Stability of Solutions of Hydrodynamic Equations Describing the Scaling Limit of a Massive Piston in an Ideal gas

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Abstract

We analyze the stability of stationary solutions of a singular Vlasov type hydrodynamic equation (HE). This equation was derived (under suitable assumptions) as the hydrodynamical scaling limit of the Hamiltonian evolution of a system consisting of a massive piston immersed in an ideal gas of point particles in a box. We find explicit criteria for global stability as well as a class of solutions which are linearly unstable for a dense set of parameter values. We present some numerical evidence that when the mechanical system (with a large number of particles) has initial conditions "close" to stationary stable solutions of the HE, then it stays close to these solutions for a long time. On the other hand, if the initial state of the particle system is close to an unstable stationary solution of the HE, then the mechanical system rapidly diverges from that solution and later appears to develop long lasting periodic oscillations. We find similar (approximately periodic) solutions of the HE that are linearly stable.

1 Introduction

The time evolution of a system consisting of a piston of mass M moving parallel to the x-axis in a cube containing non-interacting point particles of unit mass has been studied extensively [CLS1, CLS2, CL, G, GF, GP, H, KBM, LPS, Li, PG]. After some rescaling of space and time (by the length of the cube) the problem reduces to that of a one dimensional system with N_L (N_R) particles in the interval $[0, X]$ (resp., $[X, 1]$) where $X(t)$ is the position of the piston. The left (right) particles move freely between collisions with the wall at $x = 0$ ($x = 1$) and the piston at $x = X(t)$. At the walls the velocities

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of the particles get reversed while at $x = X(t)$ the outgoing velocity v' is related to the incoming velocity v by the rules of elastic collisions [CLS2, CL, LPS],

$$v' = -\frac{M-1}{M+1} v + \frac{2M}{M+1} V \tag{1.1}$$

where V is the incoming velocity of the piston. It follows from (1.1) that N_L, N_R, as well as the total kinetic energy, $\frac{1}{2} \sum_{i=1}^{N} v_i^2 + \frac{1}{2} M V^2$ ($N = N_L + N_R$) are conserved quantities. The dynamics of the system can be reduced to a billiard in a $(2N+1)$-dimensional domain (polyhedron), cf. [CL]. It was shown in [LPS, CLS1, CLS2], under certain quite restrictive conditions on the initial distribution of gas particles, that, in the limit $N \to \infty$, $M \sim N^{2/3}$, the dynamics of the piston and the gas satisfy a closed system of Euler-type hydrodynamic equations (HE) for a time interval $(0, \tau)$ in which any particle had at most two collisions with the piston.

The origin of the scaling $M \sim N^{2/3}$ is as follows. For N particles with velocities of $O(1)$ distributed with density of $O(1)$ in a parallelopiped of length L and crossectional area A the number of particles colliding with the piston per unit (unrescaled) time, and hence the pressure (from each side), is proportional to A. To ensure that, on this time scale, the acceleration of the piston stays of $O(1)$ as L, A and $N \sim O(AL)$ grow to infinity, it is necessary to make the mass of the piston grow as A. For a cube this corresponds to $M \sim N^{2/3}$. In the rescaled units, the number of collisions experienced by the piston per unit time is $O(N)$ independent of A, and the HE hold for general M as long as $M \sim N^{\alpha}$, $\alpha \in (0, 1)$, i.e. when the kinetic energy of the piston becomes negligible compared to that of the gas. The time interval (in the scaled units) during which the derivation of the HE remains valid depends on α (getting larger as $\alpha \to 1$), see Remarks 3 and 4 in Sect. 4 of [CLS2]. For $\alpha = 2/3$ this time interval is such that the piston suffers no more than two collisions with any gas particle. It is however not clear from the derivation to what extent those equations may actually approximate the real evolution of the particle system with large N, for longer times.

This led us to carry out extensive computer simulations of particle systems, with $M \sim N^{2/3}$ (precisely, $M = 2N^{2/3}$), N as large as 27×10^6 [CL], and initial conditions for which the hydrodynamic equations have a trivial stationary solution $X(t) = 0.5$ and $V(t) = 0$ for all $t > 0$. We found nevertheless that for certain initial velocity distributions (see later) the trajectory of the piston diverged greatly from these values after a few collision times. In particular, it was observed in these simulations that the motion of the piston, after experiencing large seemingly random fluctuations, quickly converges to a more stable regime, in which the piston and the gas undergo regular (slowly damped) oscillations lasting many collision times. The parameters of those oscillations (the period, the amplitude, and the rate of damping) seem to depend little on the number of point particles and some other details of the initial distributions.

It is reasonable to conjecture that the behavior of mechanical system for finite but large N, is related to the solution of the HE subjected to random fluctuations arising from the discrete nature of the gas particles. The large deviations of the motion of the

mechanical system from the solution of HE, observed in [CL], may thus be due to the instability of the HE for the initial conditions considered; c.f. discussion of hydrodynamic fluctuation theory [S].

This interpretation requires that the HE describe the limit (as $N \rightarrow \infty$) of the mechanical evolution of the system for times much longer than those for which they were derived in [CLS2]. This in turn would imply that when the solution of the hydrodynamic equations is stable, then the mechanical evolution of the system should remain close to that solution for arbitrarily long times as $N \rightarrow \infty$. We note further that by choosing α in $M \sim N^\alpha$ close to one, the time for which the rigorous derivation of the HE remains valid can be made arbitrary large. Motivated by these considerations we investigate here stability properties of a special class of stationary solution of the HE. These are of independent intrinsic interest since they represent, as far as we know, a first example of a rigorous derivation of non-trivial continuum equations from a Hamiltonian system (without any stochasticity).

The paper is organized as follows. In Section 2 we state the hydrodynamic equations and discuss their structure. In Section 3 we prove (rigorously) global stability for a class of stationary solution. We note that simulations of the mechanical systems with such distributions in [CL] indeed yield a piston trajectory that remains close to the solution of the HE, with $V(t) \sim 0$ for all $t > 0$. In Section 4, we use a perturbative analysis to find sufficient conditions for linear instability (our arguments here are heuristic, but the conclusions are in good agreement with simulations). In Section 5 we investigate a particular family of stationary solutions, which include those used in the [CL] simulations, and show that, according to the criteria developed in Section 4, it contains both linearly stable and unstable ones, alternating in a very intricate manner. We also describe there numerical solutions of the HE for two initial conditions obtained by adding the same perturbation to a linearly stable and a linearly unstable stationary state. After a short initial time in which the solutions behave similarly, they exhibit interesting differences (Figures 3–5). In Section 6 we compare a mechanical trajectory of the piston with a solution of the HE with initial conditions obtained from those used for the simulation of the mechanical system by a small perturbation. The two trajectories look startlingly similar, thus confirming our previous analysis. After an initial period of 10-15 recollision times, both trajectories evolve almost periodically for another 15-20 recollision times, which is much longer than the time for which the equations were derived in [CLS2], but sufficiently short so that the periodic motion of the mechanical system has no visible damping. These observations suggest the existence of a periodic solution of the HE, and we then carry out an approximate analytic construction of such periodic solutions.

2   Hydrodynamic equations

Let $X(t) \in (0,1)$ be the position of the piston at time t and $V(t)$ its velocity. We denote the continuum density of the gas in $[0,1] \times \mathbb{R}$ by a function $p(x,v,t)$. The HE describing the time evolution of this, continuum fluid plus piston system, are as follows.

(H1) *Free motion.* Inside the container the density satisfies the standard continuity equation for a noninteracting particle system without external forces:

$$\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right) p(x, v, t) = 0 \tag{2.1}$$

for all $x \in (0, 1)$ except $x = 0$, $x = 1$ and $x = X(t)$.

Equation (2.1) has a simple solution

$$p(x, v, t) = p(x - vs, v, t - s) \tag{2.2}$$

for $0 < s < t$ such that $x - vr \notin \{0, X(t-r), 1\}$ for all $r \in (0, s)$. Equation (2.2) has one advantage over (2.1): it applies to all points (x, v), including those where the function p is not differentiable, or even continuous.

(H2) *Collisions with the walls.* At the walls $x = 0$ and $x = 1$ we have

$$p(0, v, t) = p(0, -v, t) \tag{2.3}$$

$$p(1, v, t) = p(1, -v, t) \tag{2.4}$$

(H3) *Collisions with the piston.* At the piston $x = X(t)$ we have (this is obtained from (1.1) when $M \to \infty$)

$$\begin{aligned} p(X(t) - 0, v, t) &= p(X(t) - 0, 2V(t) - v, t) && \text{for } v < V(t) \\ p(X(t) + 0, v, t) &= p(X(t) + 0, 2V(t) - v, t) && \text{for } v > V(t) \end{aligned} \tag{2.5}$$

where v represents the velocity after the collision and $2V(t) - v$ that before the collision; and

$$X(t) = X(0) + \int_0^t V(s)\, ds \tag{2.6}$$

is the (deterministic) position of the piston.

It remains to describe the evolution of $V(t)$ (which we take to be left continuous). Suppose the piston's position at time t is X and its velocity V. The piston is affected by the fluid at (x, v) exerting pressure on it from the right ($x = X + 0$ and $v < V$) and from the left ($x = X - 0$ and $v > V$). Accordingly, we define the density of the fluid in contact with the piston before a collision ("density on the piston") by

$$P(v, t; X, V) = \begin{cases} p(X + 0, v, t) & \text{if } v < V \\ p(X - 0, v, t) & \text{if } v > V \end{cases} \tag{2.7}$$

(H4) *Piston's velocity.* The velocity $V = V(t)$ of the piston satisfies the equation

$$\int_{-\infty}^{\infty} (v - V)^2 \operatorname{sgn}(v - V)\, P(v, t; X, V)\, dv = 0 \tag{2.8}$$

The origin of eqs. (H1)–(H3) in the particle system is clear. Eq. (H4) is essentially a force balance equation—since the rate of collision of the piston with particles on either side and consequent force on piston is much larger than mass of the piston when $N/M \to \infty$, V adjusts instantaneously to make the forces from the two sides balance exactly. The system of (hydrodynamic) equations (H1)–(H4) is now closed and, given initial conditions, satisfying (2.3)–(2.8) at $t = 0$, completely determine the functions $X(t)$, $V(t)$ and $p(x, v, t)$ for $t > 0$. When the initial conditions do not satisfy these equations one has to imagine that they become satisfied instantaneously for $t = 0+$. The existence and uniqueness of solutions of (H1)–(H4) were proven, under general conditions, in [CLS1, CLS2]. We need only to assume that the $p(x, v, 0)$ is bounded, piecewise differentiable, and either has a compact support in the x, v plane or decays fast enough as $|v| \to \infty$. We also require that $\int p(x, v, 0) \, dv > 0$ for all x.

The HE, like the Vlasov equations for plasmas, are time-reversible, see [P] and [MP]. They preserve the classical integrals of motion. The mass of the fluid to the left and to the right of the piston as well as the total kinetic energy of the fluid remain constant along any solution.

