Selected Problems in Real Analysis
(with solutions)
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Contents

1 Lebesgue measure

JPE, May 2011. Are the following true of false?
(a) If A is an open subset of $[0,1]$, then $m(A) = m(\bar{A})$, where $\bar{A}$ is the closure of the set.
(b) If A is a subset of $[0,1]$ such that $m(\text{int}(A)) = m(\bar{A})$, then A is measurable. Here $\text{int}(A)$ denotes the interior of the set.

(a) False. Counterexample: the complement to a modified Cantor set. Its measure is < 1, but its closure is the entire interval $[0,1]$.

(b) True. Indeed, we have

$$\partial A = \bar{A} \setminus \text{int}(A) \qquad \Rightarrow \qquad m(\partial A) = m(\bar{A}) - m(\text{int}(A)) = 0.$$

Now

$$A = \text{int}(A) \cup (A \setminus \text{int}(A)), \qquad A \setminus \text{int}(A) \subset \partial A.$$

Hence A is the union of an open set, $\text{int}(A)$, and a subset of the null set ∂A. Since the latter is always measurable, we conclude that A is a measurable set.

JPE, Sept 2010. Is the following true of false?
Let E be a subset of $\mathbb{R}^n$, and $\text{int}(E)$ the set of all interior points of E. Then $\text{int}(E) = \emptyset$ if and only if $\mu^*(E) = 0$. (Here μ^* denotes the outer measure.)

If $\mu^*(E) = 0$, then $m(E) = 0$, so $\text{int}(E)$ is indeed empty. But the converse is not true. The set of points with irrational coordinates has infinite measure and empty interior.

JPE, May 2005. Show that if $A \subset [0, 1]$ and $m(A) > 0$, then there are x and y in A such that $|x - y|$ is an irrational number.

If $|x - y| \in \mathbb{Q}$ for any $x, y \in A$, then $A \subset x + \mathbb{Q}$ for any point $x \in A$, hence A would be a countable set and we would have $m(A) = 0$.

JPE, Sept 2004 and Jan 1989. Is the following true or false?
There is a subset A of $\mathbb{R}$ which is not measurable, but such that $B = \{x \in A \colon x \text{ is irrational}\}$ is measurable.

False. The set $A \setminus B \subset \mathbb{Q}$ is countable, hence measurable. So if B was measurable, then $A = B \cup (A \setminus B)$ would be measurable, too.

JPE, May 2001. Does there exist a non-measurable subset of $\mathbb{R}$ whose complement in $\mathbb{R}$ has outer measure zero?

No. If the outer measure of a set is zero, then its inner measure is also zero, so the set is measurable. Then its complement is measurable, too.

JPE, May 2000. Do there exist two non-measurable sets whose union is measurable?

Yes. If A is any non-measurable set, then its complement A^c is also non-measurable, but their union is the whole space (a measurable set).

JPE, May 2000. Is the following true of false?
If the boundary of $\Omega \subset \mathbb{R}^k$ has outer measure zero, then Ω is measurable.

True. Since the outer measure of $\partial\Omega$ is zero, its inner measure is zero, too, hence its Lebesgue measure is zero. Then any subset of $\partial\Omega$ is a null set, and therefore it is measurable, too. Now Ω is the union of two sets:

$$\Omega = \text{int}(\Omega) \cup (\Omega \setminus \text{int}(\Omega)).$$

Note that $\text{int}(\Omega)$ is an open set, hence it is measurable. And $(\Omega \setminus \text{int}(\Omega)) \subset \partial\Omega$ is a subset of a null set, hence it is also measurable. Therefore Ω is measurable.

JPE, May 1998. Let $A \subset [0,1]$ be a non-measurable set. Let $B = \{(x,0) \in \mathbb{R}^2 \colon x \in A\}$.
(a) Is B a Lebesgue measurable subset of $\mathbb{R}^2$?
(b) Can B be a closed subset of $\mathbb{R}^2$ for some such A?

(a) Yes. The set B is a subset of a straight line ($y = 0$), so it has outer measure zero. Thus it is Lebesgue measurable.

(b) No. If B was closed in $\mathbb{R}^2$, then A would be closed in $[0,1]$, and then it would be measurable.

JPE, Sept 1997. For a measurable subset $E \subset \mathbb{R}^n$, prove or disprove:
(a) If E has Lebesgue measure zero, then its closure has Lebesgue measure zero.
(b) If the closure of E has Lebesgue measure zero, then E has Lebesgue measure zero.

(a) False. Example: E consists of points with all rational coordinates. E is countable, hence $m(E) = 0$. On the other hand, E is dense in $\mathbb{R}^n$, hence its closure is $\mathbb{R}^n$.

(b) True. Since E is a subset of its own closure, then E also has Lebesgue measure zero.

