On a slow drift of a massive piston in an ideal gas that remains at mechanical equilibrium

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February 18, 2004

Abstract

We consider a heavy piston in an infinite cylinder surrounded by ideal gases on both sides. The piston moves under elastic collisions with gas atoms. We assume here that the gases always exert equal pressures on the piston, hence the piston remains at the so called mechanical equilibrium. However, the temperatures and densities of the gases may differ across the piston. In that case some earlier studies by Gruber, Piasecki and others reveal a very slow motion (drift) of the piston in the direction of the hotter gas. At the same time the energy is slowly transferred across the piston from the hotter gas to the cooler one. While the previous studies of this interesting phenomenon were only heuristic or experimental, we provide first rigorous proofs assuming that the velocity distribution of the ideal gas satisfies a certain "cutoff" condition.

AMS Subj. Classification: 70F45,82C21
Key words: equilibrium, ideal gas, massive piston

1 Introduction

Consider an isolated cylinder filled with an ideal gas and divided into two compartments by a large piston which is free to move along the axis of the cylinder, Fig. 1. The piston interacts with the gas atoms via elastic collisions. Assume that the gas in each compartment separately is at equilibrium with temperature and density T_-, n_- and T_+, n_+, respectively. Let the gases exert equal pressures on the piston, i.e. let

$$P_- = n_- k_B T_- = n_+ k_B T_+ = P_+$$

Then the system is at the so called *mechanical equilibrium*, and according to the laws of thermodynamics this state should be (macroscopically) stable.

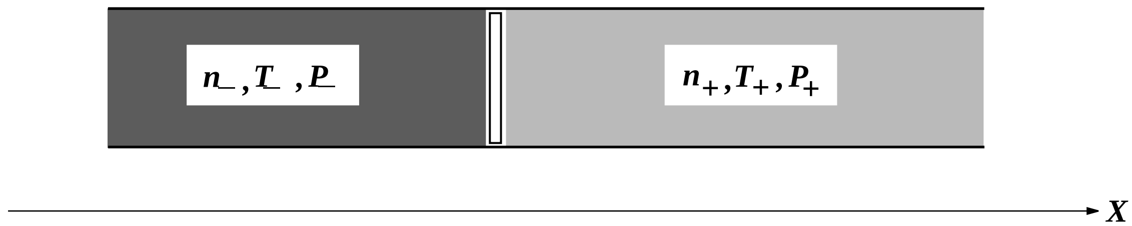

Figure 1: Piston in a cylinder filled with gas.

However, the system as a whole is not in a true equilibrium state, unless $T_- = T_+$, hence microscopically it is not stable yet and should find ways to evolve to a true, *thermal* equilibrium, in which $T_- = T_+$. This was predicted earlier by Landau and Lifshitz [LL], Feynman [F] and others. Recently Gruber, Piasecki and Frachebourg and others [GP, GF, GPL, P] derived heuristically, by means of kinetic theory and the Liouville equation, exact formulas describing the slow drift of the piston and the slow heat transfer from the hotter gas to the cooler gas.

Our goal is to derive rigorously the main formulas of [GP, GF, GPL, P] describing the slow drift of the piston and the slow heat transfer between the gases.

The piston model trivially reduces to a one-dimensional system by the projection onto the axis of the cylinder. Then one obtains an ideal gas on an interval, and the piston itself becomes a heavy point particle. The motion of a heavy particle in an infinite ideal gas of light particles is a classical example of Brownian motion studied by van Kampen [vK] and many others [L, H, DGL, GF, GP, P].

We consider a one-dimensional ideal gas on the entire line, without boundaries. The heavy piston is initially placed at the origin $X(0) = 0$ and is at rest $V(0) = 0$. The initial configuration of gas atoms and their velocities is chosen at random as a realization of a (two-dimensional) Poisson process on the (x, v)-plane with density $p(x, v)$. This means that for any domain $D \subset \mathbb{R}^2$ the number N_D of gas particles $(x, v) \in D$ at time $t = 0$ is a Poisson random variable with parameter

$$\lambda_D = \iint_D p(x, v)\, dx\, dv$$

The system evolves according to the rules of elastic collisions. Denote the mass of the piston by M and the mass of an atom by m. Since atoms have identical masses, their collisions can be ignored. When an atom with velocity v collides with the piston, whose velocity is V, their velocities after the collision are given by

$$V' = \frac{M-m}{M+m}\, V + \frac{2m}{M+m}\, v \tag{1.1}$$

$$v' = -\frac{M-m}{M+m}\,v + \frac{2M}{M+m}\,V \tag{1.2}$$

These rules preserve the total kinetic energy and the total momentum. Between collisions, all the particles and the piston move with constant velocities.

The position $X(t)$ and velocity $V(t) = dX(t)/dt$ of the piston make a random process whose characteristics are determined by the initial gas density $p(x,v)$. It is natural to assume that $p(x,v)$ is symmetric in v and spatially homogeneous, i.e. $p(x,v) = p(v)$ and $p(v) = p(-v)$. We can set $p(v) = nf(v)$, where $f(v) = f(-v)$ is a probability density and $n > 0$ is a (constant) spatial density. Then one can approximate $X(t)$ and $V(t)$ by certain Gaussian stochastic processes:

Theorem 1.1 (Holley [H]) *Let the density $f(v)$ have a finite fourth moment $\int v^4 f(v)\,dv < \infty$. Then for every finite $t_0 < \infty$, the function $V(t)\sqrt{M}$ on the interval $[0, t_0]$ converges, in distribution, as $M, n \to \infty$ and $M/n \to$ const, to an Ornstein-Uhlenbeck velocity process $\mathcal{V}_t$, while $X(t)\sqrt{M}$ converges to an Ornstein-Uhlenbeck position process $\mathcal{X}_t$.*

An Ornstein-Uhlenbeck process $(\mathcal{X}_t, \mathcal{V}_t)$ is defined by [Ne]

$$d\mathcal{X}_t = \mathcal{V}_t\,dt, \qquad d\mathcal{V}_t = -a\mathcal{V}_t\,dt + \sqrt{D}\,d\mathcal{W}_t$$

where $a > 0$, $D > 0$ are constants and $\mathcal{W}_t$ a Wiener process. The Ornstein-Uhlenbeck position process $\mathcal{X}_t$ converges in an appropriate limit (e.g. $a \to \infty$, $a^2/D =$const) to a Wiener process.