(D1) *Mass conservation*

$$\mathcal{M}_L = \int_0^{X(t)} \int p(x, v, t) \, dv \, dx, \qquad \mathcal{M}_R = \int_{X(t)}^1 \int p(x, v, t) \, dv \, dx$$

(D2) *Energy conservation*

$$2E = \int \int v^2 p(x, v, t) \, dv \, dx \tag{2.9}$$

Just like the energy, the total momentum of the fluid $\int \int v \, p(x, v, t) \, dv \, dx$ is also left unchanged by the fluid-piston interaction, but unlike the energy the momentum changes due to reflections at the walls. The reason that the piston itself does not contribute to the total momentum and energy of the system in this model is due to the fact that its mass and energy vanish, when divided by N, in the limit $N \to \infty$. (The mass, energy and momentum of the fluid all correspond to the original quantities in the particle system divided by N).

The HE define a dynamics on the domain $G := \{(x, v) : 0 \le x \le 1\}$ in which every point $(x, v) \in G$ moves freely with constant velocity and collides elastically with the walls and the piston. Denote by (x_t, v_t) the position and velocity of an arbitrary point at time $t \ge 0$. Then (H1) translates into $\dot{x}_t = v_t$ and $\dot{v}_t = 0$ whenever $x_t \notin \{0, 1, X(t)\}$, (H2) becomes $(x_{t+0}, v_{t+0}) = (x_{t-0}, -v_{t-0})$ whenever $x_{t-0} \in \{0, 1\}$, and (H3) gives

$$(x_{t+0}, v_{t+0}) = (x_{t-0}, 2V(t) - v_{t-0})$$

whenever $x_{t-0} = X(t)$. Note that the point (x_t, v_t) moves in G and reflects at the walls and the piston as if those had infinite mass.

The motion of points in G is described by a one-parameter family of transformations $F^t : G \to G$ defined by $F^t(x_0, v_0) = (x_t, v_t)$ for $t > 0$. We will also write $F^{-t}(x_t, v_t) = (x_0, v_0)$. According to (H1)–(H3), the density $p(x, v, t)$ satisfies a simple equation

$$p(x_t, v_t, t) = p(F^{-t}(x_t, v_t), 0) = p(x_0, v_0, 0)$$

for all $t \geq 0$. It is easy to see that for each $t > 0$ the map F^t is one-to-one and preserves area, i.e. $\det |DF^t(x, v)| = 1$. Hence, the family F^t describes an incompressible flow on G and consequently:

(D3) *Incompressibility.* For any $a < b$ the Lebesgue measures (areas) of the sets

$$\{(x, v) : \ a < p(x, v, t) < b, \ 0 < x < X(t)\}$$

and

$$\{(x, v) : \ a < p(x, v, t) < b, \ X(t) < x < 1\}$$

remain constant in time

A particular case in which it is possible to solve equations (H1)–(H4) analytically, is when the initial distribution is stationary. This happens when $p(x, v, 0)$ satisfies two conditions:

(S1) *Uniformity and symmetry.* The initial density $p(x, v, 0) = p(x, v)$ is of the form

$$p(x, v) = \begin{cases} p_L(|v|) & \text{for} \ \ x < X_0 \\ p_R(|v|) & \text{for} \ \ x > X_0 \end{cases}$$

for all v and $X(0) = X_0$.

(S2) *Pressure balance.* The pressure on the piston from both sides is equal:

$$P_L := 2 \int_0^\infty v^2 p_L(v) \, dv = P_R := 2 \int_0^\infty v^2 p_R(v) \, dv \tag{2.10}$$

Under conditions (S1)–(S2) the equations (H1)–(H4) have a simple solution: the system remains frozen in its initial state:

$$X(t) \equiv X_0, \quad V(t) \equiv 0, \quad p(x, v, t) \equiv p(x, v, 0) \tag{2.11}$$

for all $t > 0$.

We will analyze in the next three sections the stability of this stationary solution. Note that there is no requirement on the form of $p_L(|v|)$ or $p_R(|v|)$; all that is required is a balance of forces (2.10).

3 Globally stable solutions

Here we prove that stationary solutions $p(x, v)$ satisfying (S1)–(S2) and an additional monotonicity requirement:

$$p_L(|v_1|) \geq p_L(|v_2|) \quad \text{and} \quad p_R(|v_1|) \geq p_R(|v_2|) \tag{3.1}$$

for all $|v_1| \leq |v_2|$ globally stable. This criteria is very similar to the stability criteria for the Vlasov equation described by Penrose [P] and by Marchioro and Pulvirenti [MP].

Before we state our result, we introduce some notation. Denote by $\| \cdot \|$ the following special norm on the space of functions on G

$$\|f(x, v) - g(x, v)\| = \int \int |f(x, v) - g(x, v)| \, (1 + v^2) \, dv \, dx \tag{3.2}$$

Theorem 3.1 *Let $p(x, v)$ satisfy* (S1), (S2) *and* (3.1)*. Then for any $\varepsilon > 0$, there exists a $\delta > 0$ such that if the initial density $p(x, v, 0)$ satisfies $\|p(x, v, 0) - p(x, v)\| < \delta$, and $X(0) = X_0$ then*

   (i) $\|p(x, v, t) - p(x, v)\| < \varepsilon$;

  (ii) $|X(t) - X(0)| < \varepsilon$

for all $t > 0$.

Proof. Our theorem is an almost straightforward consequence of the stability theorem[1] by Marchioro and Pulvirenti [MP] (in particular, when $p_L = p_R$, then (i) is exactly their theorem), so we only outline the argument here.

It is clear from (2.9) that, given the position of the piston X and values of the areas of the level sets, defined in $(D3)$, the minimal possible value of the total energy for any phase-space density $\pi(x, v)$, is attained when $\pi(x, v)$ is uniform in x and monotonically decreasing in $|v|$ in each compartment.

Consider first the case where $\pi(x, v)$ has, in each compartment, the same area of the level sets as some $p(x, v)$ satisfying $(S1)$ and $(S2)$. Then the minimum of the energy when the piston position is X is attained when

$$\pi(x, v) = p_L(vX/X_0), \quad 0 < x < X$$

and

$$\pi(x, v) = p_R(v(1 - X)/(1 - X_0)), \quad X < x < 1$$

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[1] We note that the stability theorem in [MP] is stated in the L_1 norm, but is, in fact, proven in the (3.2) norm. It therefore needs additional conditions on the space of densities that make these two norms equivalent.

The minimal total energy is then

$$2E_{\min}(t) = \int_0^\infty \int_0^X v^2 \pi(x,v)\, dx\, dv + \int_0^\infty \int_X^1 v^2 \pi(x,v)\, dx\, dv$$
$$= \frac{X_0^3}{X^2} \int_0^\infty v^2 p_L(v)\, dv + \frac{(1-X_0)^3}{(1-X)^2} \int_0^\infty v^2 p_R(v)\, dv$$

(we used a change of variable $u = vX/X_0$ in the first integral and $u = v(1-X)/(1-X_0)$ in the second one). Using the pressure balance (2.10) and denoting $P = P_L = P_R$ gives

$$E_{\min}(t) = \frac{P}{2}\left(\frac{X_0^3}{X^2} + \frac{(1-X_0)^3}{(1-X)^2}\right)$$

Consider now the above expression as a function of X. Its minimum is attained at the point where $dE_{\min}/dX = 0$, i.e.

$$\frac{X_0^3}{X^3} = \frac{(1-X_0)^3}{(1-X)^3}$$

which is only possible if $X = X_0$. Therefore, the state $X = X_0$ provides a unique minimum of the total energy function under the incompressibility constraint (D3). Any deviation of X from X_0 would result in the increase of the total energy. Since only a small increase of the total energy is allowed by a δ-perturbation of the initial density then only small deviations of X from X_0 are permitted. This proves claim (ii) and also implies that E_L and E_R remain at all times very close to the values corresponding to the unperturbed density. We can then apply the stability theorem by Marchioro and Pulvirenti [MP] and obtain (i).

This proves the theorem in the case where $p(x,v,0)$ has exactly the same area of the level sets, in each compartment, as $p(x,v)$. For other perturbed initial densities $p(x,v,0)$, the above estimates only hold approximately, and our results then follow by standard approximation techniques. $\square$

Suppose, for example, that the initial density $p(x,v,0)$ satisfies (S1) and (3.1), but (S2) only holds approximately: $|P_L - P_R| = \Delta$ with a small $\Delta > 0$. This is a particular case of our theorem, hence the piston will remain ε-close to its initial position X_0 at all times. Interestingly, we can estimate, in terms of Δ, how far the piston can swing from its initial position. Indeed, the piston can move as long as

$$\frac{X_0^3}{X(t)^2} \int_0^\infty v^2 p(X(t)-0,v,t)\, dv + \frac{(1-X_0)^3}{(1-X(t))^2} \int_0^\infty v^2 p(X(t)+0,v,t)\, dv \le 2E(0)$$

where $X_0 = X(0)$, as before, and $E(0)$ is the initial total energy:

$$2E(0) = X_0 \int_0^\infty v^2 p_L(v)\, dv + (1-X_0)\int_0^\infty v^2 p_R(v)\, dv$$

By simple calculations one obtains, to the leading order of Δ, the following bound on the piston displacements:

$$|X(t) - X_0| \le \frac{2\Delta}{3P_L}\left(\frac{1}{X_0} + \frac{1}{1-X_0}\right)^{-1} + O(\Delta^2)$$

4 Perturbative analysis

Here we analyze linear stability of the solutions of the HE corresponding to initial densities $p(x, v, 0)$ satisfying the following stricter version of (S1)–(S2):

(S3) *Full uniformity and symmetry.* The initial density $p(x, v, 0)$ is uniform in x across the entire cylinder, i.e. $p(x, v, 0) = p_0(|v|)$ for all v and $0 < x < 1$.

We also assume that the piston is initially at the midpoint $X(0) = 0.5$. Of course, under the conditions (S3), the hydrodynamic equations (H1)–(H4) have a simple stationary solution (2.11). On the other hand, we no longer assume monotonicity (3.1).

We use perturbative analysis to investigate the stability of the stationary solution (2.11). Our arguments are essentially heuristic: we perturb the initial density (by ε) in the L^1 norm (or, equivalently, in the (3.2) norm) and then expand all the equations in ε up to first order and ignore terms of higher order.

¿From now on we denote by $p_0(v) = p_0(|v|)$ an initial density satisfying (S3) and by $p(x, v, 0)$ a perturbed initial density, which we write as

$$p(x, v, 0) = p_0(|v|) + \varepsilon p_1(x, v, 0)$$

where ε is small and $\|p_1(x, v, 0)\| = 1$. For $t > 0$, we decompose the density $p(x, v, t)$ as

$$p(x, v, t) = p_0(|v|) + \varepsilon p_1(x, v, t)$$

We also set $p_1(x, v, t) = p_L(x, v, t)$ for $x < X(t)$ and $p_1(x, v, t) = p_R(x, v, t)$ for $x > X(t)$.