JPE, May 1993. Let r_n be an enumeration of rational numbers in $\mathbb{R}$.
(a) Show that $\mathbb{R} \setminus \cup_{n=1}^{\infty}(r_n - \frac{1}{n^2}, r_n + \frac{1}{n^2})$ is never empty.
(b) Show that $\mathbb{R} \setminus \cup_{n=1}^{\infty}(r_n - \frac{1}{n}, r_n + \frac{1}{n})$ can be empty or non-empty, depending on how the rationals are enumerated.

(a) By the σ-subadditivity of the Lebesgue measure

$$m\big(\cup_{n=1}^{\infty}(r_n - \tfrac{1}{n^2}, r_n + \tfrac{1}{n^2})\big) \le \sum_{n=1}^{\infty} m\big((r_n - \tfrac{1}{n^2}, r_n + \tfrac{1}{n^2})\big) = \sum_{n=1}^{\infty} \tfrac{2}{n^2} < \infty,$$

thus these intervals cannot cover the entire $\mathbb{R}$.

(b) Now the above estimate gives $\sum_{n=1}^{\infty} \frac{2}{n} = \infty$, thus our previous argument would not work. However presenting specific examples of enumeration so that the above intervals cover (or do not cover) $\mathbb{R}$ is not easy. Let us not get into these complications...

JPE, May 1990. Does there exist a measure space $(X, \mathfrak{M}, \mu)$ such that there is no countable collection of subsets $X_n \in \mathfrak{M}$ satisfying $\mu(X_n) < \infty$ for all n and $X = \cup_{n=1}^{\infty} X_n$.

Yes. Example: μ is the counting measure on $\mathbb{R}$ with Borel σ-algebra.

JPE, May 1989. Does there exist an open dense subset $A \subset [0,1] \times [0,1]$ such that its complement $([0,1] \times [0,1]) \setminus A$ has positive Lebesgue measure?

Yes. The complement to a modified two-dimensional Cantor set.

2 Measurable functions

JPE, Sept 2011. Is the following true of false?
If $f\colon [0,1] \to \mathbb{R}$ is continuous a.e., then f is measurable.

True. Let $E \subset [0,1]$ be the set of points where f is discontinuous. We have $m(E) = 0$. The restriction of f to $E^c = [0,1] \setminus E$ is continuous, hence for any open set $U \subset \mathbb{R}$ its preimage $f^{-1}(U) \cap E^c$ is open in E^c, therefore $f^{-1}(U) = (V \cap E^c) \cap B$ for some open set $V \subset [0,1]$ and some subset $B \subset E$. Any subset of the null set E is measurable, hence $f^{-1}(U)$ is a measurable set.

JPE, Sept 2011 and May 2005. Let $f\colon [0,1] \to \mathbb{R}$. Is it true that if the set $\{x \in [0,1]\colon f(x) = c\}$ is measurable for every $c \in \mathbb{R}$, then f is measurable?

False. Let $A \subset [0,1]$ be a non-measurable set. Define $f(x) = x$ on A and $f(x) = -x$ on $[0,1] \setminus A$. This function is injective, hence $\{x \in [0,1]\colon f(x) = c\}$ is either empty or a one-point set (a singleton) for each $c \in \mathbb{R}$; in either case it is measurable. But $f^{-1}([0,1]) = A$ is a non-measurable set.

JPE, Sept 2009. Does there exist a sequence $\{f_k\}$ of Lebesgue measurable functions such that f_k converges to 0 in measure on $\mathbb{R}$ but no subsequence converges uniformly on any subset of positive measure?

No. In one of the homework exercises, we proved that if f_k converges in measure, then there is a subsequence $\{f_{n_k}\}$ that converges a.e. Now by Egorov's theorem the convergence must be uniform on a set of positive measure.

JPE, Sept 2007. Show that $f_n(x) = e^{-n|1-\sin x|}$ converges in measure to $f(x) = 0$ on $[a,b] \subset \mathbb{R}$.

We have

$$|f_n - f| > \varepsilon \quad \Leftrightarrow \quad e^{-n|1-\sin x|} > \varepsilon \quad \Leftrightarrow \quad |1 - \sin x| < \frac{1}{n} \ln \frac{1}{\varepsilon}$$

Note that $\sin x = 1$ whenever $x = \frac{\pi}{2} + 2k\pi$ ($k \in \mathbb{N}$). Thus the above inequality $|1 - \sin x| < \frac{1}{n} \ln \frac{1}{\varepsilon}$ specifies a neighborhood of each point $x = \frac{\pi}{2} + 2k\pi$ whose size shrinks as $n \to \infty$. Note that there can only be finitely many points $\frac{\pi}{2} + 2k\pi$ in any finite interval $[a,b]$. Thus the Lebesgue measure of the union of the above neighborhoods of these points tends to zero as $n \to \infty$.

