such that $F^n x$ and $F^n y$ lie in different elements of the partition. This completes the description of the coarse structure of $F : \Delta \to \Delta$.

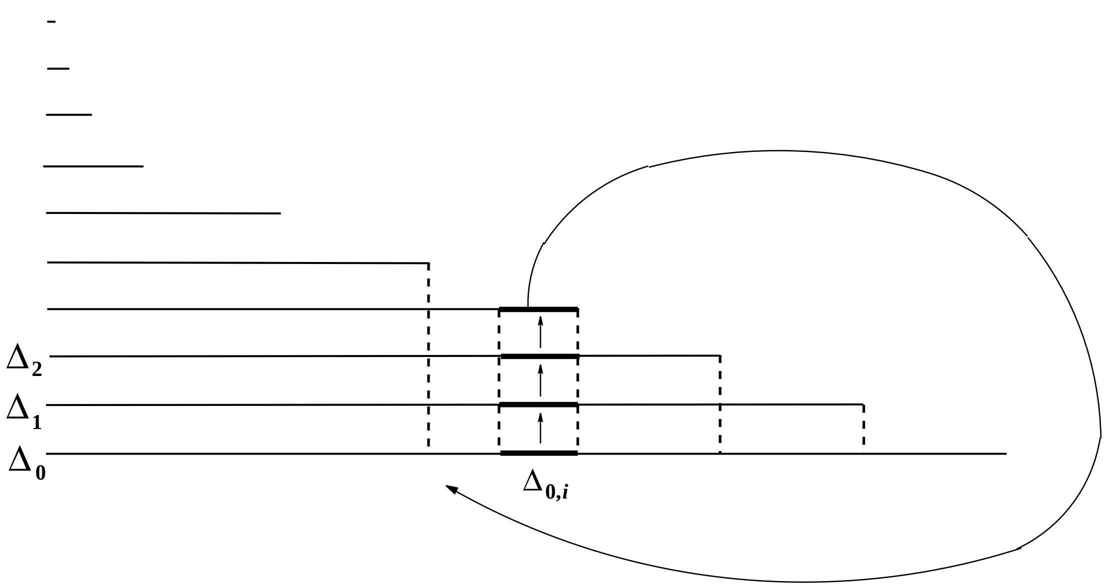

Figure 3: The tower map $F : \Delta \to \Delta$

We are interested in returns from Δ_0 to Δ_0. The return time function $R : \Delta_0 \to \mathbb{Z}^+$ is given by $R|\Delta_{0,i} = R_i$, and the return map $F^R : \Delta_0 \to \Delta_0$ is defined by $F^R|\Delta_{0,i} = F^{R_i}$.

Moving on to the finer structures of $F : \Delta \to \Delta$, we assume that there is a reference measure m on Δ, not necessarily finite and not necessarily F-invariant. The map F as well as its local inverses are assumed to be measurable and nonsingular with respect to m, so that measures with densities with respect to m are transformed under the action of F to measures with densities.

Finally, we introduce a notion of Hölderness or Lipschitzness for functions on Δ. A metric on Δ that measures symbolic distances between points can be defined as follows: Let the *separation time* between $x, y \in \Delta_0$ be given by

$$s(x, y) \;=\; \text{the smallest } n \text{ such that } (F^R)^n x \text{ and } (F^R)^n y \text{ belong in different } \Delta_{0,i}.$$

For $x, y \in \Delta_0$, we define $d(x, y) = \beta^{s(x,y)}$ where $\beta < 1$ is a number arbitrarily chosen but fixed. For $x, y \in \Delta$, we define $d(x, y) = 1$ if they do not belong in the same $\Delta_{\ell,i}$, and if they do we let $d(x, y) = d(F^{-i}x, F^{-i}y)$. Letting JF denote the Jacobian of F with respect to m, we assume that $JF = 1$ on $\Delta \setminus F^{-1}\Delta_0$, and impose the following regularity condition on $JF|F^{-1}\Delta_0$ or, equivalently, on $JF^R|\Delta_0$: we require that $\log JF^R|\Delta_{0,i}$ be uniformly Lipschitz with respect to the metric defined, i.e. there exists $C > 0$ such that for all i and for all $x, y \in \Delta_{0,i}$,

$$\left| \frac{JF^R(x)}{JF^R(y)} - 1 \right| \le C\beta^{s(x,y)}.$$

Furthermore, all the observables considered will be assumed to be uniformly Lipschitz with respect to the same β.

This completes our description of the dynamical system $F : \Delta \to \Delta$. We remark that the structure of F is chosen so that it displays clearly the information of greatest importance to us, namely the sequence of numbers $m\{R > n\}$ as $n \to \infty$.

Before stating results on the statistical properties of F, we explain more precisely how $F : \Delta \to \Delta$ is derived from the construction in Sect. 3.1. First let Φ be the cat map, and let $\Phi^R : \Lambda \to \Lambda$ be as in Sect. 3.1. Identifying points in Λ that belong to the same stable leaves, we obtain a quotient set $\bar{\Lambda}$, a quotient measure $\bar{\mu}$ on $\bar{\Lambda}$, and an induced map $\bar{\Phi}^R : \bar{\Lambda} \to \bar{\Lambda}$. Clearly, $\bar{\Lambda}$ can be thought of as an interval and $\bar{\mu}$ the 1-dimensional measure on $\bar{\Lambda}$. Moreover, $\bar{\Lambda}$ is partitioned into a countable number of intervals each one of which is mapped affinely by $\bar{\Phi}^R$ onto $\bar{\Lambda}$. The correspondence should now be transparent: Δ in the abstract model for Φ is the tower with $\Delta_0 = \bar{\Lambda}$, $F^R = \bar{\Phi}^R$ and $m = \bar{\mu}$. It follows immediately that $m\{R > n\} < C\theta^n$ for some $\theta < 1$.