According to (2.8), the velocity $V(t)$ of the piston is given by

$$\int_V^\infty (v - V)^2 \left[p_0(v) + \varepsilon p_L(X, v, t) \right] dv = \int_{-\infty}^V (v - V)^2 \left[p_0(v) + \varepsilon p_R(X, v, t) \right] dv$$

where $X = X(t)$ is the position of the piston. Expanding in ε and ignoring terms of order $o(\varepsilon)$ gives

$$V(t) = \varepsilon \, \frac{\int_0^\infty v^2 p_L(X, v, t) \, dv - \int_{-\infty}^0 v^2 p_R(X, v, t) \, dv}{4 \int_0^\infty v \, p_0(v) \, dv}$$

Integrating by parts we obtain (for piecewise smooth p_0)

$$2 \int_0^\infty v \, p_0(v) \, dv = - \int_0^\infty v^2 p_0'(v) \, dv$$

We define

$$h(v) = -p_0'(v) \qquad \text{for} \quad v > 0$$

and for the sake of completeness set $h(-v) = h(v)$. Then

$$V(t) = \varepsilon \, \frac{\int_0^\infty v^2 p_L(X, v, t) \, dv - \int_{-\infty}^0 v^2 p_R(X, v, t) \, dv}{2 \int_0^\infty v^2 h(v) \, dv} \tag{4.1}$$

When $p_0(v)$ is not differentiable we interpret $-h(v)$ as the generalized derivative of $p_0(v)$. Denote by $\langle\cdot\rangle_+$ the integration $\int_0^\infty \cdot \, dv$, and by $\langle\cdot\rangle_-$ the integration $\int_{-\infty}^0 \cdot \, dv$. Then

$$V(t) = \varepsilon \, \frac{\langle v^2 p_L(X, v, t)\rangle_+ - \langle v^2 \, p_R(X, v, t)\rangle_-}{2\langle v^2 h(v)\rangle_+}$$

The density of the gas after interaction with the piston is given by the formulas (2.5), which imply:

$$
\begin{aligned}
p(X - 0, -v, t) &= p(X - 0, v + 2V, t) \\
&= p_0(v + 2V) + \varepsilon p_L(X, v, t) \\
&= p_0(v) + 2V p_0'(v) + \varepsilon p_L(X, v, t) \\
&= p_0(v) - 2V h(v) + \varepsilon p_L(X, v, t)
\end{aligned}
$$

Here we assume $v > 0$ and ignore terms of order $o(\varepsilon)$. Hence, the "reflection rule" can be written as

$$p_L(X, -v, t) = p_L(X, v, t) - h(v) \, \frac{\langle v^2 p_L(X, v, t)\rangle_+ - \langle v^2 \, p_R(X, v, t)\rangle_-}{\langle v^2 h(v)\rangle_+}$$

This expression suggests the introduction of new functions:

$$q_{R,L}(x, v, t) = \frac{p_{L,R}(x, v, t)}{h(v)}$$

and

$$\rho(v) = \frac{v^2 h(v)}{\langle v^2 h(v)\rangle_+}$$

The above expression for $p_L(X, -v, t)$ can now be written as

$$q_L(X, -v, t) = q_L(X, v, t) - \langle q_L(X, v, t)\, \rho(v)\rangle_+ + \langle q_R(X, v, t)\, \rho(v)\rangle_- \qquad (4.2)$$

Similarly, on the other side of the piston,

$$q_R(X, v, t) = q_R(X, -v, t) + \langle q_L(X, v, t)\, \rho(v)\rangle_+ - \langle q_R(X, v, t)\, \rho(v)\rangle_- \qquad (4.3)$$

One can interpret these "reflection rules" as follows: the functions q_L and q_R "exchange" their average values with respect to the "density" $\rho(v)$.

Note that $\rho(v)$ is normalized, so that $\langle\rho(v)\rangle_+ = 1$, but it is not necessarily positive (or even nonnegative). On the other hand when $\rho(v) \geq 0$, i.e. the unperturbed density $p_0(|v|)$ is nonincreasing, thus satisfying (3.1). In this case the stationary solution (2.11) is stable, as we already know by Theorem 3.1. Here we recover this result by our perturbative analysis:

10

Theorem 4.1 *The quantity*

$$Q = \int\!\!\int \frac{q^2(x,v,t)\,\rho(v)}{|v|}\,dx\,dv$$

is constant in time, i.e. $dQ/dt = 0$. Here $q = q_L$ for $x < X$ and $q = q_R$ for $x > X$.

Proof. Clearly, Q cannot change just due to the free motion of the gas or due to collisions with the walls, so we only need to worry about collisions with the piston. The gas particles colliding with the piston during an infinitesimal interval $(t, t+dt)$ lie in the two triangles on the xv plane: $X - v\,dt < x < X$ for $v > 0$ and $X < x < X - v\,dt$ for $v < 0$. The outgoing particles lie in similar symmetric triangles. Hence, during the interval $(t, t+dt)$, the quantity Q decreases by (up to the factor of dt)

$$\int_0^\infty |v|\,\frac{q_L^2(X,v,t)\,\rho(v)}{|v|}\,dv + \int_{-\infty}^0 |v|\,\frac{q_R^2(X,v,t)\,\rho(v)}{|v|}\,dv$$

$$= \langle q_L^2(X,v,t)\,\rho(v)\rangle_+ + \langle q_R^2(X,v,t)\,\rho(v)\rangle_-$$

and it increases by

$$\int_{-\infty}^0 |v|\,\frac{q_L^2(X,v,t)\,\rho(v)}{|v|}\,dv + \int_0^\infty |v|\,\frac{q_R^2(X,v,t)\,\rho(v)}{|v|}\,dv$$

$$= \langle q_L^2(X,v,t)\,\rho(v)\rangle_- + \langle q_R^2(X,v,t)\,\rho(v)\rangle_+$$

After substituting (4.2) and (4.3) into the above expressions for q_L and q_R and some manipulations, all changes in Q cancel out and so it stays constant. $\square$

When $p_0(|v|)$ is strictly decreasing, hence $\rho(v) > 0$, then Q is a norm in the space of functions. Thus, the above theorem implies linear stability.

When $p_0(|v|)$ is decreasing, but not strictly, then $\rho(v) \geq 0$, but there may be regions where $\rho(v) = 0$. They correspond to the intervals where $p_0' = 0$, i.e. where p_0 is constant. On such intervals, the reflection rules (4.2)–(4.3) for the perturbed density $p_{L,R}$ are trivial:

$$p_L(X,-v,t) = p_L(X,v,t) \qquad \text{and} \qquad p_R(X,v,t) = p_R(X,-v,t)$$

In this case p_L and p_R cannot grow either. Therefore, we obtain linear stability for all nonincreasing $p_0(|v|)$.

Next we turn to unstable solutions. The stationary solution for an initial density $p_0(|v|)$ satisfying (S3) is linearly unstable if some small perturbations grow exponentially in time, i.e. $\|p_1(x,v,t)\| \sim \Lambda^t$ for some $p_1(x,v,0)$ and $\Lambda > 1$. This is equivalent to having a positive Lyapunov exponent in the subspace spanned by the function p_1 and its images. To investigate the existence of such perturbations we first simplify the collision rules (4.2) and (4.3). Consider the following "symmetric" and "antisymmetric" linear combinations of q_L and q_R:

$$q_+(x,v,t) = [q_L(x,v,t) + q_R(1-x,-v,t)]/2$$

and
$$q_-(x, v, t) = [q_L(x, v, t) - q_R(1 - x, -v, t)]/2$$

They are defined for $x < 1/2$. The collision rules (4.2)–(4.3) now take form

$$q_+(X, -v, t) = q_+(X, v, t) \qquad (4.4)$$

and

$$q_-(X, -v, t) = q_-(X, v, t) - 2\langle q_-(X, v, t)\, \rho(v)\rangle_+ \qquad (4.5)$$

Hence q_+ is simply a periodic function in t, so it cannot grow to infinity or decrease to zero. In other words, it cannot affect the stability or instability of the hydrodynamic equations. The latter is determined by q_- alone. So we will only consider q_- and omit "$-$" for brevity. Our collision rule then reduces to a single equation

$$q(X, -v, t) = q(X, v, t) - 2\langle q(X, v, t)\, \rho(v)\rangle_+ \qquad (4.6)$$

Next we demonstrate, by example, that densities $\rho_0(|v|)$ for which the stationary solution (2.11) is unstable do exist.

Example. Let p_0 be a rectangular function defined by

$$p_0(v) = \begin{cases} 1 & \text{if } 0.5 < |v| < 1 \\ 0 & \text{otherwise} \end{cases} \qquad (4.7)$$

This $p_0(v)$ satisfies (S3) but not (3.1). We will show that the corresponding stationary solution is linearly unstable.

First, the function $h = -p_0'$ is the sum of two delta functions:

$$h(v) = -\delta_{0.5} + \delta_1$$

(and symmetrically for $v < 0$). It is easy to compute ρ directly

$$\rho = -\frac{1}{3}\delta_{0.5} + \frac{4}{3}\delta_1$$

Now the reflection rule (4.6) implies

$$q(-1) = -\frac{5}{3}q(1) + \frac{2}{3}q(0.5)$$
$$q(-0.5) = -\frac{8}{3}q(1) + \frac{5}{3}q(0.5)$$

Note that only the values $p(x, \pm 0.5, t)$ and $p(x, \pm 1, t)$ will evolve in a nontrivial way, as specified above, since $h(v) = 0$ for all $v \notin \{1, 0.5, -0.5, -1\}$.

We now construct a linear subspace of functions $q = q_-$ that stays invariant under the above transformations and in which functions grow exponentially in time (since the

12

q_+ component of the perturbation is irrelevant, we set it to zero). We can simplify the construction further by assuming that at time $t = 0$

$$q(x, \pm 1, 0) = u_1, \qquad q(x, 0.5, 0) = u_2, \qquad q(x, -0.5, 0) = u_3$$

with some constants u_1, u_2, u_3 (the choice of indices $1, 2, 3$ is rather arbitrary). We note that the functions $p_{L,R}(x, v, 0)$ are now piecewise constant and are completely described by the values u_1, u_2, u_3. The space of such perturbations is three-dimensional.