We note that Dürr et al [DGL] extended the above theorem to arbitrary dimensions.

Our paper concerns with another physically interesting situation, where the initial density is *not* spatially homogeneous, but a spatial homogeneity is assumed separately for the gases to the right and to the left of the piston. So we assume that

$$p(x,v) = \begin{cases} p_+(v) & \text{for } x > 0 \\ p_-(v) & \text{for } x < 0 \end{cases}$$

and we also assume symmetry: $p_-(v) = p_-(-v)$ and $p_+(v) = p_+(-v)$. We can set $p_\pm(v) = n_\pm f_\pm(v)$, where $f_\pm(v) = f_\pm(-v)$ are probability densities and $n_\pm > 0$ are (constant) spatial densities.

In addition, we want to exclude a "macroscopic motion" of the piston in either direction by requiring that its velocity vanishes as $m/M \to 0$. This is equivalent to

$$n_- \int_0^\infty v^2 f_-(v)\,dv = n_+ \int_0^\infty v^2 f_+(v)\,dv \tag{1.3}$$

as it was shown heuristically in [LPS] and under some conditions rigorously in [CLS]. The physical interpretation of Eq. (1.3) is the pressure balance. If we define the pressures of the gases by

$$P_\pm = mn_\pm \int_{-\infty}^\infty v^2 f_\pm(v)\,dv$$

(which would be a proper thermodynamical pressure if the velocity distributions were Maxwellian), then the condition (1.3) means exactly that $P_- = P_+$. Similarly, we can define the temperature of the gases by

$$T_\pm = k_B^{-1} m \,\frac{\int v^2 p_\pm \, dv}{\int p_\pm \, dv}$$

(this is sometimes called the "effective temperature"), here k_B is Boltzmann's constant.

To simplify some technical considerations we assume that the initial densities of the gases satisfy the following *velocity cutoff*:

$$p_\pm(v) = 0, \quad \text{if} \quad |v| \le v_{\min} \quad \text{or} \quad |v| \ge v_{\max} \tag{1.4}$$

for some $0 < v_{\min} < v_{\max} < \infty$. Hence, the initial velocities of atoms are bounded away from zero and infinity. Under these conditions, our arguments are rigorous. We also discuss in Section 4 how to relax these assumptions, leaving this work for the future.

2 Markov approximation

Our cutoff condition (1.4) has an important implication. As long as the speed of the piston remains small enough, every gas atom collides with the piston at most once. Indeed, let $|V(t)| < V_{\max}$, where

$$V_{\max} = \frac{M-m}{3M+m}\, v_{\min}$$

Note that $V_{\max}$ is close to $v_{\min}/3$ when $M \gg m$. Then it follows directly from (1.4) and (1.2) that the atoms' velocities after collisions are at least

$$\frac{M-m}{M+m}\, v_{\min} - \frac{2M}{M+m}\, V_{\max} = V_{\max}$$

the last equality holds due to our choice of $V_{\max}$. Therefore, the atoms after collisions remain faster than the piston, so the latter cannot "catch up" with them.

Hence, as long as $|V(t)| < V_{\max}$, the velocity of the piston $V(t)$ evolves as a Markov process with piecewise constant trajectories (a jump or step process). Moreover, this is a stationary (homogeneous in time) Markov process due to our requirements on the density $p(x,v)$. We will slightly change the function $V(t)$ so that it will evolve as a stationary Markov process unconditionally. Fix some $\bar{V} \in (0, V_{\max})$ and require that whenever the velocity V' of the piston after a collision, in the notation of (1.1), exceeds $\bar{V}$ in absolute value, it gets "reflected" at $\bar{V}$, i.e. it instantaneously changes from V' to V'' by the rules

$$\begin{aligned} V' > +\bar{V} \quad &\Longrightarrow \quad V'' = +2\bar{V} - V' \\ V' < -\bar{V} \quad &\Longrightarrow \quad V'' = -2\bar{V} - V' \end{aligned} \tag{2.1}$$

These rules are in the spirit of random walks with reflecting boundary conditions.

We will denote the velocity of the piston in the so defined dynamics by $W(t)$. It is clear that $W(t) = V(t)$ for all $t < T$ such that $\sup_{0<t<T} |V(t)| < \bar{V}$. We first study the Markov process $W(t)$ and later estimate the difference $V(t) - W(t)$.