In general, we proceed as with the cat map, but when nonlinearities are present, the situation is a little more complicated: Topologically, the map $\bar{\Phi}^R : \bar{\Lambda} \to \bar{\Lambda}$ is defined, but it is *a priori* not clear that JF^R makes sense. Some technical work is needed. We refer the reader to [Y1] for details.

Results on the statistical properties of $F : \Delta \to \Delta$ and their implications for $\Phi : M \to M$, the dynamical system from which F is derived [Y1, Y2]:

- If $\int R dm < \infty$, then
  - (i) F has an invariant probability measure ν equivalent to m;
  - (ii) Φ has a smooth invariant probability measure or an SRB measure μ.

Remarks. 1. (F, ν) is automatically ergodic. In the results below, let μ be the SRB measure with $\mu(\bigcup_n \Phi^n \Lambda) = 1$. Then (Φ, μ) is also ergodic. 2. We do not claim that Φ has no other SRB measures: through information on $\Phi^R : \Lambda \to \Lambda$, one cannot possibly know about the existence of SRB measures whose supports do not intersect $\bigcup_n \Phi^n \Lambda$.

- If $\int R dm < \infty$ and the greatest common divisor of $\{R\}$ is 1, then
  - (i) (F, ν) is mixing
  - (ii) (Φ, μ) is mixing.

We assume from here on that $\gcd\{R\} = 1$. In the next three bullets, all test functions are understood to be uniformly Lipschitz as stipulated earlier.

- If $m\{R > n\} = \mathcal{O}(n^{-\alpha}),\ \alpha > 1$, then
  (i) the rate of correlation decay of (F, ν) is $\mathcal{O}(n^{-\alpha+1})$;
  (ii) the rate of correlation decay of (Φ, μ) is $\mathcal{O}(n^{-\alpha+1})$.
- If $m\{R > n\} = C\theta^n,\ \theta < 1$, then
  (i) the rate of correlation decay of (F, ν) is $< C'\theta'^n$ for some $\theta' < 1$;
  (ii) the rate of correlation decay of (Φ, μ) is $< C'\theta'^n$ for some θ'.
- If $m\{R > n\} = \mathcal{O}(n^{-\alpha}),\ \alpha > 2$, then
  (i) the central limit theorem holds for (F, ν);
  (ii) the central limit theorem holds for (Φ, μ).

3.3 Discussion

1. To what kinds of dynamical systems would the method of this section apply, and for what kinds of statistical properties? The success of this approach depends on (1) the construction of $\Phi^R : \Lambda \to \Lambda$, and (2) success in passing information for F back to the original system. (1) generally works when Φ has "enough hyperbolicity", although it is hard to axiomatically formulate what exactly that means. As mentioned at the beginning of this section, this construction has been successfully carried out for a number of the systems considered in this survey. In addition to that, it has been shown to work for various examples of interest in dynamical systems, including logistic interval maps [Y1, BLS], expanding maps with neutral fixed points [Y2], piecewise hyperbolic maps [Y1, C5], Hénon attractors [BY] and their generalizations [WY], and certain partially hyperbolic systems. As for (2), for almost-sure properties and properties expressed in terms of the expectation of a random variable (such as correlation decay), this passage is largely (though not completely) formal; for other properties such as the Poisson law for large returns, this passage is less transparent.

2. What aspects of a dynamical system determine its statistical properties? We have seen from the cat map that uniformly hyperbolic systems have exponential decay or correlations. For systems that are predominantly but not purely hyperbolic, the following phenomenon has been observed: Initially, the hyperbolic part of the system gives rise to an exponential drop-off in $m\{R > n\}$. However, as n increases, $m\{R > n\}$ is determined more and more by *the least hyperbolic* part of the system. To illustrate how this works, imagine that the phase space of Φ has certain (localized) regions of nonhyperbolicity. These regions behave like "traps" or "eddies", in which orbits may linger for arbitrarily long times. While most orbits starting from Λ return relatively quickly, a fraction of initial conditions will, by ergodicity, get into these "traps" and remain there for a long time. The ultimate decay rate of $m\{R > n\}$, therefore, is determined by how fast these orbits are able to break free. Indeed, the following message is clear: *if* a dynamical system has identifiable sources of nonhyperbolicity, then the time it takes to overcome these nonhyperbolic parts determine $m\{R > n\}$ and consequently the type of statistical properties in Sect. 3.2.

3. What are the standard methods for obtaining rates of correlation decay in dynamical systems? In dynamical systems as in statistical mechanics and probability, the existence of a spectral gap is one of the most commonly used methods of proof of exponential decay. The operator in question for dynamical systems is the Perron-Frobenius or transfer operator (see e.g. [Ru3, HK, Y1]). Equivalent to the existence of a spectral gap but technically more flexible is the existence of strictly invariant cones with projective metrics ([FS, L]). A method used by Sinai et. al. ([BS2, BSC2]) to study correlation decay of billiards is approximation by Markov chains. Yet another standard method first used by Doeblin and well known in probability but not exploited seriously in dynamical systems until recently is the coupling method, which is first used in [Y2] to prove the results stated in Sect. 3.2. This method goes as follows: Consider two processes consisting of iterating $F : \Delta \to \Delta$ with different initial distributions λ and λ'. We run these processes independently, all the while trying to "match" $F_*^n\lambda$ with $F_*^n\lambda'$. The rate at which the L^1-norms of the densities of $F_*^n\lambda - F_*^n\lambda'$ tend to zero is a measure of the speed of convergence to equilibrium, which in turn gives a bound for the speed of correlation decay.

4 Correlation decay for planar Lorentz gases

The purpose of this section is to explain the proof of exponential decay of correlations for a class of billiard maps using the method discussed in the last section.