It is easy to see that at time $t = 1$ the functions $p_{L,R}$ will again be constant on the same intervals, hence they will be described by some other constants u'_1, u'_2, u'_3. Our collision rule (4.6) implies that the vectors $\mathbf{u}' = (u'_1, u'_2, u'_3)^T$ and $\mathbf{u} = (u_1, u_2, u_3)^T$ are related by a linear transformation

$$\mathbf{u}' = \mathbf{A}\mathbf{u}$$

where $\mathbf{A}$ is a 3×3 matrix:

$$\mathbf{A} = \frac{1}{3} \begin{pmatrix} -5 & 2 & 0 \\ 0 & 0 & 3 \\ -8 & 5 & 0 \end{pmatrix}$$

After that, the evolution will proceed periodically – the vector $\mathbf{u}$ will be multiplied by the matrix $\mathbf{A}$ at times $t = 1, 2, 3, \ldots$. The matrix $\mathbf{A}$ has three real eigenvalues:

$$\lambda_{1,2} = \frac{-4 \pm \sqrt{7}}{3} \qquad \text{and} \qquad \lambda_3 = 1$$

The largest eigenvalue $\lambda = -(4 + \sqrt{7})/3 \approx -2.215$ has the following (unit) eigenvector:

$$\mathbf{u} = (0.4472, \ -0.3680, \ 0.8152)$$

This eigenvector spans a one-dimensional subspace in the space of perturbation densities, which is invariant during time intervals of period 1 and in which the corresponding perturbations are expanded by a factor $|\lambda| \approx 2.215$. Roughly, the perturbations double over one period.

To explore the above periodic growth of perturbations, we note that the piston velocity is given by

$$
\begin{aligned}
V(t) &= \frac{\varepsilon}{2} \left(\langle q_L(X, v, t)\rho(v) \rangle_+ - \langle q_R(X, v, t)\rho(v) \rangle_- \right) \\
&= \varepsilon \langle q(X, v, t) \rangle_+
\end{aligned}
$$

Hence in our example, during the time interval $0 < t < 1$

$$V = \frac{\varepsilon}{3} \left(4u_1 - u_3 \right) = 0.9736\,\varepsilon$$

During the next time interval $1 < t < 2$ we have

$$V = \frac{\varepsilon}{3} \left(4u'_1 - u'_3 \right) = -2.156\,\varepsilon$$

and so on. Hence, over a unit period of time, the piston velocity grows by a factor of $|\lambda| = 2.215$ and changes sign – the piston starts its movements back and forth (oscillations) that increase exponentially in time.

We note that the same density (4.7) was studied in [CL] where the trajectory of the piston was computed after an initial configuration of gas molecules was selected randomly from the distribution $p_0(v)$ given in (4.7). It was found [CL] that the piston indeed made oscillations which increased exponentially in time. The piston's velocity grew as const$\cdot R^t$ with $R \approx 1.6$. This estimate is to be compared with our calculation of the largest eigenvalue ≈ 2.215.

Next, we modify the unstable perturbations q found above and make them smooth (rather than piecewise constant) functions of v.

We will be looking for the function q of the form

$$q(x, v, t) = C(v) \, e^{z(t-x/v)}$$

where z is a complex constant. Note that due to (2.2) the function q (with v fixed) can only depend on $t - x/v$. We chose the exponential form in order to investigate the existence of solutions of the linear equation which grow exponentially with time. Also, for convenience, we introduce the new space coordinate y in the following way: for all $v > 0$ and $x < 0.5$ we set $y = x + 0.5$, for $v < 0$ and $x < 0.5$ we set $y = 0.5 - x$, for $v > 0$ and $x > 0.5$ we set $y = x - 0.5$, and for $v < 0$ and $x > 0.5$ we set $y = 1.5 - x$. The coordinate y assumes the value zero when a point $(x, v) \in G$ moving under F^t reflects off the piston, then grows to 0.5 when the point travels to the wall, and grows further from 0.5 to 1 when the point travels from the wall back to the piston.

In the new coordinate y, we will be looking for the function q of the form

$$q(y, v, t) = C(|v|) \, e^{z(t-y/|v|)}$$

More precisely, let

$$q(y, \pm 1, t) = C(1) \, e^{z(t-y)}$$
$$q(y, \pm 0.5, t) = C(0.5) \, e^{z(t-2y)}$$

Recall that $p_0(|v|)$ is the characteristic function of the interval $[0.5, 1]$.

Now, the reflection rule (4.6) implies

$$C(1) = -\frac{5}{3} C(1) \, e^{-z} + \frac{2}{3} C(0.5) \, e^{-2z}$$

$$C(0.5) = -\frac{8}{3} C(1) \, e^{-z} + \frac{5}{3} C(0.5) \, e^{-2z}$$

We need to find z for which the above system of equations has a nontrivial solution. Put $e^z = \lambda$ and introduce an auxiliary variable $D(0.5) = C(0.5) \, e^{-z}$. Now the above system

can be rewritten as

$$\lambda C(1) = -\frac{5}{3} C(1) + \frac{2}{3} D(0.5)$$
$$\lambda D(0.5) = C(0.5)$$
$$\lambda C(0.5) = -\frac{8}{3} C(1) + \frac{5}{3} D(0.5)$$

Hence, λ is an eigenvalue of the matrix of coefficients

$$\frac{1}{3} \begin{pmatrix} -5 & 2 & 0 \\ 0 & 0 & 3 \\ -8 & 5 & 0 \end{pmatrix}$$

which is the same matrix $\mathbf{A}$ that we encountered before. We take its leading eigenvalue $|\lambda| > 1$ and set

$$z = \ln|\lambda| + i\pi$$

The function q now takes form

$$q(y, v, t) = \pm C(|v|) \, |\lambda|^{t - y/|v|} \cos \pi(t - y/|v|)$$

where $C(0.5)$ and $C(1)$ are the coordinates of the leading eigenvector, and we only take the real part, for obvious reasons. Since $|\lambda| > 1$, we have an exponential growth of perturbations and thus linear instability. This gives us smooth unstable perturbations.

We now generalize the above construction to arbitrary nonmonotonic initial densities p_0. Let $p_0(v)$ satisfy (S3) but not (3.1). We will be looking for perturbations of the form

$$q(y, v, t) = C(|v|) \, e^{z(t - y/|v|)} \tag{4.8}$$

with the same convention on y as before. The reflection rule (4.6) leads to (cancelling e^{zt})

$$C(v) = C(v) \, e^{-z/v} - 2 \int_0^\infty C(v) \, e^{-z/v} \rho(v) \, dv$$

for all $v > 0$. Denoting

$$D = -2 \int C(v) \, e^{-z/v} \rho(v) \, dv$$

gives immediately

$$C(v) = \frac{D}{1 - e^{-z/v}}$$

Thus, we not only eliminated t but determined the function $C(v)$ up to a constant factor. The above solution exists if

$$D = -2 \int_0^\infty \frac{D \, e^{-z/v} \rho(v)}{1 - e^{-z/v}} \, dv$$

or, cancelling D,

$$\int_0^\infty \frac{\rho(v)}{1 - e^{z/v}} \, dv = \frac{1}{2} \tag{4.9}$$

If this equation has a solution z with $\mathrm{Re}(z) > 0$, we immediately obtain an unstable perturbation (4.8). Otherwise our construction of unstable perturbations does not work.

Unfortunately, it does not seem to be easy to solve equation (4.9) for particular functions $\rho(v)$ or even to determine if it has solutions with a positive real part, as we will demonstrate in the next section.

Next we mention an important property of (4.9). Let us denote

$$F(z) := \int_0^\infty \frac{\rho(v)}{1 - e^{z/v}} \, dv - \frac{1}{2} \tag{4.10}$$

Lemma 4.2 $F(z) + F(-z) = 0$ for all $z \in \mathbb{C}$.

Proof.

$$
\begin{aligned}
F(z) + F(-z) &= \int_0^\infty \frac{\rho(v)}{1 - e^{z/v}} \, dv - \int_0^\infty \frac{e^{z/v} \rho(v)}{1 - e^{z/v}} \, dv - 1 \\
&= \int_0^\infty \frac{(1 - e^{z/v})\rho(v)}{1 - e^{z/v}} \, dv - 1 \\
&= 0 \qquad\qquad \square
\end{aligned}
$$

As a result, the existence of a solution of (4.9) with $\mathrm{Re}(z) > 0$ is equivalent to that of a solution with $\mathrm{Re}(z) < 0$. The alternative is when all the solutions lie on the imaginary axis $\mathrm{Re}(z) = 0$.

5   A special family of densities

Here we investigate a family of rectangular densities

$$p_0(v) = \begin{cases} 1 & \text{if } r < |v| < 1 \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

where $0 < r < 1$ is the parameter of our family. Note that our example (4.7) is a particular case of (5.1) with $r = 1/2$. It is easy to compute

$$h(v) = -p_0'(v) = \delta_r(v) - \delta_1(v)$$

and

$$\rho(v) = \frac{v^2 h(v)}{\int_0^\infty v^2 h(v) \, dv} = \frac{1}{1 - r^2} \left[\delta_1(v) - r^2 \delta_r(v) \right]$$

Since $h(v) = 0$ for all $v \notin \{1, r, -1, -r\}$, we only consider perturbations $q(x, v, t)$ defined for $v = 1, r, -1, -r$. The reflection rule (4.6) now gives

$$
\begin{aligned}
q(X, -1, t) &= -\alpha\, q(X, 1, t) + \beta\, q(X, r, t) \\
q(X, -r, t) &= -\gamma\, q(X, 1, t) + \alpha\, q(X, r, t)
\end{aligned}
$$

where

$$
\alpha = \frac{1 + r^2}{1 - r^2}, \qquad \beta = \frac{2r^2}{1 - r^2}, \qquad \gamma = \frac{2}{1 - r^2} \tag{5.2}
$$
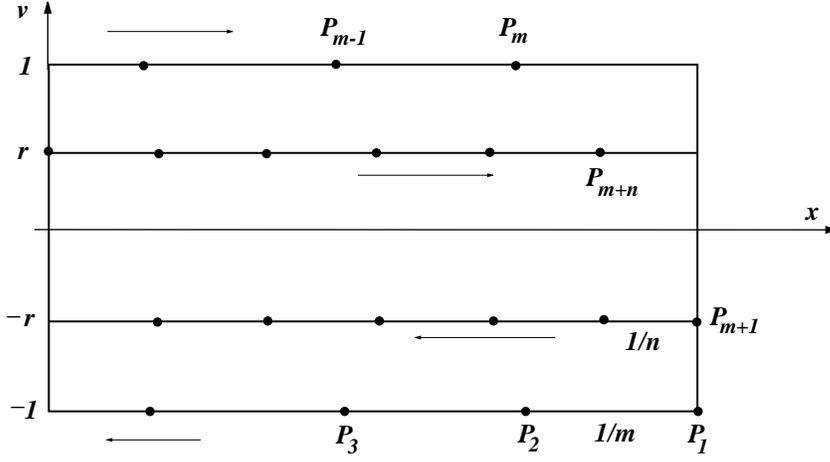
Figure 1: The construction of points P_i. Here $m = 7$ and $n = 12$.