Denote by $\mathcal{P}(u, dw; \Delta t)$ for every $\Delta t > 0$ the transition probability for the process $W(t)$, i.e.

$$P\left(W(t+\Delta t) \in A / W(t) = u\right) = \int_A \mathcal{P}(u, dw; \Delta t)$$

for every Borel set $A \subset \mathbb{R}$. It is clear that the piston does not experience any collisions with the atoms during the interval $(t, t+\Delta t)$ if and only if the trapezoidal domain

$$D = D(t, \Delta t) = \left\{(x, v) : \ \frac{v-u}{x - X(t)} < -\frac{1}{\Delta t}, \ \ v_{\min} < |v| < v_{\max}\right\} \tag{2.2}$$

does not contain gas atoms. Therefore, the probability that $W(t+\Delta t) = W(t) = u$ is

$$\mathcal{P}(u, \{u\}; \Delta t) = \exp\left(-\int_D p(x, v)\, dx\, dv\right) \tag{2.3}$$

To evaluate this integral, we partition the domain $D = D(t, \Delta t)$ into two trapezoids, $D = D^- \cup D^+$ as shown on Fig. 2. Under our assumptions (1.4)

$$\int_D p(x, v)\, dx \ = \Delta t \left[\int_{v_{\min}}^{v_{\max}} (v-u)\, p_-(v)\, dv + \int_{-v_{\max}}^{-v_{\min}} (u-v)\, p_+(v)\, dv\right]$$

We introduce the following notation: for each $k \geq 0$ let

$$\int_{v_{\min}}^{v_{\max}} v^k p_-(v)\, dv = F_k^- \quad \text{ and } \quad \int_{-v_{\max}}^{-v_{\min}} v^k p_+(v)\, dv = F_k^+$$

and

$$Q_k = F_k^- - F_k^+$$

Then we obtain

$$\begin{aligned} \int_D p(x, v)\, dx \ &= \ \Delta t \left[F_1^- - F_1^+ - (F_0^- - F_0^+) u\right] \\ &= \ \Delta t \left[Q_1 - Q_0 u\right] \end{aligned} \tag{2.4}$$

where $u = W(t)$.

It is clear that for every u and $\Delta t > 0$ the probability measure $\mathcal{P}(u, dw; \Delta t)$, besides having an atom at u with probability given by (2.3), has an absolutely continuous component with a positive density on the interval $(-\bar{V}, \bar{V})$ (in fact, that density is bounded away from zero for every fixed u and $\Delta t > 0$).

Proposition 2.1 *The stationary Markov process $W(t)$ has a unique stationary measure μ_0, which is absolutely continuous on the interval $(-\bar{V}, \bar{V})$. Any other initial distribution converges to μ_0 uniformly exponentially fast in time.*

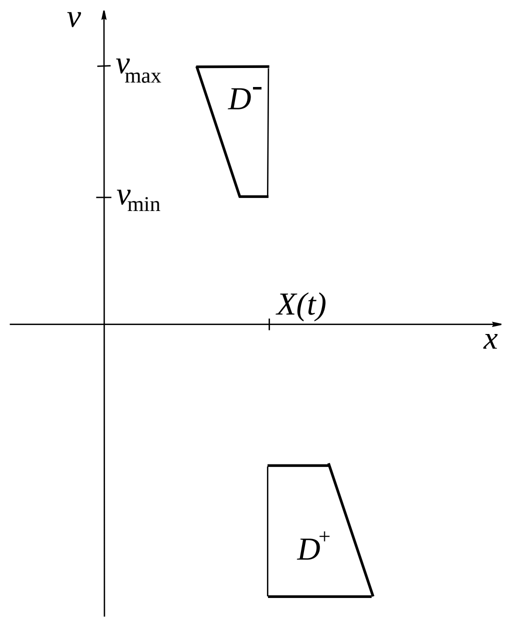

Figure 2: Region $D = D^- \cup D^+$.

Proof. The process $W(t)$ satisfies the so called Doeblin condition [D]: there exist $\varepsilon > 0$ and $\Delta t > 0$ such that for every Borel set $A \subset (-\bar{V}, \bar{V})$ and every $u \in (-\bar{V}, \bar{V})$ we have

$$m(A) < \varepsilon \Longrightarrow \int_A \mathcal{P}(u, dw; \Delta t) < 1 - \varepsilon$$

where $m(A)$ is the Lebesgue measure of A. Then our result follows from general theorems [D]. But we also outline a simple direct argument.

Since the transition probability $\mathcal{P}(u, dw; \Delta t)$ defines, for each $\Delta t > 0$, a continuous map on the convex space of probability distributions on the interval $(-\bar{V}, \bar{V})$, the existence follows from a general Schauder-Tychonoff theorem [DS], p. 456. Since $\mathcal{P}(u, dw; \Delta t)$ has an absolutely continuous component whose density is bounded away from zero, the stationary distribution μ_0 is absolutely continuous with density $f_0(w) \geq c_0 > 0$.

To prove the uniqueness of μ_0, suppose $\mu_0' \neq \mu_0$ is another stationary measure with density $f_0'(w) \geq c_0' > 0$. Then $(1+c)f_0 - cf_0'$ for small $c > 0$ is also a stationary density. Let

$$\bar{c} = \sup\{c > 0 : \inf_{|w|<\bar{V}} [(1+c)f_0(w) - cf_0'(w)] \geq 0\}$$

Then $(1+\bar{c})f_0 - \bar{c}f_0'$ is a stationary density, hence it must also be bounded away from zero on $(-\bar{V}, \bar{V})$. But this contradicts our choice of $\bar{c}$.

