MA 485-1D (Probability), Dr. Chernov | Final Exam

Show your work. 9 problems. Total is 100 points. | Fri, Dec 5, 2008

1. (10 pts) Let X be an exponential random variable with mean 100.

(a) Use Markov's inequality to obtain an upper bound on $\mathbb{P}(X \geq 300)$.

Answer: $\mathbb{P}(X \geq 300) \leq \frac{100}{300} = \frac{1}{3}$.

(b) Use Chebyshev's inequality to obtain an upper bound on $\mathbb{P}(X \geq 300)$.

Answer: $\mathbb{P}(X \geq 300) \leq \frac{10000}{(200)^2} = \frac{1}{4}$.

(c) Compute $\mathbb{P}(X \geq 300)$ precisely.

Answer: $\mathbb{P}(X \geq 300) = e^{-3} = 0.0498$.

2. (10 pts) The average price of gasoline is currently \$1.85 per gallon. Assume that each week it goes up 5 cents with probability 40% or down 5 cents with probability 60%.

(a) Find the probability that the average price per gallon goes down to \$1 before it comes back to \$2.

Answer: $a = 20$, $b = 40$, $x = 37$,

$$P_{20} = \frac{(0.6/0.4)^{17} - (0.6/0.4)^{20}}{1 - (0.6/0.4)^{20}} = 0.704.$$

(b) Find the average number of weeks the price will stay between \$1 and \$2.

Answer:

$$\mathbb{E}(T) = \frac{20 \times 0.704 + 40 \times 0.296 - 37}{0.4 - 0.6} = 55.4.$$

3. (10 pts) An insurance company has 50,000 automobile policyholders. The expected yearly claim per person is \$220 with a standard deviation of \$1,800.

(a) Compute the probability that the total yearly claim will be below \$10 mln.

Answer: $S \approx \mathcal{N}(11000000, 16200000000)$,

$$\mathbb{P}(S < 10000000) \approx \Phi\Big(\frac{10000000 - 11000000}{\sqrt{16200000000}}\Big) = \Phi(-2.48) = 0.0066.$$

(b) Use the 3σ-rule to predict the total yearly claim with probability 99.7%.

Answer: $9790000 < S < 1220000$.

4. (10 pts) Let $X = \mathcal{N}(-1, 4)$ and $Y = \mathcal{N}(-2, 3)$ be two independent normal random variables and $V = X - 2Y + 1$.

(a) Find the type and parameters of the variable V.

Answer: V is normal, $V = \mathcal{N}(4, 16)$.

(b) Find $\mathbb{P}(X > 2Y)$.

Answer: $\mathbb{P}(X > 2Y) = 1 - \Phi(-0.75) = 0.7734$.

(c) Compute $\mathsf{Cov}(V, Y)$.

Answer: $\mathsf{Cov}(V, Y) = -6$.

5. (15 pts) An investor has a stock that each week goes up \$5 with probability 0.5 or down \$5 with probability 0.5. She bought the stock when it costed \$150 and will sell it when it reaches \$250 or falls to \$100.

(a) What is the probability that she will end up with \$250?

Answer: if the unit is 5 cents, then $a = 20$, $b = 50$, $x = 30$,

$$P_{50} = \frac{30 - 20}{50 - 20} = \frac{1}{3}.$$

(b) Find the mean number of weeks the investor keeps the stock.

Answer: $\mathbb{E}(T) = 200$.

(c) Find the probability that the stock will again cost \$150 before the investor sells it.

Answer:

$$P(x, x) = 1 - \frac{50 - 20}{2(30 - 20)(50 - 30)} = \frac{37}{40} = 0.925.$$

(d) Find the average number of times the price of the stock goes back to \$150 before the investor sells the stock.

Answer:

$$G(x, x) = \frac{0.925}{0.075} = 12.3.$$

6. (10 pts) Assume that accidents on a 1000 miles long highway occur at a rate of one accident per 25 miles (on average). Justin is driving on this highway.

(a) Justin covers the first 50 miles and notices three accidents. What is the probability that in the next 50 miles, Justin will notice at least one accident? Give the formula for the probability that in the next 50 miles Justin will notice exactly k accidents.