It is relatively easy to investigate the case of rational $r = m/n$ with $1 \le m < n$. Now (5.2) takes form

$$
\alpha = \frac{n^2 + m^2}{n^2 - m^2}, \qquad \beta = \frac{2m^2}{n^2 - m^2}, \qquad \gamma = \frac{2n^2}{n^2 - m^2} \tag{5.3}
$$

To investigate the evolution of perturbations $q(x, v, t)$ as t grows, we consider $n + m$ points $P_i \in G$, $1 \le i \le n + m$, shown on Fig. 1. The points P_i are defined as follows:

$$
P_i = \begin{cases}
(-1, 0.5 - (i - 1)/m) & \text{for } 1 \le i < m/2 + 1 \\
(1, (i - 1)/m - 0.5) & \text{for } m/2 + 1 \le i \le m \\
(-r, 0.5 - (i - m - 1)/n) & \text{for } m < i < m + n/2 + 1 \\
(r, (i - m - 1)/n - 0.5) & \text{for } m + n/2 + 1 \le i \le m + n
\end{cases}
$$

It is crucial to observe that the points P_i move under the dynamics in a periodic fashion. In a time period $\Delta t = 1/m$, the point P_i is mapped to P_{i+1} for all $1 \le i < m$ and all $m + 1 \le i < m + n$. Also, P_m moves to the piston, gets reflected off it and lands on P_1. Likewise, P_{m+n} moves to the piston, gets reflected off it and lands on P_{m+1}. Therefore,

the time shift Δt permutes the points P_i, $1 \le i \le m+n$, in two independent cycles. The reason why we combine the two cycles together is that they are linked by the reflection rule, as we will see shortly.

For each i, denote $q_i(t) = q(P_i, t)$. Then we have

$$q_i(t + \Delta t) = q_{i-1}(t)$$

for all $2 \le i \le m$ and $m+2 \le i \le m+n$. The reflection rule now implies

$$
\begin{aligned}
q_1(t + \Delta t) &= -\alpha\, q_m(t) + \beta\, q_{m+n}(t) \\
q_{m+1}(t + \Delta t) &= -\gamma\, q_m(t) + \alpha\, q_{m+n}(t)
\end{aligned}
$$

Thus, the vector $\mathbf{q}(t) = (q_1(t), \ldots, q_{n+m}(t))$ is updated at time $t + \Delta t$ by the rule

$$\mathbf{q}(t + \Delta t) = \mathbf{B}\, \mathbf{q}(t)$$

where $\mathbf{B}$ is an $(n+m) \times (n+m)$ matrix,

$$
\mathbf{B} =
\begin{pmatrix}
0 & \cdots & & -\alpha & & \cdots & & \beta \\
1 & \ddots & & & & & & \\
\vdots & \ddots & 0 & \vdots & \vdots & & & \vdots \\
& & 1 & 0 & & \cdots & & \\
0 & \cdots & & -\gamma & 0 & \cdots & & \alpha \\
& & & & 1 & 0 & & \\
\vdots & & & \vdots & & \ddots & \ddots & \vdots \\
0 & \cdots & & 0 & & \cdots & 1 & 0
\end{pmatrix}
$$

We conclude that the existence of unstable perturbations $q(t)$ is equivalent to the existence of an eigenvalue λ of $\mathbf{B}$ such that $|\lambda| > 1$. The characteristic polynomial of the matrix $\mathbf{B}$ is

$$P(\lambda) = \lambda^{m+n} + \alpha\lambda^n - \alpha\lambda^m - 1 \tag{5.4}$$

where $\alpha = (n^2 + m^2)/(n^2 - m^2)$ is defined in (5.3).

Remark. Interestingly, the equation (4.9) can be reduced to $P(\lambda) = 0$ as well. Indeed, it is easy to see that

$$\int_0^\infty \frac{\rho(v)}{1 - e^{z/v}}\, dv = \frac{1}{1 - r^2}\left[\frac{1}{1 - e^z} - \frac{r^2}{1 - e^{z/r}}\right]$$

Now the substitution $\lambda = e^{z/m}$ and some algebraic manipulations show that Eq. (4.9) is equivalent to $P(\lambda) = 0$.

It is easy to see that if λ is a root of $P(\lambda)$, then so is $1/\lambda$ (this reciprocability also follows from Lemma 4.2). Thus, the existence of unstable perturbations is equivalent to the existence of eigenvalues of $\mathbf{B}$ that do not lie on the unit circle $|\lambda| = 1$.

If an eigenvalue $|\lambda| > 1$ of $\mathbf{B}$ exists, then the perturbations in the corresponding eigenspace grow by the factor of $|\lambda|$ over the time period $\Delta t = 1/m$. Hence, the expansion factor per unit time would be $\Lambda = |\lambda|^m$.

Theorem 5.1 *Let $r = m/n$ be a rational number with an even denominator n (hence, m is odd). Then there is a unique eigenvalue of $\mathbf{B}$ such that $\lambda < -1$. This eigenvalue has multiplicity one. The expansion factor per unit time $\Lambda_r = |\lambda|^m$ depends on r continuously, and we have, asymptotically,*

$$\Lambda_r = 1 + \text{const} \cdot r^{3/2} + \mathcal{O}(r^2) \qquad \text{as} \quad r \to 0 \tag{5.5}$$

and

$$\Lambda_r \sim \frac{\text{const}}{1 - r} \qquad \text{as} \quad r \to 1 \tag{5.6}$$

Proof. One can easily check that, under the conditions of the theorem, $P(-1) > 0$ and $P(-\infty) < 0$, hence a root $\lambda < -1$ exists. Next,

$$P'(\lambda) = \left[(n + m)\lambda^n + \alpha n \lambda^{n-m} - \alpha m \right] \lambda^{m-1}$$

and so $P'(-1) < 0$ and $P'(-\infty) > 0$. Now let $Q(\lambda) = (n + m)\lambda^n + \alpha n \lambda^{n-m} - \alpha m$, then

$$Q'(\lambda) = \left[(n + m)\lambda^m + \alpha(n - m) \right] n \lambda^{n-m-1}$$

and so clearly $Q'(\lambda) < 0$ for all $\lambda < 1$. Putting these facts together proves the uniqueness and the simplicity of the root $\lambda < 0$.

The equation $P(\lambda) = 0$ can be rewritten in terms of $\Lambda_r = |\lambda|^m$ as follows:

$$-\Lambda_r^{1+1/r} + \frac{1 + r^2}{1 - r^2} \Lambda_r^{1/r} + \frac{1 + r^2}{1 - r^2} \Lambda_r - 1 = 0 \tag{5.7}$$

Now the continuity of Λ_r, as a function of r, is obvious. Note that our argument is only valid when $r = m/n$ with an even n and an odd m, because this parity condition dictates the signs in (5.7).

To prove (5.5), one can substitute $\Lambda_r = 1 + \varepsilon$ into (5.7) and expand all the terms in Taylor series, the calculation is then straightforward and we omit it. The proof of (5.6) is similar. $\square$

Figure 2 presents the graph of the Lyapunov exponent $\log \Lambda_r$ as a function of r.

Lemma 5.2 *Let z be a solution of (4.9) such that $|e^z| \neq 1$ and $e^z \in \mathbb{R}$. Then $dF/dz \neq 0$ (in fact, dF/dz is a real negative number).*
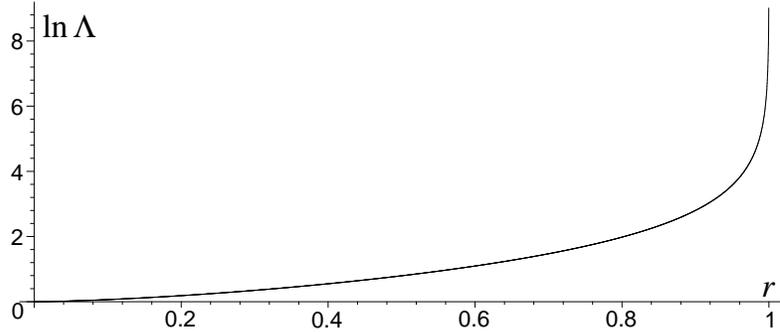
Figure 2: $\log \Lambda_r$ as a function of r.

Proof. A direct calculation shows that

$$\frac{dF}{dz} = -\frac{(1 + r + r^2)(1 + e^z)^2 + r^3(1 - e^z)^2}{4r^2(1 + r)(1 - e^z)^2}$$

which proves the lemma. □

For any $r = m/n$ with an even n and odd m, the corresponding solution $e^z = -\Lambda_r$ satisfies the conditions of the above lemma. Hence, this solution changes continuously with r, and so we get the following

Corollary 5.3 *For every $r = m/n$ with an even n and an odd m there is an interval $(r - \varepsilon, r + \varepsilon)$ in which all parameter values have unstable perturbations.*

Therefore, unstable perturbations exist for an open and dense set of parameter values $0 < r < 1$. Moreover, one can show [C] that Eq. (4.9) has solutions with $\mathrm{Re}(z) \neq 0$, implying the existence of unstable perturbations, for all irrational $r \in (0, 1)$. One would naturally wonder if all r's have unstable perturbations. The answer is, surprisingly, negative:

Fact 5.4 *For the density (5.1) with $r = 1/3$, there are no solutions z of (4.9) with $\mathrm{Re}(z) > 0$, hence there are no solutions of the linearized equation which grow exponentially with time.*

Proof. The characteristic equation

$$\lambda^4 + \frac{5}{4}\lambda^3 - \frac{5}{4}\lambda - 1 = 0$$

has two real roots ($\lambda = \pm 1$) and two complex roots. The complex roots are, on the one hand, conjugate and, on the other, satisfy the reciprocability rule: $P(\lambda) = 0$ if and only if $P(1/\lambda) = 0$. Hence, they must belong to the unit circle $|\lambda| = 1$. □

It is interesting to know if other rational parameter values $r = m/n$ with odd n are also stable. We have examined the values $r = 1/n$ for small odd values of $n = 5, 7, \ldots, 31$

numerically (by using MATLAB) and always found that all the roots of $P(\lambda)$ belonged to the unit circle. Therefore, we conjecture that the values $r = 1/n$ with odd n are stable.

On the other hand, the values $r = m/n$ with an odd n but $m > 1$ appear to be unstable. For $r = 2/3, 2/5, 3/5, 3/7$ we found, again numerically (by using MATLAB), roots λ such that $|\lambda| > 1$. All those roots are complex, for example for $r = 2/3$ they are $\lambda = -0.3778 \pm 1.7173\,i$. It remains to determine theoretically whether all rational values $r = m/n$ with $m > 1$ are unstable, we leave this question open.

Fact 5.4 seems to disagree with Theorem 5.1. Indeed, let $p_0(v)$ be the rectangular density (5.1) corresponding to $r = 1/3$ and $p(x, v, 0) = p_0(v) + \varepsilon p_1(x, v, 0)$ an arbitrary perturbation with an infinitesimally small ε. According to Fact 5.4, this perturbation cannot grow exponentially in time. On the other hand, let us approximate $1/3$ by a rational number $r = m/n$ with an even n. Denote by $p_0^*(v)$ the corresponding rectangular density (5.1) for the chosen $r = m/n$. Then we have

$$p(x, v, 0) = p_0^*(v) + \varepsilon p_2(x, v, 0) \quad \text{with} \quad p_2 = p_1 + (p_0 - p_0^*)/\varepsilon \qquad (5.8)$$

Hence, if $|r - 1/3| < \varepsilon$, then $(p_0 - p_0^*)/\varepsilon$ is of order one (in the L^1 metric), and $p(x, v, 0)$ becomes an ε-perturbation of the density $p_0^*(v)$. As such, it "must" grow exponentially in time according to Theorem 5.1. This apparent disagreement requires an explanation, which we provide next.