To prove the convergence, we take an arbitrary initial distribution ν_0 and consider its image ν_t at time t. Fix a $t > 0$. The measure ν_t has an absolutely continuous component whose density is bounded away from zero. Hence $\nu_t = (1-\alpha)\nu^{(1)} + \alpha\mu_0$ with some small

$\alpha > 0$ and some measure $\nu^{(1)}$. The value of α depends on t but not on ν_0. This implies, by induction, that $\nu_{kt} = (1-\alpha)^k \nu^{(k)} + [1-(1-\alpha)^k]\mu_0$ for all $k \geq 1$, which proves the exponential convergence of ν_t to μ_0. $\square$

We now derive one useful equation. Fix a $\Delta t > 0$ and for every $u \in (-\bar{V}, \bar{V})$ denote by

$$E(u, \Delta t) = \int w \, \mathcal{P}(u, dw; \Delta t)$$

the conditional expectation of the process $W(t+\Delta t)$ given that $W(t) = u$. The invariance of the measure μ_0 implies

$$\int E(u, \Delta t) \, d\mu_0(u) = \int w \, d\mu_0(w) \tag{2.5}$$

Now suppose that

$$E(u, \Delta t) = u + g(u) \, \Delta t + R(u, \Delta t) \tag{2.6}$$

so that

$$|R(u, \Delta t)| \leq \text{ const} \cdot (\Delta t)^2$$

Substituting this into (2.5) and taking the limit as $\Delta t \to 0$ gives

$$\int g(u) \, d\mu_0(u) = 0 \tag{2.7}$$

Our next step is to derive (2.6) for small $\Delta t > 0$ and compute $g(u)$ explicitly. Let $X(t) = X$ and $W(t) = u$. Due to our assumption (1.4) the probability that the domain

$$D_1 = \Big\{(x, v) : \;\; 0 < (X - x) \cdot \operatorname{sgn} v < \bar{V} \Delta t, \quad v_{\min} < |v| < v_{\max}\Big\}$$

contains more than one atom is $\mathcal{O}((\Delta t)^2)$, hence we can ignore it and assume that D_1 contains at most one atom. Note that D_1 is much larger than D defined by (2.2). Now it is easy to see that the piston experiences at most one collision during the interval $(t, t+\Delta t)$. Precisely, a collision occurs if and only if the domain $D = D^+ \cup D^-$ contains an atom. When it does, and the atom has velocity v at time t, then the piston's velocity at time $t + \Delta t$ is computed by (1.1):

$$W(t + \Delta t) = u + \frac{2m}{M + m} \, (v - u) \tag{2.8}$$

Therefore, the conditional expectation $E(u, \Delta t)$ is

$$E(u, \Delta t) = u + \frac{2m}{M + m} \, \Big[\int_D vp \, dx \, dv - u \int_D p \, dx \, dv\Big] + \mathcal{O}((\Delta t)^2) \tag{2.9}$$

(the last term comes from our assumption that D contains at most one atom). While the second integral in (2.9) is already computed in (2.4), the calculation of the first one is similar. For every $k \geq 0$ we have

$$\begin{aligned}\int_D v^k\, p(x,v)\, dx &= \Delta t \left[\int_{v_{\min}}^{v_{\max}} v^k (v-u)\, p_-(v)\, dv + \int_{-v_{\max}}^{-v_{\min}} v^k (u-v)\, p_+(v)\, dv\right] \\ &= \Delta t \left[F^-_{k+1} - F^+_{k+1} - (F^-_k - F^+_k) u\right] \\ &= \Delta t \left[Q_{k+1} - Q_k u\right] \qquad (2.10)\end{aligned}$$

Thus,

$$E(u, \Delta t) = u + \frac{2m}{M+m}\left[Q_2 - 2Q_1 u + Q_0 u^2\right] \Delta t + \mathcal{O}((\Delta t)^2)$$

This gives us

$$g(u) = \frac{2m}{M+m}\left[Q_2 - 2Q_1 u + Q_0 u^2\right]$$

We now apply the key identity (2.7) and arrive at

$$Q_2 - 2Q_1 \langle u \rangle + Q_0 \langle u^2 \rangle = 0$$

where $\langle \cdot \rangle$ means averaging with respect to the stationary measure μ_0 (we note that Q_i do not depend on u). Thus, the average velocity of the piston is given by

$$\langle W \rangle = \frac{Q_0 \langle W^2 \rangle + Q_2}{2Q_1} \qquad (2.11)$$

The mechanical equilibrium is defined by the equality of pressures on both sides of the piston, i.e. by

$$Q_2 = F_2^- - F_2^+ = 0$$

and we will restrict ourselves to this case in the rest of the paper. Then (2.11) reduces to

$$\langle W \rangle = \frac{Q_0 \langle W^2 \rangle}{2Q_1} \qquad (2.12)$$

which is still different from zero, albeit very small.

It is well known that at mechanical equilibrium the average value of $\langle W^2 \rangle$ is $\mathcal{O}(m/M)$, see, e.g. [H, DGL]. But we will estimate it more accurately below.

Similarly to $E(u, \Delta t)$, we define

$$E_2(u, \Delta t) = \int w^2\, \mathcal{P}(u, dw; \Delta t)$$

the conditional expectation of the square velocity $W^2(t + \Delta t)$ given that $W(t) = u$. The stationarity of the distribution μ_0 implies

$$\int E_2(u, \Delta t)\, d\mu_0(u) = \int u^2\, d\mu_0(u)$$

Assume that

$$E_2(u, \Delta t) = u^2 + g_2(u)\, \Delta t + R_2(u, \Delta t)$$

with $|R_2(u,\Delta t)| \le$ const $\cdot (\Delta t)^2$. Then, as in (2.7)

$$\int g_2(u)\, d\mu_0(u) = 0 \tag{2.13}$$

Now, squaring the equation (2.8) we obtain

$$W^2(t+\Delta t) = u^2 + \frac{4m}{M+m}\,(v-u)u + \frac{4m^2}{(M+m)^2}\,(v-u)^2$$

Then, as in (2.9),

$$\begin{aligned} E_2(u,\Delta t) = u^2 \quad &+ \quad \frac{4m}{M+m}\int_D (v-u)up\,dx\,dv \\ &+ \quad \frac{4m^2}{(M+m)^2}\int_D (v-u)^2 p\,dx\,dv + \mathcal{O}((\Delta t)^2) \end{aligned}$$