The class we will focus on is the 2-dimensional periodic Lorentz gas, which is a model for electron gases in metals. Mathematically it is represented by the motion of a point mass in $\mathbb{R}^2$ bouncing off a (fixed) periodic configuration of convex scatterers. This model was first studied by Sinai around 1970 [Si5]; it is sometimes called the Sinai billiard. Putting the dynamics on the torus, we assume that the billiard flow takes place on $\Omega = \mathbb{T}^2 \setminus \cup_{i=1}^k \Omega_i$ where the Ω_i's are disjoint convex regions with C^3 boundaries (see Figure 4). The section map Φ is defined on $M = \partial\Omega \times [-\frac{\pi}{2}, \frac{\pi}{2}]$. We denote points in M by $p = (x, \theta)$ where $x \in \partial\Omega$ is the footpoint of the arrow indicating the direction of the flow and θ is the angle this arrow makes with the normal pointing into Ω. It is straightforward to check that Φ leaves invariant the probability measure $\mu = c \cos\theta \, dx \, d\theta$ where c is the normalizing constant. For simplicity, we will assume in this section the *finite horizon* condition, which requires that the time between collisions be uniformly bounded.

This entire section is devoted to explaining the ideas behind the following result. Let C^α denote the class of Hölder functions on M with Hölder exponent α.

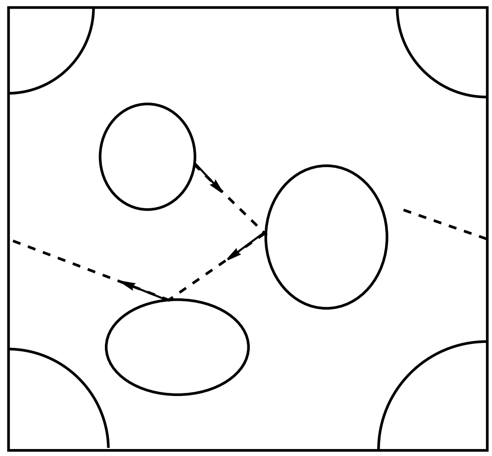

Figure 4: Billiard on $\mathbb{T}^2$ with convex scatterers

Theorem [Y1] Let (Φ, μ) be as above. Then correlation decays exponentially fast for observables in C^α. More precisely, there exists $\beta = \beta(\alpha) > 0$ such that for every $f, g \in C^\alpha$,

$$|C_{f,g}(n)| \leq C \exp(-\beta n)$$

for some $C = C(f, g)$.

As mentioned in the introduction, a weaker version of this result, namely that of "stretched exponential decay", is first proved for this class of billiards in [BSC2] along with other statistical properties. The finite horizon condition in the theorem above is dropped in [C6]. This and other results on correlation decay for billiards based on a similar approach are discussed in Section 5.

4.1 Geometric properties of the map $\Phi : M \to M$

The following geometric properties of Φ play important roles in determining its statistical properties. For more background information, we refer the reader to [Si5] and [BSC1, BSC2].

(1) *The discontinuity set S* It is easy to see that Φ is discontinuous at a point $(x, \theta) \in M$ if and only if the straight line segment starting at x and going in the direction determined by θ meets $\partial\Omega$ tangentially at the first point of intersection. We claim that the geometry of S as a subset of (x, θ)-space is as shown in Figure 5: (i) S is the union of a finite number of smooth curve segments (this number is infinite in the infinite horizon case). (ii) Fixing an orientation for $\partial\Omega$, the slopes of these segments all have the same sign; let us assume they are negatively sloped. (iii) Some of the segments in S run from the top edge to the bottom edge of M, whereas others end abruptly as they join one of the "main" branches.

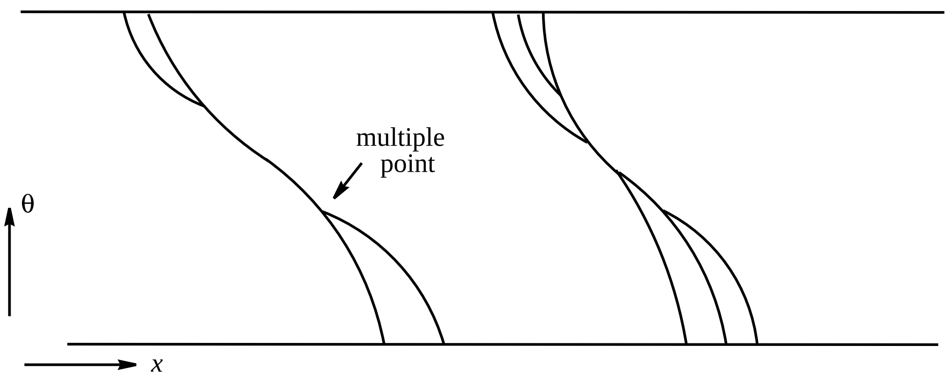

Figure 5: The singularity set S in phase space

To understand these assertions better, let us imagine starting from a fixed component of $\partial\Omega$, aiming in a certain general direction and having an obstacle in front of us. (See Figure 6.) It is easy to see that the set of directions that give rise to a tangency forms a smooth curve. Moreover, if the obstacle in question is the "nearest" obstacle, meaning there is no other obstacle in front of it, then this curve extends from $\theta = -\frac{\pi}{2}$ to $\theta = \frac{\pi}{2}$. If, on the other hand, the obstacle in question is the "second" obstacle, then the curve of tangencies cannot be extended beyond the point of "double tangency". Points of multiple tangency, therefore, are points in S at which two or more smooth segments of S meet. They are called *multiple points*. We will return to them later on in the discussion.