We recall that smooth unstable perturbations are given by the general formula (4.8). For the rectangular density (5.1), the velocity v in this formula only takes two values, $|v| = r$ and $|v| = 1$, hence the factor $C(|v|)$ takes two values, as well, and so plays little role. For simplicity, we set $|v| = 1$ and ignore the constant factor $C(|v|) = C(1)$. Now the (real part of) unstable perturbations is described by

$$q(y, 1, t) = \operatorname{Re} e^{z(t-y)} = e^{(\operatorname{Re} z)(t-y)} \cos[(\operatorname{Im} z)(t-y)] \qquad (5.9)$$

A similar formula holds for $|v| = r$, and we omit it. Now recall that for any rational $r = m/n$ we have $e^{z/m} = \lambda$, where $\lambda < -1$ is the eigenvalue of $\mathbf{B}$ described by Theorem 5.1. Therefore, $\operatorname{Re} z = m \log|\lambda| = \log \Lambda$ and $\operatorname{Im} z = \pm m\pi$.

We see that the real part of z changes continuously with $r = m/n$ but the imaginary part does not. In particular, when $r = m/n$ is close to $1/3$ and n is even, both m and n have to be large, so that $m \to \infty$ and $|\operatorname{Im} z| \to \infty$ as $r \to 1/3$. In terms of the perturbation (5.9), the growth of $|\operatorname{Im} z|$, as r approaches $1/3$, implies that the function $q(y, 1, t)$ becomes highly oscillatory, and so does the corresponding initial unstable perturbation $p(x, v, 0) = h(v)q(x, v, 0)$. Thus, the linear subspace of unstable perturbations (along which exponential growth takes place) becomes nearly orthogonal to any given function, in particular to $p_2(x, v, 0)$ defined in (5.8).

This explains the above "disagreement". The density p_2 does grow exponentially in time for any $r = m/n$ with an even n, but, as $r \to 1/3$, the projection of p_2 onto the unstable subspace corresponding to the positive Lyapunov exponent $\log \Lambda_r > 0$ becomes small and vanishes in the limit, hence the exponential growth is not visible during a long

initial interval of time. In the limit $r \to 1/3$ that "initial interval" becomes infinite and the instability evaporates.

One can also reverse this line of argument. Indeed, when ε is not infinitesimally small but finite, the representation (5.8) implies that any perturbation $p(x, v, 0)$ of the rectangular density (5.1) for *any* $0 < r < 1$ will eventually grow exponentially fast in time (because any $r \in (0, 1)$ can be approximated by rational numbers m/n with even n). We checked this conclusion experimentally and found that it was indeed correct.
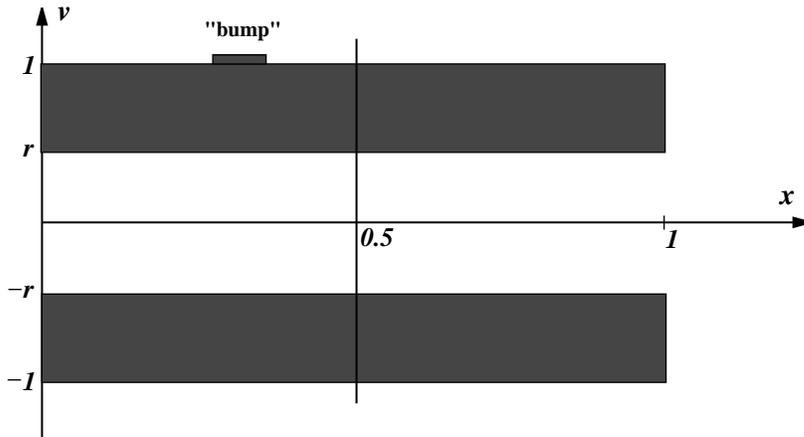
Figure 3: Initial rectangular density (5.1) perturbed by a "bump".

To investigate the instability experimentally, we solved the hydrodynamic equations (H1)–(H4) numerically starting with a perturbed rectangular density (5.1) shown on Fig. 3. The initial density $p(x, v, 0)$ takes the value one on the black region and zero elsewhere. The small 'bump" on the top left edge of the upper rectangle represents the perturbation. The area of the bump in our experiments was less than 1% relative to the

total area of each black rectangle.
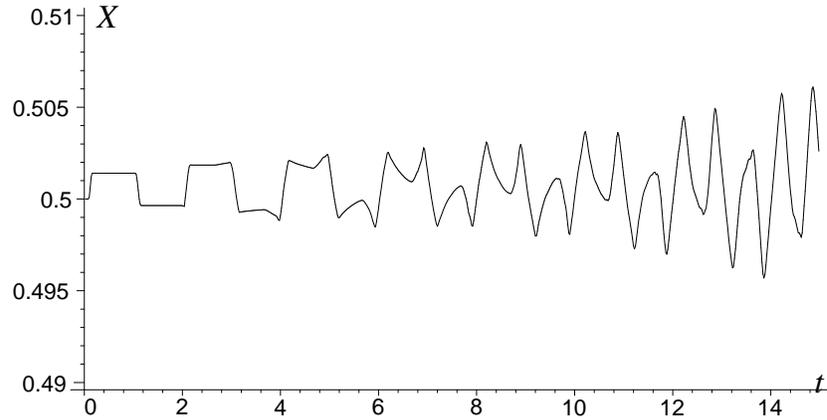
Figure 4: Piston's trajectory from the solution of the HE for a perturbed rectangular density with $r = 1/3$.
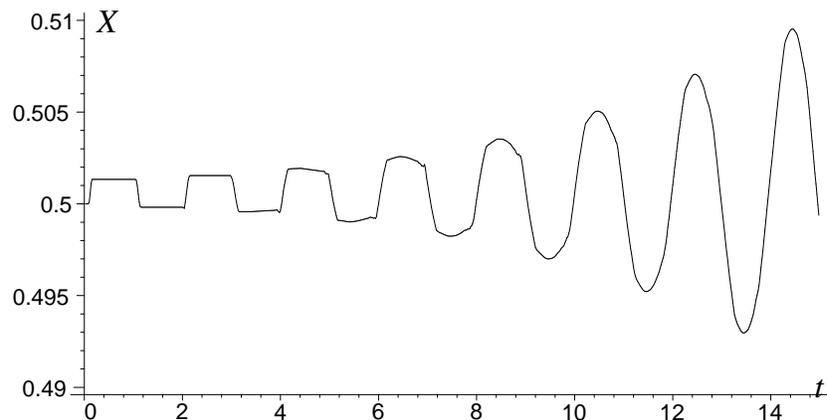

Figure 5: Piston's trajectory from the solution of the HE for a perturbed rectangular density with $r = 1/4$.

Figures 4 and 5 show typical trajectories of the piston $X(t)$ for $r = 1/3$ and $r = 1/4$, respectively. Note that their shape during the first 4-5 units of time only reflects the size and position of the "bump", hence the shape is almost identical. For times $\tau > 5$, though, interesting differences develop. In the $r = 1/4$ unstable case, the oscillations grow exponentially in a pronounced steady fashion. In the $r = 1/3$ linearly stable case, the oscillations distort their shape first, then start growing slowly, and only pick up pace at $\tau \sim 12$. Eventually, they also grow exponentially. These observations indicate that the linearly stable stationary solution with $r = 1/3$ is nonlinearly (globally) unstable.

6 Periodic solutions of the hydrodynamic equations

Here we discuss the long-term behavior of our system in the unstable regime.

In our previous work [CL] we reported the results of computer simulations of the piston and particle dynamics in an ideal gas with many (up to 27 million) particles. The initial configuration of particles was selected randomly with the average density (4.7), see [CL] for details. A typical trajectory of the piston $X(t)$ found in our experiments is shown here on Fig. 6. One can see that during the initial interval of time $0 < t < 8$ the piston moves back and forth with an exponentially increasing amplitude, which is consistent with our analysis in Section 4, where the density (4.7) was proven to be unstable.
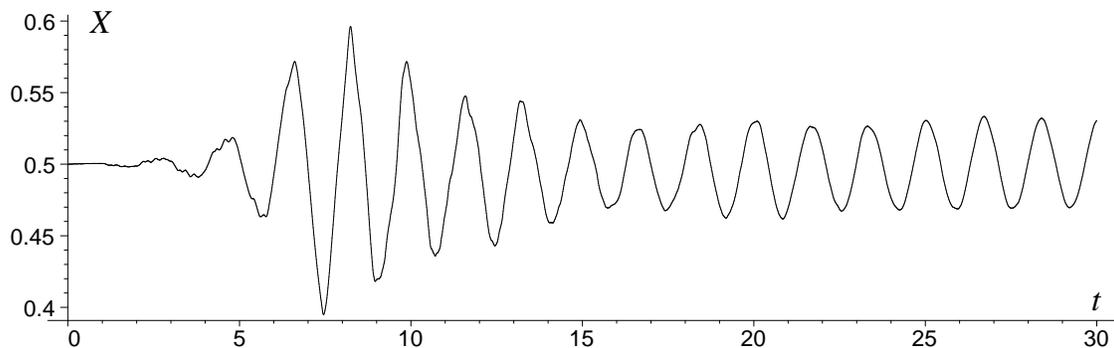
Figure 6: Piston's trajectory in the mechanical model with 10^6 particles.

Later on, however, at times $8 < t < 15$, the amplitude of the piston's oscillations decreases to a certain constant value (nearly a half of its maximum attained at $t = 8$). Then the piston's oscillations become very stable and continue almost unchanged for a very long time, up to $t = 50$ or 100, with a very slowly decreasing amplitude.

On the other hand, we have solved the hydrodynamic equations (H1)–(H4) numerically, starting with the same initial density (4.7) perturbed by a bump shown on Fig. 3. Figure 7 presents the resulting trajectory of the piston. One can see that it behaves almost identically to the simulated trajectory of the piston shown on Fig. 6. Thus, not only the initial instability, but also the longer term behavior of the simulated piston trajectory approximately match those of perturbed solutions of the hydrodynamic equations.