Computing these integrals as before gives

$$\begin{aligned} g_2(u) \quad = \quad & \frac{4m}{M+m}\,(Q_2u - 2Q_1u^2 + Q_0u^3) \\ & + \frac{4m^2}{(M+m)^2}\,(Q_3 - 3Q_2u + 3Q_1u^2 - Q_0u^3) \end{aligned}$$

We included the terms with Q_2 for the sake of clarity, even though we had assumed that $Q_2 = 0$. In the subsequent expressions we remove all Q_2's. The equation (2.13) gives

$$2Q_1\langle W^2\rangle - Q_0\langle W^3\rangle = \frac{m}{M+m}\,(Q_3 + 3Q_1\langle W^2\rangle - Q_0\langle W^3\rangle)$$

For brevity, we denote $\varepsilon = m/M$. By making one step further and analyzing $E_3(u,\Delta t) = \int w^3\,\mathcal{P}(u,dw;\Delta t)$ in the same manner one can see that $\langle W^3\rangle = \mathcal{O}(\varepsilon^2)$, we omit details. Hence we arrive at

$$\langle W^2\rangle = \frac{Q_3}{2Q_1}\,\varepsilon + \mathcal{O}(\varepsilon^2) \tag{2.14}$$

Substituting this into (2.12) yields

$$\langle W\rangle = \frac{Q_0Q_3}{4Q_1^2}\,\varepsilon + \mathcal{O}(\varepsilon^2) \tag{2.15}$$

3 Thermodynamical analysis

In this section we express our rigorous results in statistical mechanical terms, such as temperature, density, pressure, and heat transfer.

First, we consider the heat flow across the piston, from right to left. Even though both gases are infinite, the heat transfer from one to the other is well defined and its rate can be computed. The thermodynamical definition of the heat transfer [GF] is

$$R_{+\to -} = \langle d\mathcal{E}_-/dt\rangle - \langle d\mathcal{M}_-/dt\rangle\langle W\rangle \tag{3.1}$$

where $d\mathcal{E}_-/dt$ is the rate of change of kinetic energy and $d\mathcal{M}_-/dt$ the rate of change of momentum on the left hand side. By the conservation of energy and momentum during collisions, it is enough to compute the average change of energy and momentum of the piston when it collides with an atom on the left hand side.

According to (1.1), the momentum of the piston at one collision changes by

$$\delta\mathcal{M}_M = \frac{2mM}{M+m}\,(v-W)$$

The conditional average change of momentum during the interval $(t, t+\Delta t)$, given that $W(t) = u$, is

$$\begin{aligned} E_u(\delta\mathcal{M}_M) &= \frac{2mM}{M+m}\int_{D^-}(v-u)\,dx\,dv + \mathcal{O}((\Delta t)^2) \\ &= \frac{2mM}{M+m}\,(F_2^- - 2F_1^- u + F_0^- u^2)\,\Delta t + \mathcal{O}((\Delta t)^2) \end{aligned}$$

where only collisions with atoms on the left are taken into account. Negating and averaging with respect to the stationary measure μ_0 gives the average momentum transfer to the left gas:

$$\langle d\mathcal{M}_-/dt\rangle = -2m[F_2^- - 2F_1^-\langle W\rangle + F_0^-\langle W^2\rangle] + \mathcal{O}(m^2/M) \tag{3.2}$$

By squaring (1.1) we find the change of the kinetic energy of the piston at one collision:

$$\delta\mathcal{E}_M = 2m\varepsilon v^2 + 2mVv - 2mV^2 + \cdots$$

where only essential terms are shown (the rest will eventually end up in $\mathcal{O}(m^3/M^2)$ in the expression (3.3) below, so we omit them). The conditional average change of kinetic energy during the interval $(t, t+\Delta t)$, given that $W(t) = u$, is (again only collisions with atoms on the left are taken into account)

$$E_u(\delta\mathcal{E}_M) = 2m(\varepsilon F_3^- - \varepsilon F_2^- u + F_2^- u - 2F_1^- u^2 + F_0^- u^3)\,\Delta t + \cdots + \mathcal{O}((\Delta t)^2)$$

Negating and averaging with respect to the stationary measure μ_0 gives the average kinetic energy transfer to the left gas:

$$\langle d\mathcal{E}_-/dt\rangle = -2m[\varepsilon F_3^- + F_2^-\langle W\rangle - 2F_1^-\langle W^2\rangle] + \mathcal{O}(m^3/M^2) \tag{3.3}$$

Combining (3.2) and (3.3) we obtain the expression for the heat transfer

$$R_{+\to -} = -2m[\varepsilon F_3^- - 2F_1^-\langle W^2\rangle] + \mathcal{O}(\varepsilon^2) \tag{3.4}$$

Substituting (2.14) gives

$$R_{+\to -} = \frac{2m\varepsilon}{Q_1}(F_1^+ F_3^- - F_1^- F_3^+) + \mathcal{O}(\varepsilon^2) \tag{3.5}$$

Now, even though the thermodynamical temperatures of the gases are not defined, unless their velocity distributions are Maxwellian, we can define the "effective temperatures" via the second moment of the velocity distributions:

$$T_\pm = k_B^{-1} m \, \frac{\int v^2 p_\pm \, dv}{\int p_\pm \, dv} = k_B^{-1} m \, \frac{F_2^\pm}{F_0^\pm}$$

where k_B is Boltzmann's constant. We recall that the pressures of the gases are given by $P_\pm = 2mF_2^\pm$ and their spatial densities by $n_\pm = 2F_0^\pm$, hence the classical law $P_\pm = n_\pm k_B T_\pm$ holds.

