MA 486/586-1G (Statistics), Dr Chernov — Midterm test #2

Show your work. — Fri, Mar 21, 2008

Note: the total is 120 points. To get full credit for the test you need to earn 100 points. If you get more than 100, it will be for extra credit.

1. (20 pts) A sample $x_1, \dots, x_{29}$ from $N(\mu, \sigma^2)$ yields $\bar{x} = 217$ and $s^2 = 5.5$. Test the hypothesis $H_0 : \sigma = 2$ against $H_1 : \sigma \neq 2$ at the 10% significance level.

Solution: the test statistics is

$$\chi^2 = 28 \times 5.5/4 = 38.5.$$

The critical region is

$$\chi^2 > \chi^2_{0.05}(28) = 41.34$$

or

$$\chi^2 < \chi^2_{0.95}(28) = 16.93.$$

Since neither is true, we accept H_0.

2. (20 pts) Let p_1 and p_2 be the proportions of babies with a low birth weight in the countries A and B, respectively. We shall test the hypothesis $p_1 = p_2$ against $p_1 \neq p_2$. In random samples of sizes $n_1 = 500$ and $n_2 = 1000$ babies in the two countries, $y_1 = 46$ and $y_2 = 104$ babies were found to have low birth weight. Test the hypothesis at the significance level $\alpha = 0.02$. Find the p-value of the test.

Solution: the pooled estimate of p is

$$\hat{p} = \frac{46 + 104}{500 + 1000} = 0.1.$$

The test statistics is

$$Z = \frac{46/500 - 104/1000}{\sqrt{0.1 \times 0.9 \times (1/500 + 1/1000)}} = -0.73.$$

The critical region is $|Z| > z_{0.01} = 2.326$. We accept H_0. The p-value is

$$2[1 - \Phi(0.73)] = 0.4654.$$

3. (20 pts) In a school fund-raising lottery, each ticket wins one of four prizes, call them A, B, C, D, which are supposed to occur in the ratio 1:2:4:8. In one day, 90 students have bought tickets (one each). Out of those students, 4 won prize A, 7 won prize B, 18 won prize C, and the rest – prize D. Are these data consistent with the announced ratio of chances to win prizes? Let $\alpha = 0.10$. What can you say about the p-value of the test?

The theoretically expected frequencies are 6, 12, 24, 48. The experimentally observed frequencies are 4, 7, 18, 61. The test statistic is

$$Q = \frac{2^2}{6} + \frac{5^2}{12} + \frac{6^2}{24} + \frac{13^2}{48} = 7.771$$

The critical region is $Q > \chi^2_{0.1}(3) = 6.251$. We accept H_1, i.e. conclude that the data are **not** consistent with the announced ratio of chances to win prizes.

The p-value is between 0.1 and 0.05.

4. (30 pts) Let $x_1, \ldots, x_{100}$ be a sample taken from $N(\mu, 4)$. We shall test the hypothesis $H_0 : \mu = 1$ against $H_1 : \mu > 1$.
(a) Let the critical region be $\bar{x} > 1.5$. Find α and write down formulas for β and $K(\mu)$.
(b) Compute $K(1.3)$, $K(1.5)$, $K(1.7)$, $K(1.9)$ and $K(2.1)$. Sketch the graph of the power function.
(c) Assume that the sample yields $\bar{x} = 1.522$. What is the p-value of the test?
(d) Suppose we want $\alpha = 0.001$ and $K(1.6) = 0.9975$. How large a sample will be necessary? What would be the critical region?

Solution: for (a) we have

$$\alpha = 1 - \Phi\left(\frac{1.5 - 1}{0.2}\right) = 1 - \Phi(2.5) = 0.0062,$$

and

$$\beta = \Phi\left(\frac{1.5 - \mu}{0.2}\right), \quad \text{so} \quad K(\mu) = 1 - \Phi\left(\frac{1.5 - \mu}{0.2}\right).$$

For (b) we have $K(1.3) = 0.1587$, $K(1.5) = 0.5$, $K(1.7) = 0.8413$, $K(1.9) = 0.9772$, $K(2.1) = 0.9987$. The sketch is not given here.

For (c) we have $Z = (1.522 - 1)/0.2 = 2.61$ and the p-value$= 0.0045$.

For (d) we have

$$n > \frac{(3.090 + 2.807)^2 \times 4}{(1.6 - 1)^2} = 386.38,$$

hence $n \geq 387$, and

$$c = \frac{1 \times 2.807 + 1.6 \times 3.090}{2.807 + 3.090} = 1.314.$$

The critical region is $\bar{x} > 1.314$.

5. (30 pts) Let $x_1, \ldots, x_{31}$ and $y_1, \ldots, y_{41}$ be two independent samples taken from $N(\mu_X, \sigma_X^2)$ and $N(\mu_Y, \sigma_Y^2)$, respectively. The test results are $\bar{x} = -6$, $s_x^2 = 9$, $\bar{y} = -8$ and $s_y^2 = 10$.
(a) Construct a 98% confidence interval for σ_X^2/σ_Y^2.
(b) Test the hypothesis of equality of these two normal random variables at the level $\alpha = 2\%$.

Solution: for (a) we have

$$\frac{9}{10} \times \frac{1}{2.20} < \frac{\sigma_X^2}{\sigma_Y^2} < \frac{9}{10} \times \frac{2.30}{1},$$

hence

$$0.409 < \frac{\sigma_X^2}{\sigma_Y^2} < 2.07.$$

For (b) we need to make two steps. In Step 1, we test the hypothesis $\sigma_X^2 = \sigma_Y^2$. The test statistic is

$$F = \frac{s_y^2}{s_x^2} = \frac{10}{9} = 1.111.$$

The critical region is

$$F > F_{0.01}(40, 30) = 2.30$$

or

$$F < 1/F_{0.01}(30, 40) = 1/2.20 = 0.455$$

Since neither is true, we conclude that $\sigma_X^2 = \sigma_Y^2$.

In Step 2, we test the hypothesis $\mu_X^2 = \mu_Y^2$. The test statistic is

$$T = \frac{-6 - (-8)}{\sqrt{\frac{30\times 9 + 40\times 10}{70}} \times \sqrt{\frac{1}{31} + \frac{1}{41}}} = 2.716.$$

The critical region is $|T| > t_{0.01}(70) = 2.326$ (use the 'infinite' number of degrees of freedom). So we accept H_1, i.e. conclude that these two random variables are **not** equal.