24
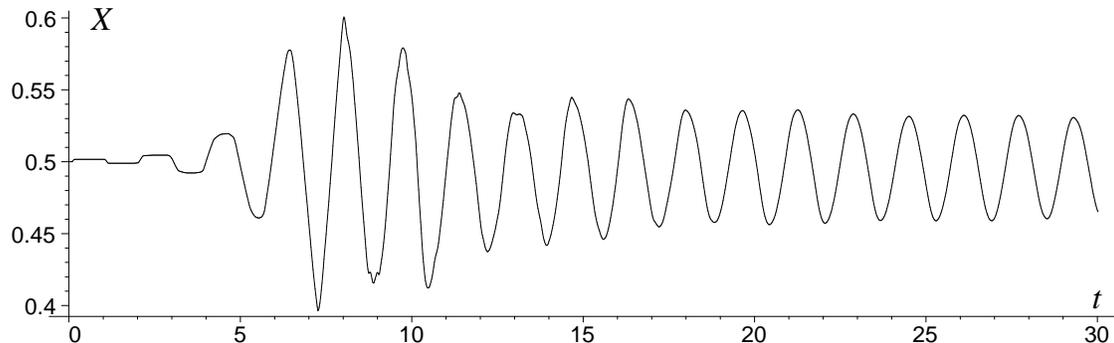
Figure 7: Piston's trajectory from the HE for a perturbed rectangular density with $r = 1/2$.

The behavior shown on Fig. 7 persists when various perturbations of the initial density (4.7) are applied. It seems like there is a periodic cycle or an invariant manifold of quasi-periodic solutions of (H1)–(H4) that acts as an attractor. Of course, due to the time-reversibility of the hydrodynamic equations there can be no attractors in the strict sense. It is more likely that there is an invariant manifold of periodic or quasi-periodic solutions that acts as a saddle point in the phase space: typical trajectories approach that manifold temporarily and then slowly move away. We cannot rigorously prove the existence of periodic or quasi-periodic solutions, but we construct such solutions by using perturbative analysis.

We will be looking for solution of the hydrodynamic equations (H1)–(H4) such that the piston makes harmonic oscillations

$$X(t) = \frac{1}{2} + \varepsilon \cos \omega t, \qquad \dot{X}(t) = -\varepsilon\omega \sin \omega t, \tag{6.10}$$

with some fixed $\omega > 0$ and small $\varepsilon > 0$. We will approximate such solutions up to the first order in ε, i.e. ignoring terms of higher order.

The construction is done in two steps. First, we assume that the piston moves as prescribed by (6.10) and consider the motion of a fluid point bouncing against the moving piston $X(t)$ and the fixed wall $x = 0$. Second, we define the density $p(x, v, t)$ which, coupled with the piston's oscillations (6.10), satisfies equations (H1)–(H4).

Let the piston move according to Eqs. (6.10). Then fluid points in the left compartment $0 < x < X(t)$ bounce against the wall $x = 0$ and the piston, the latter simply acts on them as a moving wall. It is known that the phase space of gas particles bouncing against a periodically moving wall necessarily contains many invariant curves. Moreover, the region corresponding to high velocities $|v| > v_0$ is densely filled by such invariant curves, the larger v_0 the higher the density of invariant curves. This fact is a consequence of KAM theory, it was first proved by R. Douady in his thesis [Do] and later independently by S. Laederich and M. Levi [LL]. We describe these invariant curves approximately, up to the first order in ε, by equation

$$v + \varepsilon F(t, V) = V + \mathcal{O}(\varepsilon^2) \tag{6.11}$$

where v denotes the velocity of the particle when it kicks the piston, t the the collision time, and V is the parameter of the curve. In fact, we will construct invariant curves for all $V > V_0$ with some $V_0 > 0$. Here we only consider particles to left to the piston, the particles to the right of the piston are completely symmetric.

Let us consider successive collisions of a gas particle with the piston. Denote by $v_n > 0$ the velocity of the particle before its n-th collision and by t_n the time of that collision. Then the law of elastic impact reads

$$v_{n+1} = v_n - 2\dot{X}(t_n) = v_n + 2\varepsilon\omega \sin \omega t_n \tag{6.12}$$

Let $t_{n+1/2}$ denote the time at which the particle bounces off the wall $x = 0$ between its n-th and $(n+1)$-st collisions with the piston. Obviously, $t_{n+1/2} = t_n + X(t_n)/v_{n+1}$ and $v_{n+1}(t_{n+1} - t_{n+1/2}) = X(t_{n+1})$. Since we are interested in knowing v_n up to terms $\mathcal{O}(\varepsilon)$, it is sufficient to find t_n up to terms $\mathcal{O}(1)$. This is easy:

$$t_{n+1} = t_n + \frac{1}{v_{n+1}} + \mathcal{O}(\varepsilon) = t_n + \frac{1}{V} + \mathcal{O}(\varepsilon) \qquad (6.13)$$

where we used (6.11).

Now let us look for an invariant curve of the form

$$v + \varepsilon F(t, V) = V + \mathcal{O}(\varepsilon^2)$$

We have to impose the constraint

$$v_{n+1} + \varepsilon F(t_{n+1}, V) = v_n + \varepsilon F(t_n, V) + \mathcal{O}(\varepsilon^2) \qquad (6.14)$$

By equations (6.12), (6.13) and (6.14) we get

$$v_n + 2\varepsilon\omega \sin \omega t_n + \varepsilon F(t_n + 1/V + \mathcal{O}(\varepsilon), V) = v_n + \varepsilon F(t_n, V)$$

Cancelling v_n and ε and removing the index n gives a general equation for an invariant curve:

$$2\omega \sin \omega t + F(t + 1/V, V) - F(t, V) = 0 \qquad (6.15)$$

We construct solutions of this equation in the form

$$F(t, V) = a \cos \omega t + b \sin \omega t \qquad (6.16)$$

where a and b depend on V. By substituting this expression into (6.15) we find that (6.15) can only hold if

$$\begin{aligned} a(\cos(\omega/V) - 1) + b \sin(\omega/V) &= 0 \\ a \sin(\omega/V) - b(\cos(\omega/V) - 1) &= 2\omega \end{aligned}$$

The solution of the above system is

$$\begin{aligned} a &= \frac{\omega \sin(\omega/V)}{1 - \cos(\omega/V)} \\ b &= \omega \end{aligned}$$

Remark. Notice that a (and hence the invariant curve) is not defined for $V = \frac{\omega}{2\pi k}$, $k = \pm 1, \pm 2, \ldots$ To avoid these singularities, we will not use invariant curves corresponding to $V \le \frac{\omega}{2\pi}$. In particular, the density $p(x, v, t)$ that we define below will be constant for $|v| \le \frac{\omega}{2\pi}$.

Thus, for any $V > V_0 > \frac{\omega}{2\pi}$ we can define an invariant curve $u(x, t; V)$ in the phase space of gas particles, where V is the parameter of the curve and $u(x, t; V)$ is the velocity of the particle on the curve at point x at time t. The curve is made by two branches: the upper branch u^+ and the lower branch u^-. Obviously, we have

$$
\begin{aligned}
u^+(X(t), t; V) &= V - \varepsilon F(t, V) + \mathcal{O}(\varepsilon^2) \\
u^-(X(t), t; V) &= -[V - \varepsilon F(t, V) - 2\dot{X}(t)] + \mathcal{O}(\varepsilon^2) \\
u^+(t, 0, V) &= -u^-(t, 0, V)
\end{aligned}
$$

Note that the last equation here is equivalent to (6.14).

Now we define a density $p(x, v, t)$ so that its value on each invariant curve $u(x, t; V)$, $|V| > V_0$, is a constant denoted by $\rho(V)$. Between the curves $u^+(x, t; V_0)$ and $u^-(x, t; V_0)$ we set the density to a constant equal to one. Therefore

$$
p(x, u^+(t, x, V), t) = p(x, u^-(t, x, V), t) = \rho(V) \quad \text{if} \quad V > V_0
$$

and

$$
p(x, v, t) \equiv 1 \quad \text{if} \quad u^-(x, t; V_0) < v < u^+(x, t; V_0)
$$

The function $\rho(V)$ and the "cutoff" value $V_0 > \frac{\omega}{2\pi}$ will be specified below.

Example. Let us set $\rho(V) \equiv 0$ for $V > V_0$, i.e.

$$
p(x, v, t) = \begin{cases} 1 & \text{for} \quad u^-(x, t; V_0) < u < u^+(x, t; V_0) \\ 0 & \text{elsewhere} \end{cases}
$$

In order to compute the pressure on the piston we only need to know the density $p(x, v, t)$ at the point $x = X(t)$, i.e. we need to know the function

$$
v(t, V) := u^+(X(t), t, V) = V + \varepsilon F(t, V) + \mathcal{O}(\varepsilon^2)
$$

In our example the density on the piston (on the left hand side) is 1 up to $v^+ = V_0 - \varepsilon F(V_0, t)$. The density on the piston on the right hand side is 1 up to a similar invariant curve, which is phase shifted by $\Delta t = \pi/\omega$. Therefore the density on the right hand side is 1 down to $v^- = -V_0 - \varepsilon F(t, V_0)$. Since $F(t + \pi/\omega) = -F(t)$ by (6.16), the velocity of the piston is exactly the average of v^+ and v^- and therefore is $-\varepsilon F(t, V_0)$.

Thus our density and the piston satisfy the hydrodynamic equations (H1)–(H4) if $\dot{X} = -\varepsilon F(t, V_0)$, which gives

$$
-\varepsilon\omega \sin\omega t = -\varepsilon\omega \sin\omega t - \frac{\varepsilon\omega \sin(\omega/V_0)\cos\omega t}{1 - \cos(\omega/V_0)}
$$

In our example, the only possible choice is $V_0 = \omega/\pi$.

Now let us consider the case of a generic function $\rho(V)$. The pressure on the piston on the left hand side is equal to

$$
P_L = \int_{\dot{X}}^\infty p_L(v)\,(v - \dot{X})^2\,dv = \int_0^\infty p_L(v)\,(v^2 - 2v\dot{X})\,dv + \mathcal{O}(\varepsilon^2)
$$

27

where $p_L(v) = p(X(t) - 0, v, t)$ is the density on the piston (we have used the fact that $\dot{X} = \mathcal{O}(\varepsilon)$). Recall that the density is $p_L(v) = \rho(V) = \rho(v + \varepsilon F(t, v)) + \mathcal{O}(\varepsilon^2)$. ¿From now on we neglect terms of order $\mathcal{O}(\varepsilon^2)$. Then we get

$$P_L = \int_0^\infty (v^2 - 2v\dot{X})\, \rho(v + \varepsilon F(t, v))\, dv$$

The pressure on the right hand side is given, by analogy,

$$P_R = \int_{-\infty}^0 p_R(v)\, (v^2 - 2v\dot{X})\, dv = \int_0^\infty p_R(-v)\, (v^2 + 2v\dot{X})\, dv$$

Note that for $v > 0$ we have $p_R(v) = \rho(V) = \rho(v + \varepsilon F(v, t + \pi)) = \rho(v - \varepsilon F(t, v))$. Therefore

$$P_R = \int_0^\infty (v^2 + 2v\dot{X})\, \rho(v - \varepsilon F(t, v))\, dv$$

We now conclude that $P_L = P_R$ iff

$$\dot{X} = \varepsilon\, \frac{\int_0^\infty \rho'(v) F(t, v) v^2\, dv}{\int_0^\infty \rho(v)\, 2v\, dv} = -\varepsilon\, \frac{\int_0^\infty \rho'(v) F(t, v) v^2\, dv}{\int_0^\infty \rho'(v) v^2\, dv}$$

which is analogous to our early formula (4.1).