The piston's velocity distribution μ_0 is, generally, not Maxwellian, but its effective temperature can be computed similarly:

$$T_M = k_B^{-1} M(\langle W^2 \rangle - \langle W \rangle^2) \simeq k_B^{-1} m \frac{Q_3}{2Q_1}$$

where we used (2.14) and (2.15). Hence T_M is of the same order of magnitude as $T_\pm$, but for generic distributions $p_\pm$ there is no simple relation between the exact values of these temperatures.

However, we can derive some interesting relations assuming that the distributions $p_\pm$ are Maxwellian. Since this would violate our assumptions (1.4), our further analysis is heuristic. Assume that the densities $p_\pm$ satisfy

$$p_\pm(x, v) = n_\pm f_\pm(v)$$

where $n_\pm$ are (constant) spatial densities and

$$f_\pm(v) = \frac{1}{\sqrt{2\pi\sigma_\pm^2}} \, \exp\left(-\frac{v^2}{2\sigma_\pm^2}\right)$$

the Maxwellian distributions with temperatures $T_\pm = k_B^{-1} m \sigma_\pm^2$. As before, we assume that the pressures are equal, i.e. $n_- T_- = n_+ T_+$. It is then easy to compute

$$Q_0 = \frac{1}{2}\,(n_- - n_+)$$

$$Q_1 = \frac{1}{\sqrt{2\pi}}\,(n_-\sigma_- + n_+\sigma_+) = \frac{\sqrt{k_B}}{\sqrt{2\pi m}}\,\left(n_- T_-^{1/2} + n_+ T_+^{1/2}\right)$$

$$Q_2 = \frac{1}{2}\,(n_-\sigma_-^2 - n_+\sigma_+^2) = \frac{k_B}{2m}\,(n_- T_- - n_+ T_+) = 0$$

$$Q_3 = \sqrt{\frac{2}{\pi}}\,(n_-\sigma_-^3 + n_+\sigma_+^3) = \frac{\sqrt{2k_B^3}}{\sqrt{\pi m^3}}\,\left(n_- T_-^{3/2} + n_+ T_+^{3/2}\right)$$

Now the effective temperature of the piston is

$$T_M = \frac{mQ_3}{2k_B Q_1} = \frac{n_- T_-^{3/2} + n_+ T_+^{3/2}}{n_- T_-^{1/2} + n_+ T_+^{1/2}} = \sqrt{T_- T_+} \tag{3.6}$$

where we used the equality of the pressures. The average velocity of the piston is

$$\begin{aligned}\langle W \rangle &= \frac{\sqrt{2\pi k_B m}}{4M}\,\frac{(n_- - n_+)(T_- T_+)^{1/2}}{n_- T_-^{1/2} + n_+ T_+^{1/2}} + \mathcal{O}(\varepsilon^2)\\ &= \frac{\sqrt{2\pi k_B m}}{4M}\,\left(\sqrt{T_+} - \sqrt{T_-}\right) + \mathcal{O}(\varepsilon^2) \end{aligned}\tag{3.7}$$

Lastly, the heat transfer across the piston (from right to left) given by (3.5) is now

$$\begin{aligned}R_{+\to-} &= \frac{2m\varepsilon\sqrt{2k_B^3}}{\sqrt{\pi m^3}} \times \frac{n_- n_+ (T_+^{3/2} T_-^{1/2} - T_-^{3/2} T_+^{1/2})}{n_- T_-^{1/2} + n_+ T_+^{1/2}} + \mathcal{O}(\varepsilon^2)\\ &= \frac{k_B}{M}\sqrt{\frac{8k_B m}{\pi}}\,\frac{n_- n_+ (T_- T_+)^{1/2}}{n_- T_-^{1/2} + n_+ T_+^{1/2}}\,(T_+ - T_-) + \mathcal{O}(\varepsilon^2) \end{aligned}\tag{3.8}$$

Hence, the heat flow is proportional to the temperature gradient, $T_+ - T_-$, and the heat conductivity is thus

$$\kappa = \frac{k_B}{M}\sqrt{\frac{8k_B m}{\pi}}\,\frac{n_- n_+ (T_- T_+)^{1/2}}{n_- T_-^{1/2} + n_+ T_+^{1/2}} \tag{3.9}$$

The formulas (3.6)–(3.9) were obtained earlier in [GP, GF, GPL] by means of kinetic theory and the Liouville equation.

Let us examine closely what happens when the pressures are equal, $P_- = P_+$, but the temperatures are different. Without loss of generality, assume that $T_- < T_+$, i.e. the gas on the left is cooler but denser and the gas on the right is hotter but sparser. Then $\langle W \rangle > 0$, i.e. the piston slowly drifts in the direction of the hotter gas. At the same time $R_{+\to-} > 0$, i.e. the heat slowly flows from the hotter gas to the cooler gas. This demonstrates that the mechanical equilibrium is not a stable state. If our gases were finite, there would be a continuing evolution toward *thermal equilibrium*, in which the temperatures become equal, too. Of course, this conclusion makes no sense in the idealized model of infinite gases.

We note that the evolution of the system is somewhat counterintuitive. When the piston moves to the right, i.e. $V(t) > 0$, then by (1.2) the atoms on the right bounce off it with a higher speed and so gain energy, while the atoms on the left collide with the piston and slow down, hence lose some energy. When the piston moves to the left, i.e. $V(t) < 0$, it is vice versa. Since, on the average, the heat flows from right to left, one may conclude that the piston's movements to the left dominate. On the other hand, the piston slowly drifts to the right, so its displacements in that direction actually dominate.