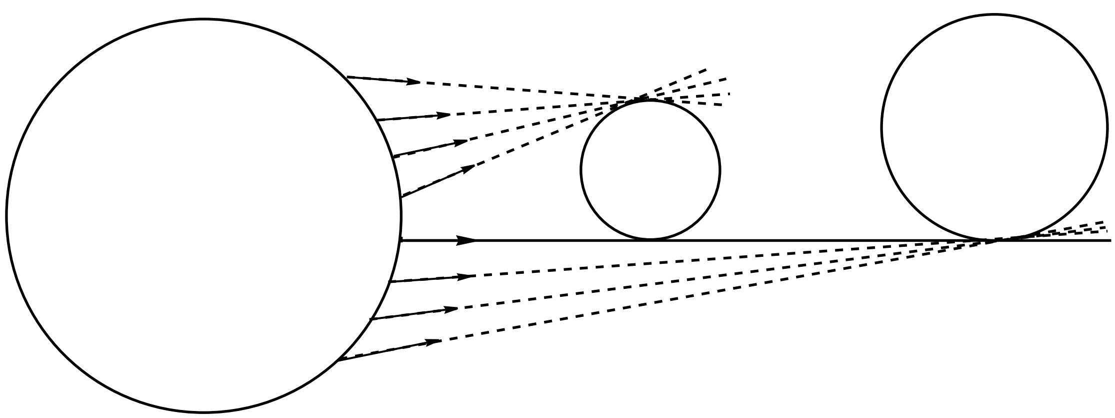

Figure 6: Curves of singularity. The top set of arrows represents a branch of S that extends from top to bottom of M. The bottom set of arrows represents a branch of S that ends abruptly at a double tangency

(2) *Hyperbolicity* Hyperbolicity of Φ is first proved in [Si5]. Intuitively, it is easy to see from the strict convexity of the scatterers that $D\Phi$ has strictly invariant cones ([W]), the existence of which is a standard way or proving hyperbolicity:

Tangent vectors at $(x, \theta) \in M$ are represented by curves in M passing through (x, θ), and curves in M are parametrized families of "arrows" in Ω. Consider the cone corresponding to all 1-parameter families of arrows that are divergent. Since divergent families upon reflection at a convex curve become even more divergent, we see that the cones in question are strictly invariant. This proves that on the projective level, at least, $D\Phi$ is uniformly hyperbolic. A little more work shows that E^u, the unstable directions of $D\Phi$, are positively sloped and transversal to the curves in S.

(3) *Unbounded derivatives* In addition to having discontinuities, another aspect of Φ that complicates its analysis is that $D\Phi$ is unbounded as (x, θ) tends to S. The technical difficulties that arise from this have been taken care of in [BSC2] and will not be discussed in this article.

4.2 Comparison with the cat map

According to the scheme discussed in Section 3, it suffices to pick a reference set Λ, construct an acceptable return map $\Phi^R : \Lambda \to \Lambda$, and show that $\mu\{R > n\} < C\theta^n$ for some $\theta < 1$. The exponential decay of correlations and Central Limit Theorem for Φ will then follow automatically.

Instead of repeating the construction from scratch, let us remember what is done for the cat map (Sect. 3.1) and focus on the differences. First, Φ here is nonlinear and $D\Phi$ is unbounded, but as we have said earlier, we will not concern ourselves in this article with these technical aspects of the problem. There are two conceptual differences between the cat map and the situation here, both caused by the presence of discontinuities:

(1) Stable and unstable curves can be arbitrarily short. Points that come arbitrarily close to the singularity set S in forward (respectively backward) time have their local stable (resp. unstable) curves cut arbitrarily short. By the ergodicity of Φ, which is proved in [Si5] and taken for granted in this argument, points with arbitrarily short stable or unstable curves are dense in M. The best that we can do, therefore, is to choose Λ to be a positive Lebesgue measure set that is homeomorphic to a product of two Cantor sets.

(2) Connected components of unstable curves may remain short for a long time. Imagine the following worse-case scenario: Suppose a piece of unstable leaf γ is, under the action of Φ, magnified by a factor of $\sim \frac{3}{2}$ and cut into two pieces of roughly equal lengths. Suppose also that each one of these segments is dealt the same hand, i.e. it is again magnified by $\sim \frac{3}{2}$ and cut into two equal pieces. Suppose this goes on indefinitely, so that after n iterates, $\Phi^n(\gamma)$ has 2^n components each of length $\sim (\frac{3}{4})^n$. This is detrimental to our construction of $\Phi^R : \Lambda \to \Lambda$, for in order for a segment to make an acceptable return, it must grow to the size of Λ. While this scenario exactly as described is extremely

unlikely, long stretches of time during which the images of γ are cut faster than they have a chance to grow may have an effect on the tail of the return times.

Of these two concerns, (2) has an obvious bearing on $\mu\{R > n\}$; it will be the center of our attention in the rest of this section. It turns out that once (2) is resolved, (1) can be handled as well. We first discuss briefly some problems that arise from (1): Since it is virtually impossible to track the evolution of Cantor sets, we track instead the rectangle Q spanned by Λ. An immediate question is: when the Φ^n-image of an s-subrectangle Q^s of Q is mapped to a u-subrectangle of Q, does $\Phi^n(\Lambda \cap Q^s)$ coincide with $\Lambda \cap \Phi^n(Q^s)$?

It turns out that in the unstable direction, $\Phi^n(\Lambda \cap Q^s)$ is bigger, and bits of this Cantor set fall through the gaps of Λ without "returning". This leads to more complicated estimates on return times. We remark that technical as they may seem, these problems are far from purely technical in nature: The main difference between Φ and an Anosov map is the presence of discontinuities, and the gaps in Λ are precisely the handiwork of the discontinuity set.

4.3 Growth of unstable curves

To prevent the phenomenon in (2) above from happening, consider first a condition of the following type:

(*) There exists $N \in \mathbb{Z}^+$ and $\delta > 0$ such that for all unstable curves γ with $\ell(\gamma) < \delta$, the number of connected components of $\Phi^N(\gamma)$ is $< \lambda^N$ where $\lambda > 1$ is the minimum expansion on unstable curves.