Answer: if the unit is one mile, then $\lambda = 0.04$,

$$\mathbb{P}(N \geq 1) = 1 - e^{-2} = 0.865.$$

(b) What is the distribution of the intervals between accidents? Write down a formula for the distribution function, give its mean value and variance. State clearly which unit of length you are using.

Answer: Exponential; if the unit is one mile, then $F_W = 1 - e^{0.04x}$, $\mathbb{E}(W) = 25$, $\mathsf{Var}(W) = 625$.

7. (10 pts) Assume that accidents on a 1000 miles long highway occur at a rate of one accident per 25 miles (on average).

(a) Let X be the total number of accidents on the entire 1000 miles of the highway. What is the type of the random variable X? What is its parameter?

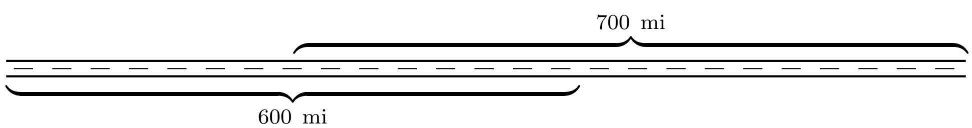
Answer: $X = $poisson(40).

(b) Use normal approximation to compute $\mathbb{P}(35 < X < 44)$. Apply the histogram correction if necessary.

Answer: $X \approx \mathcal{N}(40, 40)$,

$$\mathbb{P}(35 < X < 44) \approx \Phi\Big(\frac{43.5 - 40}{\sqrt{40}}\Big) - \Phi\Big(\frac{35.5 - 40}{\sqrt{40}}\Big) = \Phi(0.55) - \Phi(-0.71) = 0.4699.$$

(Bonus) Let V be the number of accidents in the first 600 miles of the highway and W the number of accidents in the last 700 miles of the highway (see the sketch below). Note that these two intervals of the highway overlap. Compute $\mathsf{Cov}(V, W)$ and $\rho_{V,W}$.



Answer: $\mathsf{Cov}(V, W) = 12$ and $\rho_{V,W} = 0.463$.

8. (15 pts) Suppose you buy cereal of a certain brand, and 75% of boxes with cereal contain a coupon. When you collect 50 such coupons, you can order a free backpack by mail. In the following, use De Moivre-Laplace theorem.

(a) What is the probability that you will need to buy at least 65 boxes of cereal to collect 50 coupons?

Answer: 64 is not enough, $X = b(64, 0.75) \approx \mathcal{N}(48, 12)$,

$$\mathbb{P}(X < 50) \approx \Phi\Big(\frac{49.5 - 48}{\sqrt{12}}\Big) = \Phi(0.43) = 0.6664.$$

(b) What is the probability that if you buy 100 boxes of cereal, then you get exactly 75 coupons?

Answer: $X = b(100, 0.75) \approx \mathcal{N}(75, 18.75)$,

$$\mathbb{P}(X = 75) \approx \Phi\Big(\frac{75.5 - 75}{\sqrt{18.75}}\Big) - \Phi\Big(\frac{74.5 - 75}{\sqrt{18.75}}\Big) = \Phi(0.12) - \Phi(-0.12) = 0.0956.$$

9. (10 pts) A system has 10 components. The lifetime (time to failure) of each component is a uniform random variable taking values between 1 and 3 (years), and their lifetimes are independent. The system works as long as at least two components are functioning.

(a) Find the **distribution function** $F_T(x)$ and the **density function** $f_T(x)$ of the lifetime T of the system.

Answer: $F_X = \frac{x-1}{2}$ for $1 < x < 3$, so

$$F_T(x) = \left(\frac{x-1}{2}\right)^{10} + 10\left(1 - \left(\frac{x-1}{2}\right)\right)\left(\frac{x-1}{2}\right)^{9}$$

and

$$f_T(x) = 45\left(1 - \left(\frac{x-1}{2}\right)\right)\left(\frac{x-1}{2}\right)^{8}$$

for $1 < x < 3$.

(b) Let $X_1, \ldots, X_{10}$ denote the lifetimes of all 10 components. The lifetime of the system is an order statistic, $X_{(k)}$. What is the value of k?

Answer: $k = 9$, so $T = X_{(9)}$.