To explain this "paradox", we first recall that the average speed of the piston is of order $\langle W^2\rangle^{1/2} \sim \varepsilon^{1/2}$, which is much larger than the speed of the drift $\langle W\rangle \sim \varepsilon$. Hence the piston jiggles back and forth much faster than it drifts in one direction. The heat flow is due to "jiggling" rather than "drifting". Indeed, when the piston jiggles to the right, the slow atoms on the left have less chance to collide with the piston, since the relative velocity is small. Most of the collisions between the slow atoms on the left and the piston occur when the latter jiggles to the left. This is why the cooler atoms speed up and gain energy, on the average. On the other hand, the hotter atoms on the right are less sensitive to the variations in the piston's velocity, and the reason why they cool down is different, as we explain next.

The piston's vibrations are not spatially symmetric. During excursions to the right, relatively few collisions with atoms on the right occur, since they are more energetic and able to quickly reverse the velocity of the piston. On the contrary, when the piston drifts to the left, the atoms on the left are weak and it takes them longer to turn the piston back to the right. During these intervals, the fast atoms on the right continue hitting the piston and lose energy. This explains why the hotter atoms mostly slow down. We must admit though that the precise mechanism of the heat flow across the piston in our model remains unclear. Some physicists call it a "conspiracy" between the microscopic vibrations of the piston and the incoming atoms of the gases [GF, GP].

4 Justification of Markov approximation

Here we estimate the difference between the true velocity of the piston $V(t)$ and our Markov approximation $W(t)$.

Recall that $\langle W^2\rangle \sim \varepsilon = m/M$, i.e. the typical speed of the piston (in the Markov approximation) is of order $\sqrt{\varepsilon}$. The following theorem, which is essentially proved in [CLS] (see remarks below), estimates large deviations of the true velocity $V(t)$:

Theorem 4.1 ([CLS]) *Let $|V(0)| < \varepsilon^{1/2}$. There are constants $C, d > 0$ such that for every $T > 0$ we have*

$$P\left(\sup_{0<t<T} |V(t)| > C\,\varepsilon^{1/2}\,\ln\varepsilon^{-1}\right) < T^2\varepsilon^{-d\ln\ln\varepsilon}$$

This gives a good bound on large deviations during time intervals of length $T \sim \varepsilon^{-A}$ for any large but fixed constant $A > 0$: it shows that the true velocity $V(t)$ of the piston remains $\mathcal{O}(\varepsilon^{1/2}\,\ln\varepsilon^{-1})$ with "overwhelming" probability.

In our construction of the Markov approximation $W(t)$ we can choose any small but positive cutoff value $\bar{V} > 0$, and then assume that ε is so small that $C\,\varepsilon^{1/2}\,\ln\varepsilon^{-1} < \bar{V}$. Then we arrive at

Corollary 4.2 *For all sufficiently small ε and all $T > 0$ we have*

$$P\Big(V(t) = W(t) \quad \forall t \in (0,T)\Big) \geq 1 - T^2 \varepsilon^{-d \ln \ln \varepsilon}$$

Therefore, during time intervals of length $T \sim \varepsilon^{-A}$ for any fixed constant $A > 0$, the true velocity $V(t)$ of the piston coincides with its Markov approximation $W(t)$ with "overwhelming" probability. If $V(t)$ and $W(t)$ differ, though, then we have an obvious bound: $|V(t)| \leq v_{\max}$. So we obtain analogues of our main formulas (2.15) and (2.14):

Corollary 4.3 *Let $\varepsilon > 0$ be small and $|V(0)| < \varepsilon^{1/2}$. Then for any fixed $A > 0$ and all $0 < t < T = \varepsilon^{-A}$ we have*

$$E(V(t)) = \frac{Q_0 Q_3}{4Q_1^2}\, \varepsilon + \mathcal{O}(\varepsilon^2)$$

and

$$E(V^2(t)) = \frac{Q_3}{2Q_1}\, \varepsilon + \mathcal{O}(\varepsilon^2)$$

where $E(\cdot)$ is the mean value.

It follows from the results of [CLS] that whenever the true velocity $V(t)$ exceeds $C\,\varepsilon^{1/2} \ln \varepsilon^{-1}$, then with "overwhelming" probability it will be driven back toward zero by further collisions with the gas atoms. Hence, the estimates of the last corollary should remain true for all $t > 0$, but we do not pursue such a goal, because the drift of the piston becomes evident on the time scale of order ε^{-A} already for $A > 2$, as we show next, so the estimates in Corollary 4.3 are sufficient for practical purposes.

Indeed, let $X(0) = 0$ and consider the random position of the piston $X(t)$ at time t. Also, let $Y(t) = \int_0^t W(s)\, ds$ be the corresponding function in the Markov approximation model. We obviously have

$$\langle Y(t) \rangle = t \langle W \rangle = \frac{Q_0 Q_3}{4Q_1^2}\, \varepsilon t + \mathcal{O}(\varepsilon^2 t) \tag{4.1}$$

We also need to estimate the variance of $Y(t)$, which is a little harder. We claim that

$$\mathrm{Var}[Y(t)] \leq Dt \tag{4.2}$$

with some constant $D > 0$ independent of ε. Below we outline the argument, suppressing some technical details that are based on the estimates obtained in [CLS].

It is standard that

$$\mathrm{Var}[Y(t)] = \int_0^t \int_0^t \mathrm{Cov}(W(s), W(u))\, ds\, du$$

By the stationarity of the process $W(t)$, the covariance here equals

$$\mathrm{Cov}(W(s), W(u)) = \rho_{s-u}\, \mathrm{Var}(W)$$

where ρ_{s-u} is the correlation between $W(s)$ and $W(u)$, which only depends on $|s-u|$. Also recall that

$$\mathrm{Var}(W) = \langle W^2\rangle - \langle W\rangle^2 = \frac{Q_3}{2Q_1}\,\varepsilon + \mathcal{O}(\varepsilon^2)$$

Therefore,

$$\mathrm{Var}[Y(t)] \le \mathrm{const}\cdot t\,\varepsilon \int_0^t \rho_s\,ds \tag{4.3}$$

It remains to estimate the decay of the velocity autocorrelation function ρ_s.