Condition (*) has the following interpretation: Thinking of cutting as introducing a form of local complexity, (*) says that the growth in local complexity is dominated by the rate of expansion. When this condition is satisfied, we have immediately that *on average*, each of the components of $\Phi^n(\gamma)$ grows exponentially in length.

We first explain why Sinai billiards have exponential decay of correlations *assuming* that Φ satisfies Condition (*). Justification of this condition is postponed to later. We will maintain throughout that the number δ in (*) is the relevant length scale to consider, referring to segments shorter than δ as "short" and those longer than δ as "long".

Let γ be an unstable curve. On γ we introduce a stopping time

$$T(x) := \text{ the smallest } n \text{ such that the component of } \Phi^n\gamma \text{ containing } \Phi^n x \text{ has length} > \delta.$$

Here is what we propose to do: We run Φ until time T, that is to say, we run Φ on each component of $\Phi^n\gamma$ (for as long as it takes) until it becomes "long". Then we stop. The following is an estimate on the distribution of T.

Lemma 1 *Let m_γ denote the Lebesgue measure on γ. Assuming (*), there exists $\alpha < 1$ independent of γ such that*

$$m_\gamma\{T > n\} < \alpha^n.$$

Proof Let us assume for simplicity that $N = 1$ in (*), and let K be the maximum number of components into which a "short" unstable curve can be cut by Φ. By (*), $K < \lambda$. The number of components that have remained "short" up until time n is $\leq CK^n$ where $C = [\ell(\gamma)/\delta] + 1$. Thus the total measure of the pull-back of these short components is $\leq CK^n\lambda^{-n}$. Taking $\alpha = K\lambda^{-1} < 1$ completes the proof.

When a component of $\Phi^i\gamma$ becomes "long", i.e. when it reaches a length $> \delta$, we start up the process again. This defines on γ a sequence of stopping times

$$T_1 \;<\; T_2 \;<\; T_3 \;<\; \cdots$$

with $T_1 = T$ and T_n is the first time after T_{n-1} when the component in question becomes "long". **The key idea is to look at the images of γ under Φ^{T_n}, not Φ^n.** Observe that $\Phi^{T_1}(\gamma)$ is the union of curves all of which are "long", and that the same is true of $\Phi^{T_2}(\gamma)$, $\Phi^{T_3}(\gamma)$, and so on. Moreover, *pretending* for the moment that $T_1 < T_2 < T_3 < \cdots$ represents *real time*, one would conclude that γ grows exponentially, and there are never any short curves around. The situation, therefore, is entirely similar to that for the cat map (see Sect. 3.1). Using the mixing property of Φ (proved in [Si5]) in the place of topological transitivity, we see that the proof in Sect. 3.1 continues to work, giving $\mu\{R > n\} < C\theta^n$.

To reconcile the stopping times T_n with real time, observe that the distribution of $(T_n - T_{n-1})|T_{n-1}$ is essentially the same as that of T, which is estimated in the Lemma above to have exponentially decaying tails.

To summarize, then, we first confuse T_n with real time, and conclude exponential decay of correlations for reasons similar to those for the cat map. Under Condition (*), the distributions of $T_n - T_{n-1}$ have exponentially decaying tails. A little bit of work shows that these two estimates together give exponential decay of correlations with respect to real time.

It remains to explain why Condition (*) holds for Φ, the billiard map in question. Let $S(\Phi^n)$ denote the singularity set of Φ^n. It is not hard to see that $S(\Phi^n)$ has the same structure as $S(\Phi)$, except that it contains more curve segments and becomes denser as n increases. For $z \in M$, let $\mathcal{N}(\Phi^n, z)$ denote the number of smooth segments of $S(\Phi^n)$ meeting at z. If z is not a multiple point of $S(\Phi^n)$, then $\mathcal{N}(\Phi^n, z) = 1$. Also, let $\mathcal{N}(\Phi^n) = \sup_{z\in M} \mathcal{N}(\Phi^n, z)$. The verification of Condition (*) is a rephrasing of the following observation due to Bunimovich:

Lemma 2 [BSC1] *There exists K depending only on Ω such that for all $n \geq 1$,*

$$\mathcal{N}(\Phi^n) \leq Kn.$$

Verification of Condition (*) assuming Lemma 2 Recall that unstable curves are transversal to $S(\Phi^n)$, so that for each n, if an unstable curve γ is sufficiently short, then it will not meet $S(\Phi^n)$ in more than $\mathcal{N}(\Phi^n)$ points. This implies that $\Phi^n\gamma$ will have not more than $\mathcal{N}(\Phi^n)+1$ components. We choose our parameters in the following order: First choose N so that $KN+1<\lambda^N$ where $\lambda>1$ is the minimum expanion along unstable curves. Then we choose $\delta>0$ such that if γ has length $<\delta$, then γ meets $S(\Phi^N)$ in $\leq KN$ points. This δ is the relevant length scale for Condition (*).

We conclude this section with an explanation for Bunimovich's observation. Consider a (straight) billiard trajectory with multiple tangencies, starting at $z\in M$, ending in $z'\in M$, with no regular collisions and at least one tangency in between. We stress that z and z' are points in the phase space M, not on the billiard table.

Starting from a small neighborhood of z, there is a finite number of ways of reaching a neighborhood of z': it is possible for the billiard trajectory not to touch any of the scatterers in between, or to bounce off the first but not the second, or the second but not the first, and so on. From this one deduces the following picture (see Figure 7):

- there are small neighborhoods U of z and U' of z' such that U is the disjoint union of a finite number of sectors $V_1,\cdots,V_k$ and U' is the disjoint union of a finite number of sectors $V_1',\cdots,V_k'$;
- for each j, there exits n_j such that Φ^{n_j} maps V_j diffeomorphically onto V_j'.

