MA 587/687 (Advanced Probability), Dr. Chernov — Final Exam
10 problems. — April, 26, 2012

587 students: do 8 problems for full credit. (If you do more, you get extra points.)
687 students: do 9 problems for full credit. (If you do more, you get extra points.)

1. Let X and Y be continuous random variables with joint density function

$$f_{XY}(x,y) = \begin{cases} cy & \text{for} \;\; x \le y \le \sqrt{x}, \;\; 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of the constant c. Answer: $c = 12$
(b) Find the marginal density function of X. Answer: $f_X(x) = 6x(1-x)$ for $0 < x < 1$
(c) Find the marginal density function of Y. Answer: $f_Y(y) = 12y^2(1-y)$ for $0 < y < 1$

2. Let X and Y be independent random variables with density functions

$$f_X(x) = \begin{cases} 2x & \text{for} \;\; 0 \le x \le 1, \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 3y^2 & \text{for} \;\; 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

Find the density function for $X+Y$ by using the convolution formula. Answer:
$f(a) = a^4/2$ for $0 < a < 1$ and $f(a) = 3a^2 - 2a - a^4/2$ for $1 < a < 2$

3. The conditional distribution of X, given Y, is uniform on the interval $[Y, 2Y]$. The marginal density of Y is

$$f_Y(y) = \begin{cases} 3y^2 & \text{for} \;\; 0 \le y \le 1, \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the joint density function $f_{XY}(x,y)$. Answer:
$f_{XY}(x,y) = 3y$ for $0 < y < 1$ and $y < x < 2y$
(b) Find the conditional density of Y, given $X = x > 0$. Answer:
If $0 < x < 1$, then $f_{Y|X}(y|x) = \frac{8y}{3x^2}$ for $x/2 < y < x$.
If $1 < x < 2$, then $f_{Y|X}(y|x) = \frac{8y}{4-x^2}$ for $x/2 < y < 1$.

4. In a math tournament, 30 teams from different schools participate. Each team consist of four members. All the 120 participants are to be seated in 10 rooms, with 12 participants in each room. Suppose the seating is done at random. We say that a team is **lucky** if all its members end up seating in the same room. What is the expected number of lucky teams in this tournament? Answer: $30 \cdot \frac{\binom{4}{4} \cdot \binom{116}{8}}{\binom{120}{12}} \approx 0.018$

5. Independent random variables X, Y and Z are identically distributed. Let $W = X+Y$. The moment generating function of W is $M_W(t) = (0.2 + 0.8e^{-t})^8$.
(a) Find the moment generating function of $V = X + Y + Z$. Answer:

$$M_V(t) = (0.2 + 0.8e^{-t})^{12}$$

(Bonus) Find possible values of X and the respective probabilities. Answer:
values: $0, -1, -2, -3, -4$; respective probabilities: $0.0016, 0.0256, 0.1536, 0.4096, 0.4096$

6. Let $X_1, \ldots, X_n$ be independent and identically distributed random variables, each of them is exponential with a common parameter $\lambda > 0$. Let Y_n be the minimum of $X_1, \ldots, X_n$, i.e., $Y_n = \min\{X_1, \ldots, X_n\}$. .
(a) Find the cumulative distribution of Y_n. Answer: $F_{Y_n}(x) = 1 - e^{-\lambda x n}$
(b) Show that Y_n converges in probability to 0 by showing that for arbitrary $\varepsilon > 0$

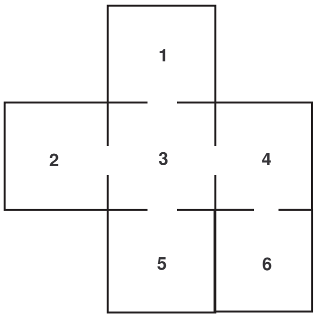
$$\lim_{n\to\infty} \mathbb{P}(|Y_n - 0| \leq \varepsilon) = 1.$$

(Bonus) Show that the same is true when $X_1, \ldots, X_n$ are independent and identically distributed random variables that are nonnegative and have a common density function $f(x)$ such that $f(x) > 0$ for all $0 < x < 1$.

7. Let X and Y be the number of hours that a randomly selected person watches movies and sporting events, respectively, during a three-month period. The following information is known about X and Y:

$$\begin{aligned} \mathbb{E}(X) &= 30 \\ \mathbb{E}(Y) &= 20 \\ \sigma_X &= 5 \\ \sigma_Y &= 4 \\ \rho_{X,Y} &= 0.2 \end{aligned}$$

Four hundred people are randomly selected and observed for these three months. Let T be the total number of hours that these four hundred people watch movies or sporting events during this three-month period. Approximate the value of $\mathbb{P}(T > 21000)$. Answer: T is approx. normal with $\mu = 20,000$ and $\sigma = 140$. Now $\mathbb{P}(T > 21000) = 1 - \Phi(7.13) = 0$.



8. A rat runs through the maze shown in the above figure. At each step it leaves the room it is in by choosing at random one of the doors out of the room.
(a) Give the transition matrix P for this Markov chain.
(b) Is it an irreducible (i.e., ergodic) chain? Answer: yes
(c) Is it a regular (i.e., irreducible and aperiodic) chain? Answer: no
(d) Find the fixed (i.e., stationary) probability vector.
Answer: $w = (0.1, 0.1, 0.4, 0.2, 0.1, 0.1)$

9. An absorbing Markov chain has the following transition matrix:

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.2 & 0.4 & 0.1 & 0.3 \\ 0.2 & 0.2 & 0.5 & 0.1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(a) Draw the corresponding graph, identify transient and absorbing states.
(b) Find the fundamental matrix $\mathbf{F}$. Answer: $\mathbf{F} = \begin{bmatrix} 25/14 & 25/14 \\ 5/7 & 15/7 \end{bmatrix}$
(c) Find the product $\mathbf{FR}$ and interpret its values. Answer: $\mathbf{FR} = \begin{bmatrix} 3/7 & 4/7 \\ 4/7 & 3/7 \end{bmatrix}$
(d) Compute the product $\mathbf{Fu}$, where $\mathbf{u}$ is a vectors of ones, and interpret its values. Answer: $\mathbf{Fu} = \begin{bmatrix} 15/7 \\ 20/7 \end{bmatrix}$

10. Let X and Y have a bivariate normal distribution with parameters $\mu_X = 1$, $\mu_Y = -1$, $\sigma_X = 3$, $\sigma_Y = 2$, and $\rho = 2/5$.
(a) Find the joint density function $f_{XY}(x, y)$. Answer:

$$f(x,y) = \frac{1}{12\pi\sqrt{0.84}} e^{-\frac{25}{42}\left[\left(\frac{x-1}{3}\right)^2 + \left(\frac{y+1}{2}\right)^2 - \frac{4}{5}\left(\frac{x-1}{3}\right)\left(\frac{y+1}{2}\right)\right]}$$

(b) Compute the probability $\mathbb{P}(|Y| < 1)$. Answer: 0.3413
(c) Compute the conditional probability $\mathbb{P}(|Y| < 1 | X = 0)$. Answer: 0.3329
(Bonus) Find a constant c such that the variable $U = X + cY$ will be independent of Y.
Answer: $c = -0.6$