Suppose during the interval $(u, u+s)$ the piston experiences k collisions with gas atoms with velocities $v_1, \ldots, v_k$ numbered in the order in which the collisions occur. Then (1.1) implies

$$W(u+s) = (1-\alpha)^k W(u) + \alpha \sum_{j=1}^{k} (1-\alpha)^{k-j}\cdot v_j \tag{4.4}$$

where $\alpha = 2m/(M+m) \simeq 2\varepsilon$. This equation can be easily verified by induction on k.

Now, the number of collisions k grows as const$\cdot s$, on the average, in fact

$$P(|k - c_1 s| \ge c_2\sqrt{s}\ln s) \le s^2 \varepsilon^{-d\ln\ln\varepsilon} \tag{4.5}$$

with some positive constants c_1, c_2, see [CLS]. For $s < \varepsilon^{-A}$ this gives a good bound on large deviations, so we can safely assume that $k \ge c_1\varepsilon/2$. Next, the velocities v_i are almost independent of $W(u)$. In fact, there is a dependence, since the time of collision between a particular atom and the piston depends on the velocity of the latter, but the correlation between $W(u)$ and velocities v_j for every fixed j is negative. This follows from a simple observation: if $W(u) > 0$, the piston is more likely to collide with atoms on the right, for which $v_j < 0$, and vice versa. Therefore, the correlation ρ_s is determined by the first term in (4.4), hence

$$|\rho_s| \le (1-\alpha)^{c_1 s/2}$$

This implies

$$\int_0^t \rho_s\,ds \le \int_0^\infty (1-\alpha)^{c_1 s/2}\,ds = -\frac{2}{c_1\ln(1-\alpha)} \simeq \frac{1}{c_1\varepsilon}$$

and combining with (4.3) proves (4.2).

The estimates (4.1) and (4.2) show that the piston's displacement due to the drift exceeds its typical random fluctuations at times $t = \varepsilon^A$ for all $A > 2$. On such time scales the drift is "physically evident" and it is possible to verify it experimentally. Numerical experiments of this sort were done by [GF].

We now comment on the proofs of Theorem 4.1 and the estimate (4.5). The corresponding results were proven in [CLS] in a slightly different context. That paper dealt with a heavy piston in a cubical container of side L filled by an ideal gas. That model, too, reduced by a trivial projection onto the axis perpendicular to the piston surface to a one-dimensional gas on an interval $[-L/2, L/2]$. The velocity cutoff assumptions in [CLS] were identical to our (1.4). The piston had mass $M = aL^2$ ($a > 0$ was a constant) and the spatial density of the one-dimensional gases was proportional to $M = aL^2$, but a simple rescaling of time and space by M reduces the spatial density to values $\mathcal{O}(1)$ and makes it independent of M. Hence, we arrive exactly at the same initial conditions as in the present paper, except the gases in [CLS] were finite – they evolved on an interval $[-aL^3/2, aL^3/2]$ with the piston initially placed at $X(0) = 0$.

Due to the velocity cutoff (1.4) the gas atoms in [CLS] cannot interact with the piston more than once unless they travel across the half interval $[-aL^3/2, aL^3/2]$, reflect at an endpoint $x = \pm aL^3/2$ and then travel back to the piston in the middle. This takes time $T = \mathcal{O}(L^3)$, as it was proven in [CLS]. Thus, during the initial interval $(0, T)$ with $T = \mathcal{O}(M^{3/2})$ the evolution of the piston studied in [CLS] is identical to ours. Hence, all the results and estimates derived in Section 3 of [CLS] directly apply to our present context. Furthermore, with some minor obvious changes those results can be extended to arbitrary time intervals as we did in Theorem 4.1 and (4.5).

Lastly, we have assumed the velocity cutoff (1.4) here and in the paper [CLS] mostly for convenience. It might be true that the upper bound on velocities is quite essential in [CLS], since arbitrarily fast atoms could bounce back and forth between the piston and the wall many times during short time intervals thus making significant impact on the piston, as it was remarked in Section 5 of [CLS]. But in the present work, if we had arbitrarily fast particles, they would hit the piston once and get away never to come back, so their impact would be quite limited.

As for the lower bound on velocities, it seems to be less essential in both models. It was conjectured in Section 5 of [CLS] that the results of that paper could be extended to velocity distributions without lower bounds (i.e. with $v_{\min} = 0$). If so, then the estimate in Theorem 4.1 implies that the typical velocity of the piston remains of order $\varepsilon^{1/2} \ln \varepsilon^{-1}$. As long as this is true, only atoms with smaller velocities can experience two or more collisions with the piston, thus violating the Markovian character of the piston's velocity process $V(t)$. It is customary to redefine the dynamics so that the collisions with slow atoms are ignored altogether, and then the resulting process $W(t)$ will be Markovian. The difference $|V(t) - W(t)|$ should be estimated separately and presumably is negligible. Such a strategy was successfully implemented in earlier works [H, DGL] in somewhat different context, and it is likely to go through in our case as well. This is the subject of our future work.

Acknowledgement. The author thanks J. Lebowitz, J. Piasecki and Ya. Sinai for many useful discussions and encouragement. The author was partially supported by NSF grant DMS-0098788. This work was completed during the author's stay at the Institute for Advanced Study with partial support by NSF grant DMS-9729992.

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