Here each V_j represents one "type" of trajectories from U to U', and n_j is the number of (nontangential) bounces in between. Clearly, the number of sectors, k, depends only on the configuration of scatters.

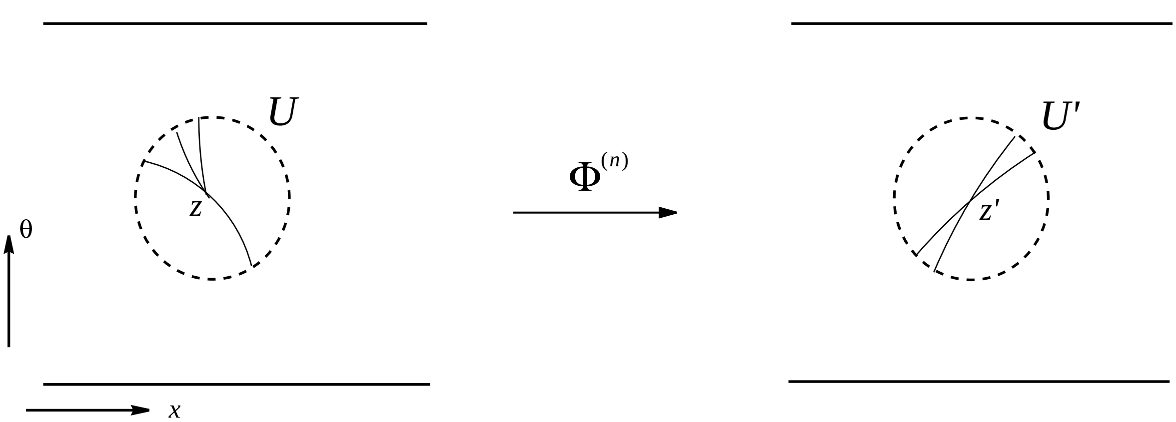

Figure 7: Neighborhoods of multiple points
($\Phi^{(n)}=\Phi^{n_j}$ when restricted to the sector V_j)

An upper bound for $\mathcal{N}(\Phi^n,z)$ can be estimated as follows. Let $\mathcal{N}(\Phi^i|V_j',z')$ denote the number of smooth segments of $S(\Phi^i)$ passing through z' that lie in V_j'. Then pulling back the picture from z' to z, we have

$$\mathcal{N}(\Phi^n, z) \leq \; k \; + \; \sum_j \mathcal{N}(\Phi^{n-n_j}|V_j', z').$$

Since $\mathcal{N}(\Phi^{n-n_j}|V_j', z') \leq \mathcal{N}(\Phi^{n-1}|V_j', z')$, which we assume inductively to be $\leq K(n-1)$, we have argued that $\mathcal{N}(\Phi^n, z)$ grows linearly with n, completing the proof of Bunimovich's lemma and the hence the proof of exponential decay for this class of billiards.

5 Correlation decay in related billiard models

Here we describe other physical models with elastic collisions for which estimates on correlation decay rates have been proven or conjectured. In Sects. 5.1 –5.5, the discussion pertains to the billiard *map* (or relevant section map) Φ. Results for the corresponding flows are considerably more delicate and are discussed in Sect. 5.6, under the headline "Real time dynamics".

5.1 Sinai billiard tables

Imagine a Lorentz gas whose scatterers are so large that they overlap and trap the particle in a bounded diamond-like region as shown in Fig. 8. This defines a billiard system on a table whose sides are convex inward. The resulting models are called Sinai billiards. They differ from the Lorentz gases discussed earlier in two ways.

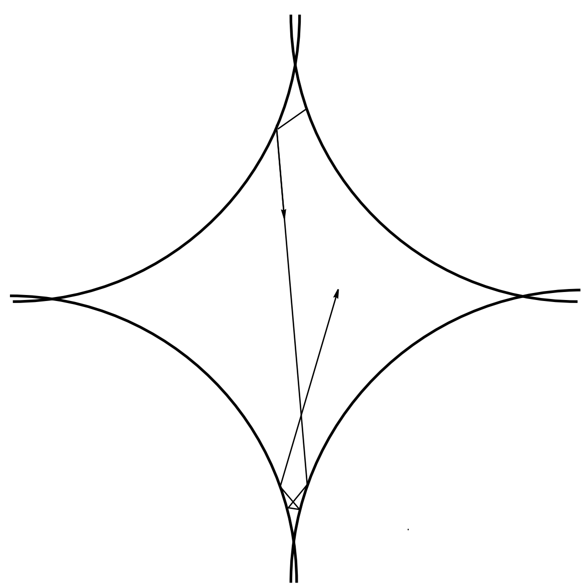

Figure 8: Sinai billiards in a diamond-type table

A. Traps at the corners. If a trajectory comes close to a corner point (where two scatterers intersect), it may experience two or more rapid collisions within a very short time. Between those rapid collisions, the unstable curves do not have a chance to grow. More precisely, the expansion factor along unstable curves under the billiard map Φ is

$$D = 1 + \tau B \tag{5.1}$$

where τ is the time between consecutive collisions and B the geometric curvature of the outgoing wave front made by the unstable curve. The quantity B is positive and usually bounded; since τ can be arbitrarily close to 0, D can be arbitrarily close to 1. As a result, Φ is not uniformly hyperbolic. For more details, see [BSC1, BSC2, C6]).