Using (6.16) and the subsequent equations we find

$$\dot{X} = -\varepsilon\omega \sin\omega t - \varepsilon\, \frac{\int_0^\infty \rho'(v)\, \frac{\sin(\omega/v)}{1 - \cos(\omega/v)}\, v^2\, dv}{\int_0^\infty \rho'(v)\, v^2\, dv}\, \omega\cos\omega t$$

Our density, coupled with the piston oscillations (6.10), satisfies the hydrodynamic equations (H1)–(H4) if and only if $\dot{X} = -\varepsilon\omega \sin\omega t$. This implies

$$\int_{\omega/2\pi}^\infty dv\, \rho'(v)\, v^2\, \frac{\sin(\omega/v)}{1 - \cos(\omega/v)} = 0 \tag{6.17}$$

where we have imposed $\rho' = 0$ for $v \leq \omega/2\pi$.

Interestingly, (6.17) is related to our early equation (4.9). Precisely, let z in (4.9) be a purely imaginary number, $z = \omega i$. Also note that $\rho(v)$ in (4.9) is just proportional to $v^2 \rho'(v)$ here. Then (6.17) becomes equivalent to $\operatorname{Im} F(z) = 0$, with $F(z)$ defined by (4.10). In other words, (6.17) expresses the "imaginary part" of the equation (4.9). We already observed in the previous section that $\operatorname{Im} z$ characterized the frequency of oscillations of unstable perturbations, and here $\omega = \operatorname{Im} z$ is the frequency of oscillations of the piston. We note that for $z = \omega i$ one always has $\operatorname{Re} F(z) = 0$, as it follows from (4.10), hence in our case (6.17) is equivalent to $F(z) = 0$. Therefore solving (6.17) correspond to finding pure imaginary eigenvalues of the HE linearized around a stationary solution ρ.

This analysis is only relevant when $\rho(v)$ is stable, otherwise the motion will be dominated by other exponentially increasing modes, so from now on we restrict ourselves to the case in which ρ is a non-increasing function of $|v|$.

In this case there are some quantitative restrictions on the period of oscillations of the piston. The period $T = 2\pi/\omega$ can be bounded from below by a function of the average kinetic energy $\langle \mathcal{K} \rangle = \mathcal{K}/\mathcal{M}$, where

$$\mathcal{K} = \int_0^\infty \rho(v)\frac{v^2}{2}\,dv \qquad \text{and} \qquad \mathcal{M} = \int_0^\infty \rho(v)\,dv$$

Proposition 6.1 *Let $\rho' \leq 0$ be supported on the interval $[\omega/2\pi, \infty)$ and satisfy Eq. (6.17). Then the period of oscillations T is bounded by*

$$T \geq \sqrt{\frac{2}{3\langle \mathcal{K} \rangle}}$$

The equality holds when $\rho' = (\pi/\omega)\delta(v - \omega/\pi)$, i.e. when $\rho(|v|)$ is constant on $(0, \omega/\pi)$ and 0 elsewhere.

Notice that Eq. (6.17) can be satisfied in many different ways all of them leading to different periodic solutions of the hydrodynamic equations, but it is not clear which of them, if any, describe the long term behavior of the system for a given initial condition.

Nevertheless it is possible to verify Proposition 6.1 numerically. For instance in the case of our unstable density (4.7) we have $\mathcal{K} = 7/24$ and $T = \sqrt{16/7} \simeq 1.51186$. The experimentally determined period of oscillations of the piston is $T \simeq 1.62$, see [CL]. We also simulated the piston trajectory with other unstable densities (5.1) with $r \to 0$ and observed that the period of oscillations approached 2, which is exactly the value of the lower bound given by the above proposition (because in this case $\mathcal{K} = 1/6$, hence $T = 2$). The fact that the period converges to 2, as $r \to 0$, also follows from our early example (with $V_0 = 1$), in which $\omega = \pi V_0 = \pi$, hence $T = 2\pi/\omega = 2$.

Proof of Proposition 6.1. Consider a function

$$G(v) := v\,\frac{\sin(\omega/v)}{1 - \cos(\omega/v)}$$

Then equation (6.17) reads

$$\mathcal{C} := \int_{\omega/2\pi}^\infty -\rho' v G\,dv = 0$$

Note that in the interval $[\omega/2\pi, \infty)$ the function $G(v)$ is strictly increasing and that $G(\omega/2\pi) = -\infty$, $G(\omega/\pi) = 0$, and $G(\infty) = \infty$.

Introducing a new function $R(v) = -\rho'(v)v \geq 0$ and integrating by parts yields

$$\mathcal{M} = \int_{\omega/2\pi}^\infty R(v)\,dv, \quad \mathcal{K} = \int_{\omega/2\pi}^\infty R(v)\frac{v^2}{6}\,dv, \quad \mathcal{C} = \int_{\omega/2\pi}^\infty R(v)G(v)\,dv$$

It is useful to replace v by a new variable $u = G(v)$, $-\infty < u < \infty$. Since G is strictly increasing, we can write

$$\mathcal{M} = \int_{-\infty}^\infty S(u)\,du, \quad \mathcal{K} = \int_{-\infty}^\infty S(u)\eta(u)\,du, \quad \mathcal{C} = \int_\infty^\infty S(u)u\,du$$

where $S(u) = R(G^{-1}(u))/G'(G^{-1}(u))$ and $\eta(u) = (G^{-1}(u))^2/6$ (here G^{-1} denotes the inverse of the function G).

We have to solve the following variational problem: minimize $\mathcal{K}/\mathcal{M}$ under the constraint $\mathcal{C} = 0$. As we shall see in the sequel the function η turns out to be convex. This easily implies Proposition 6.1.

Also, the convexity of η implies that the solution of the variational problem $\bar{S}$ is a delta-function centered at $u = 0$, i.e. at $v = \omega/\pi$ (so that $\mathcal{C}$ vanished).

So it only remains to prove that η is a convex function of u. By direct computation we get

$$6\eta' = \frac{d}{du}(G^{-1}(u))^2 = 2G^{-1}(u)\frac{dG^{-1}(u)}{du} = 2v/G'(v)$$

Hence it is sufficient to prove that the function $G'(v)/v$ is strictly decreasing in the interval $v > \omega/2\pi$. Without loss of generality we set $\omega = 1$, then

$$G(v) = \frac{v\sin(1/v)}{1 - \cos(1/v)}$$

Consider a new function

$$H(v) := \frac{G'(v)}{v} = \frac{1 + v\sin(1/v)}{v^2(1 - \cos(1/v))}$$

then

$$H(v)' = \frac{-v - (-1 + v^2)\sin(1/v) + v\cos(1/v)(1 + v\sin(1/v))}{v^4(1 - \cos(1/v))^2}$$

The denominator of H' being positive, we only need to prove that the numerator of H' is negative in the interval $v > 1/2\pi$.

If we replace v by $1/x$ and multiply by the numerator by x^2 we find the expression

$$\begin{aligned} h(x) &= -x + (x^2 - 1)\sin(x) + \cos(x)(x + \sin x) \\ &= (\cos x - 1)(\sin x + x) + x^2\sin x \end{aligned}$$

We need to show that $h(x) < 0$ in the interval $x \in (0, 2\pi)$. First of all, $h(0) = h(2\pi) = 0$ and for any $x \in (\pi, 2\pi)$ the expression is clearly negative.

It only remains to prove that $h(x)$ is negative in $(0, \pi]$. By computing the Taylor expansion of h about $h = 0$ one finds

$$h(x) = \sum_{k=3}^{+\infty}(-1)^k \frac{2^{2k} - 4k^2}{(2k + 1)!}x^{2k+1}$$

It is easy to prove that for any $x \in (0, \pi]$ this is an alternating series, the absolute values of its terms being strictly decreasing.

The first few terms of the above expansion are

$$h(x) = -\frac{x^7}{180}\left(1 - \frac{2}{21}x^2 + \frac{1}{240}x^4 - \frac{19}{166320}x^6\right) + \mathcal{O}(x^{15})$$

Therefore

$$-\frac{x^7}{180} < h(x) < -\frac{x^7}{180}\left(1 - \frac{2}{21}x^2\right)$$

which implies that $h(x) < 0$ for any $x \in (0, \pi]$. $\square$

7   Conclusions and open problems

The instability of a massive piston in a box filled with an ideal gas that is initially in a mechanical (but not thermal) equilibrium was observed in [CL]. It was found there that sometimes this state rapidly breaks down, and then the piston and the gas tend to oscillate in an amazingly regular fashion. Our aim here was to understand this phenomenon by analyzing the stability of HE which govern the time evolution of the system in an appropriate space-time scaling limit.

We have presented several results of our studies. Section 3 contains our only rigorous (mathematical) theorem, which describes conditions under which the phenomenon in question does **not** occur in the HE. Section 4 presents a semi-heuristic "perturbative" analysis of the HE that gives a plausible (but not rigorous) criterion (4.9) for the instability of the piston dynamics. This criterion amounts to solving the highly nonlinear equation (4.9), which presents a difficult problem by itself. In fact, in Section 5 we try to apply this criterion to a particularly simple family of density functions (5.1), and only obtain partial results. We could not analyze the entire family, but our criterion did yield densities of both sorts: those for which the piston is stable and those for which it is unstable. The densities of both types alternate in a very intricate manner. The conclusions of Section 5, though derived heuristically, are in good agreement with computer simulations on the HE.

Lastly, a related but separate question is discussed in Section 6 – what may be the long-term behavior of oscillations of the piston in the case where the HE are unstable? We conjecture that there are saddle-type periodic solutions in the phase space that temporarily attract typical unstable trajectories, the latter stay close to those periodic solutions for a long time, but eventually spin off to chaotic regions in the phase space. In Section 6 we construct approximate periodic solutions using heuristic arguments similar to the perturbative analysis in Section 4.

We are clearly far from a complete understanding of the phenomenon in question. Here we only obtain some partial results and outline possible approaches to the problem that require further studies. Indeed, the paper raises more open questions than it answers, and we hope the work in this direction continues.

One possible extension of our studies is to describe the original mechanical system by hydrodynamic equations with an added fluctuation term. This is a result of the granularity of the particle system and should be of $O(N^{-\alpha})$ with some $\alpha > 0$. Such fluctuation terms may be derived as a high order corrections to the hydrodynamic equations, see [S]. With such a hydrodynamic fluctuation term, stable stationary solutions of the HE will

31

remain close to the stationary solutions without fluctuations for a long time, but eventually they will slowly drift away to a thermal equilibrium, i.e. they will behave similarly to the particle system.

Concerning the HE discussed in this work, we would like to know if they have periodic or asymptotically periodic solutions, and determine their period, amplitude and structure.

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