MA 485-1D (Probability), Dr. Chernov — Midterm test #2

5 problems, each is worth 20 points. Show your work. — Fri, Oct 31, 2008

1. A continuous random variable X has distribution function

$$F_X(x) = \frac{x^4 - 1}{80} \qquad \text{for} \qquad 1 \le x \le 3$$

(a) Find the density function $f_X(x)$.

Answer: $f_X(x) = F_X'(x) = x^3/20$, for $1 \le x \le 3$.

(b) Find $\mathbb{E}(X)$. Answer: $\frac{121}{50} = 2.42$.

(c) Find $\mathbb{E}(X^2)$. Answer: $\frac{91}{15} \approx 6.07$.

(d) Find $\mathsf{Var}(X)$. Answer: ≈ 0.2106.

(e) Find σ_X. Answer: ≈ 0.459.

(f) Find a formula for the kth moment of X for all $k \ge 1$. Answer:

$$\mathbb{E}(X^k) = \frac{3^{k+4} - 1}{20(k+4)}.$$

2. Let $X = \mathcal{N}(-2, 4)$ be a normal random variable.

(a) Find $\mathbb{P}(-3.3 < X < 5.2)$. Answer: 0.742.

(b) Find $\mathbb{P}(X^2 > 49)$. Answer: 0.0062.

(c) What is the type (and parameters) of the random variable $Y = 3(2 - X) - 15$?

Answer: Y is normal $\mathcal{N}(-3, 36)$.

(d) Suppose $x_1, \ldots, x_5$ are five random values of X observed independently. Let $V = \max\{x_1, \ldots, x_5\}$ and $W = \min\{x_1, \ldots, x_5\}$. Find $\mathbb{P}(-3.3 < W < V < 5.2)$.

Answer: 0.742^5.

3. Let $X = U(-1,1)$. Find the distribution function and the density function for the random variable $Y = 2 - \frac{4}{3-X}$.

Answers: the range is $0 < Y < 1$. The distribution function is

$$F_Y(y) = \mathbb{P}(Y \leq y) = \mathbb{P}\Big(X > 3 - \frac{4}{2-y}\Big) = \frac{2}{2-y} - 1$$

for $0 < y < 1$. The density function is

$$f_Y(y) = \frac{2}{(2-y)^2}$$

for $0 < y < 1$.

4. Two random variables X and Y have a constant joint density function $f(x, y) = c$ in the domain $\{x < 4,\ y > 0,\ x > y\}$ (and zero elsewhere).

(a) Find c. Answer: $c = \frac{1}{8}$.

(b) Find $\mathbb{P}(X - Y < 2)$. Answer: $= \frac{3}{4}$.

(c) Find $\mathbb{P}(X - Y = 2)$. Answer: 0.

(d) Find $\mathbb{P}(X^2 + Y^2 < 1)$. Answer: $= \frac{\pi}{64}$.

(Bonus) Are X and Y independent? Explain.

Answer: No. The domain where they take values is not rectangular.

5. Let X be an exponential random variable with half-life $t_{1/2} = 2$.

(a) Find its parameter λ (give an exact formula and an approximate numerical value).

Answer: $\lambda = \frac{\ln 2}{2} \approx 0.347$.

(b) Find the median m and the mean $\mathbb{E}(X)$. Which one is larger?

Answers: median is $m = t_{1/2} = 2$, mean is $\mathbb{E}(X) = \frac{1}{\lambda} \approx 2.89$.

(c) Compute $\mathbb{P}\big(X > \mathbb{E}(X) + 2\sigma_X\big)$. (You can use the formula $\sigma_X = \frac{1}{\lambda}$).

Answer: $e^{-3} \approx 0.05$.

(d) Find the conditional probability

$$P\big(X > 4\sigma_X / X > \mathbb{E}(X)\big)$$

Compare to the answer in (c). Explain.

Answer: $e^{-3} \approx 0.05$. Same as in (c) due to the memoryless property.

For an extra credit, find the quartiles $q_1 = \pi_{1/4}$ and $q_3 = \pi_{3/4}$ of X.

Answers:

$$\pi_{1/4} = \frac{\ln(4/3)}{\lambda} \approx 0.83$$

$$\pi_{3/4} = \frac{\ln 4}{\lambda} = 4.$$