Here is one way around this problem. It is a simple geometric fact (see e.g. [Re]) that if two scatterers meet at an angle $\alpha > 0$, then there can be at most $1 + \pi/\alpha$ rapid collisions of the type described above. After that the particle must leave the corner. Hence if all the angles $\alpha_1, \dots, \alpha_k$ made by the scatterers at their intersections are positive, then the maximum number of consecutive rapid collisions is bounded above by $m = \max_i\{1 + \pi/\alpha_i\}$. The map Φ^m is therefore uniformly hyperbolic, and if one proves exponential decay of correlations for Φ^m, then the same property for Φ will follow. This is done in [C5, C6] (see paragraph B. below).

On the other hand, if one of the angles α_i is zero, i.e. if two of the scatterers intersect tangentially, forming a *cusp*, then the number of consecutive rapid collisions is easily seen to be unbounded. In this case, the hyperbolicity of Φ is very nonuniform, and correlations are believed to decay slowly, at the rate

$$|C_{f,g}(t)| \sim t^{-1} \ .$$

There is a strong numerical and analytical evidence of this asymptotic behavior [MR], but a rigorous proof is not yet available.

B. Condition (*). The proof of exponential decay for the periodic Lorentz gas given in Section 4 relies on a property of Φ that expresses the fact that expansion dominates local complexity along unstable curves. This property, which we called Condition (*), has not been verified for general Sinai billiards with corner points. Indeed, Lemma 2 is likely to be false. In [BSC2, C6], Condition (*) is assumed in order to obtain bounds on correlations. We refer the reader to [Bu] for recent advances in this direction.

5.2 Lorentz gases with infinite horizon

Infinite horizon in a periodic Lorentz gas refers to the property that there is no finite upper bound for the lengths of free runs between collisions. This is equivalent to the existence of corridors along which the particle can move indefinitely without colliding with a scatterer (see Fig. 9).

In contrast to the finite horizon case, the singularity set here is the union of infinitely many smooth curves. Figure 9 shows how they are generated. It is now possible for an unstable curve, however short, to be cut in one iterate by the singularity set into an arbitrarily large, possibly infinite, number of disconnected components. Hence Condition (*) in Section 4 fails. Nevertheless, the map Φ is known to have exponential decay of correlations [C6]). This can be explained as follows. The regions in the phase space where the singularity curves accumulate correspond to long intercollision flights (Fig. 9). During those flights, the unstable curves expand very strongly, according to (5.1), where τ in now very large (in fact, B is also large, as $B \sim \tau^{1/2}$; see [C6]). This strong expansion is sufficient for overcoming the effect of the cutting. Careful analyses of these issues are carried out in [BSC2, C6].

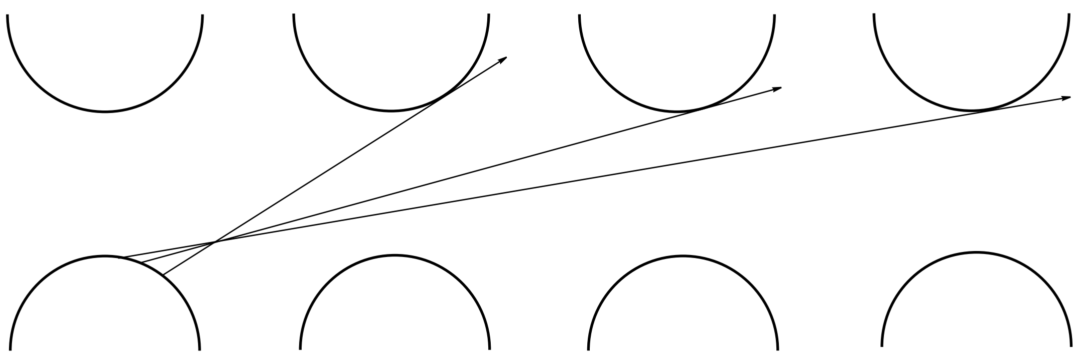

Figure 9: The periodic Lorentz gas without horizon:
long free runs generate infinitely many singularity lines

5.3 Lorentz gases under external forces

Consider the situation where the particle in a periodic Lorentz gas is subjected to an external force $\mathbf{F}$. The equations of motion (between collisions) are now

$$\dot{\mathbf{q}} = \mathbf{p}, \qquad \dot{\mathbf{p}} = \mathbf{F}(\mathbf{p}, \mathbf{q})$$

where $\mathbf{q} = (x, y)$ is the position and $\mathbf{p}$ the velocity vector of the particle. We assume there is an integral of motion $\mathcal{E}(\mathbf{p}, \mathbf{q})$, so that the dynamics can be reduced to motions on 3-D surfaces $\mathcal{E}(\mathbf{p}, \mathbf{q}) =$const. We assume that the force $\mathbf{F}$ is small, so that the particle trajectories between collisions are nearly straight. For simplicity, we assume also the finite horizon condition.

If the force $\mathbf{F}$ is given by the gradient of a potential, i.e. if $\mathbf{F} = -\nabla U(\mathbf{q})$ for some U, then the dynamics is Hamiltonian. In particular, total energy $U(q) + \frac{1}{2}||\mathbf{p}||^2$ is preserved, as is a measure compatible with volume called the Liouville measure.

For more general forces, the system admits no smooth invariant measures, and one looks for the existence of a physically meaningful invariant measure, i.e. an SRB measure (see Sect. 2). The techniques described in Sections 3-4 apply to this model to provide an

SRB measure which also has strong statistical properties: exponential decay of correlations and the CLT. For a detailed analysis of this model, see [C7]. A separate argument [C7] shows that this SRB measure is unique and its support is the entire phase space. Moreover, the system has the Bernoulli property in both discrete and continuous time.

5.4 Multi-dimensional Lorentz gases

Consider the motion of a particle in $\mathbb{R}^d$, $d \geq 2$, bouncing off a periodic array of fixed convex scatterers. This is the d-dimensional version of the periodic Lorentz gas. We again assume finite horizon. It is shown in [C2] that the correlation function decays at least faster than a stretched exponential function, i.e.

$$|C_{f,g}(t)| \leq a\, e^{-b\sqrt{t}}$$

for some constants $a, b > 0$. It is believed that the actual rate of decay is exponential, and the work to prove that is underway.

5.5 Multiple correlations

A natural generalization of the correlation function (1.2) is:

$$C_{f_1,\ldots,f_k} = \langle (f_1 \circ \Phi^{t_1}) \cdots (f_k \circ \Phi^{t_k}) \rangle - \langle f_1 \rangle \cdots \langle f_k \rangle$$

where $f_1, \ldots, f_k$ are functions on M and $t_1, \ldots, t_k$ are moments of time. This is called a multiple correlation function.

Certain physical constants are expressed in terms of multiple correlation functions. For example, the so-called super-Burnett coefficient (arising in higher-order expansions of the diffusion equation) is given by an expression that involves the sum

$$\begin{aligned} \sum_{m,n,k=-\infty}^{\infty} \quad [\ & C_{a,b,c,d}(0,m,n,k) - C_{a,b}(0,m)C_{c,d}(n,k) \\ & - C_{a,c}(0,n)C_{b,d}(m,k) - C_{a,d}(0,k)C_{b,c}(m,n)\ ] \end{aligned} \tag{5.2}$$

where a, b, c, d are certain functions.

The facts about correlation functions obtained in [BSC2, C3] allow us to prove, in the case of Lorentz gases, that the sum in (5.2) converges absolutely for any four Hölder continuous functions a, b, c and d. For a proof and the discussion of the Burnett coefficient, see [CD].

5.6 Real time dynamics

The discussions in Sects. 5.1–5.5 are for discrete time dynamics; more precisely, the statements there pertain to certain section maps corresponding to the flows in question.

Physically interesting time is, of course, real (or continuous). In finite horizon situations, the central limit theorem and invariance principle for real time follow from the corresponding results for discrete time; see [BS2, BSC2, DP]. The correlation function $C_{f,g}(t)$, however, may behave quite differently in discrete and continuous times.

In a periodic Lorentz gas without horizon, one has exponential decay of correlations in discrete time as we have seen in Sect. 5.2. There is strong evidence, however, that correlations decay slowly in real time [FM1, FM2]:

$$|C_{f,g}(t)| \sim t^{-1} \tag{5.3}$$

The reasons behind this estimate are as follows. For a particle in a corridor as shown in Fig. 9, a collision-free flight of length $\geq t$ occurs with probability $\sim 1/t$. Hence, over a period of time t, a fraction of the phase space of measure $\sim 1/t$ does not have the chance to mix with the rest of the phase space. From this one deduces the estimate (5.3). In short, the slow-down in correlation decay is caused by the presence of long corridors, which act as traps for the otherwise uniformly hyperbolic dynamics.

In contrast to the previous situation, in Sinai billiards with cusps (caused by tangential intersections of the scatterers), even though correlations in discrete time seem to decay like $1/t$ as we have discussed earlier, it is reasonable to expect that their decay rate in continuous time is much faster, possibly as fast as exponential. The difference between discrete and continuous times can be explained by the fact that in a succession of rapid collisions near a cusp, the total time elapsed is very short even though the number of collisions may be large.

In a Lorentz gas with finite horizon, correlations in real time are expected to decay exponentially as in discrete time, but the situation is considerably more involved. Mathematical proofs are not yet available, but research is underway.

5.7 Gases of hard balls

As dynamical systems, hard balls are considerably more complicated than Lorentz gases. Full hyperbolicity for an arbitrary number of balls in a torus has only been proved very recently [SS2]. Rigorous studies of correlation functions are not within reach at the present time. The only facts that are clear are that there are many traps in the phase space, they are of various kinds, and that it is possible for trajectories to be caught in them for very long periods of time without achieving full hyperbolicity.

The following trap is well known: certain phase trajectories may interact only with a proper subset of the balls. Even though the set of initial conditions with this property is of zero Lebesgue measure, nearby phase trajectories take arbitrarily long times to get to the rest of the phase space, slowing down the rate of correlation decay. We remark that this phenomenon of clusters is one of the main obstacles in the proofs of full hyperbolicity [SS2] and ergodicity for gases of hard balls.

For infinite gases assuming that the particles are at local equilibrium (i.e. they are distributed randomly and uniformly in space), one can heuristically estimate the correlation function $C_f(t)$ as follows [PR]:

Let the function f in $C_f(t)$ be the x-component of the velocity vector of a selected (tagged) ball. We fix that component $v_x(0)$ at time $t = 0$. The tagged particle interacts with its neighbors and, after time t, its initial velocity $v_x(0)$ is shared by all the particles in a volume V_t around it. Hence the average velocity of the tagged particle at time t is $\langle v_x(t) \rangle \sim v_x(0)/N_t$ where $N_t \approx \rho V_t$ is the estimated number of particles in the volume V_t and ρ is the density.

Further decay of the function $C_f(t) \approx \langle v_x(t) \rangle$ can only occur because V_t grows with t. For simplicity, assume that V_t is a round ball of radius R_t. Then it is standard in hydrodynamics that $R_t \sim \sqrt{t}$. This gives the estimate

$$|C_f(t)| \sim a\, t^{-d/2} \tag{5.4}$$

where d is the dimension of physical space, and the coefficient a depends on the density ρ.

The estimate (5.4) was arrived at more accurately by a variety of empirical and theoretical methods in statistical mechanics. B. Adler and T. Wainwright demonstrated it by molecular-dynamics calculations. R. Dorfman and E. Cohen established this estimate using kinetic theory. M. Ernst, E. Hauge, and J. van Leeuwen, then K. Kawasaki, and then Y. Pomeau derived (5.4) from hydrodynamic mode-coupling theory. We refer the reader to [EW, PR] for references and for further discussion of the topic.

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