If we replace a finite interval $[a,b]$ with an infinite interval, such as (a,∞) or $(-\infty,b)$, then there would be infinitely many of the above points $\frac{\pi}{2} + 2k\pi$ and their neighborhoods within the given interval, and then their union would have an infinite measure. In that case the convergence in measure would fail.

JPE, Sept 2005. Is the following true of false?
If $f\colon [0,\infty) \to \mathbb{R}$ is differentiable, then f' is measurable.

True. The derivative can be computed as the following limit

$$f'(x) = \lim_{n\to\infty} \frac{f(x+\frac{1}{n}) - f(x)}{\frac{1}{n}},$$

which exists because f is assumed to be differentiable at every point $x \in \mathbb{R}$. Thus f' is a limit of measurable functions, hence it is measurable.

JPE, May 2001. Does there exist a non-measurable function $f \geq 0$ such that $\sqrt{f}$ is measurable?

No. Indeed, $f = (\sqrt{f})^2$ is a composition of a measurable function $\sqrt{f}$ and a continuous (and thus Borel) function, x^2, thus f is measurable.

JPE, May 1994 Let $\{f_n\}$ be a sequence of measurable functions on a measurable space $(X, \mathfrak{M})$. Define the set

$$E = \{x \in X\colon \lim_{n\to\infty} f_n(x) \ \text{ exists}\}$$

Show that E is a measurable set.

Let $g(x) = \limsup_{n\to\infty} f_n(x)$ and $h(x) = \liminf_{n\to\infty} f_n(x)$. We know that both functions $g(x)$ and $h(x)$ are measurable. Also recall that $\lim_{n\to\infty} f_n(x)$ exists if and only if $g(x) = h(x)$. Now $E = \{x \in X\colon g(x) = h(x)\}$, hence E is measurable.

3 Lebesgue integral: definition via simple functions

JPE, May 2008. Is the following true or false?
For every non-negative, bounded and measurable function f on $[0,1]$,

$$\int_{[0,1]} f\,dm = \inf \int_{[0,1]} \varphi\,dm$$

, where the infimum is taken over all simple measurable functions φ with $f \leq \varphi$.

True. Since f is bounded, let $M = \sup f$ be its upper bound. Now for every simple function $\varphi \geq f$ there is a simple function ψ such that $\psi \leq M$ and $f \leq \psi \leq \varphi$. Indeed, it is enough to take $\psi = \min\{\varphi, M\}$. Note that $\int_{[0,1]} \psi\,dm \leq \int_{[0,1]} \varphi\,dm$, hence

$$\inf_{f\le\varphi}\int_{[0,1]}\varphi\,dm=\inf_{f\le\psi\le M}\int_{[0,1]}\psi\,dm,$$

i.e., it is enough to use only simple functions ψ satisfying $f\le\psi\le M$.

Now $g=M-f$ is a nonnegative measurable function bounded by M, thus

$$M-\int_{[0,1]}f\,dm=\int_{[0,1]}g\,dm=\sup_{0\le s\le g}\int_{[0,1]}s\,dm$$

where the infimum is taken over all simple functions s such that $0\le s\le g$. For each such s we have $\psi=M-s$ a simple function satisfying $f\le\psi\le M$. Conversely, for every simple function ψ satisfying $f\le\psi\le M$ we have $s=M-\psi$ a simple function satisfying $0\le s\le g$. Thus the above identity gives

$$\int_{[0,1]}f\,dm=-\sup_{f\le\psi\le M}\int_{[0,1]}(-\psi)\,dm=\inf_{f\le\psi\le M}\int_{[0,1]}\psi\,dm.$$

JPE, May 2004. Is the following true or false?
Let $f\ge 0$ be bounded and measurable on $\mathbb{R}$. Then

$$\int_{\mathbb{R}}f\,dm=\inf\int_{\mathbb{R}}\phi\,dm$$

where the infimum is taken over all simple measurable functions ϕ with $f\le\phi$?

False. This would be true if the measure of the whole space was finite (like in the previous problem). But here $m(\mathbb{R})=\infty$, in which case the claim is false.

A counterexample is any bounded function $f(x)\in L^1_m(\mathbb{R})$ such that $f(x)>0$ for all $x\in\mathbb{R}$. For example, $f(x)=e^{-x^2}$ or $f(x)=\frac{1}{1+x^2}$, which you might remember from Calculus or Probability Theory. If you do not remember any, you can construct f as follows:

$$f=\sum_{n=-\infty}^{\infty}2^{-|n|}\,\chi_{[n,n+1)}$$

For this function we have

$$\int_{\mathbb{R}}f\,dm=\sum_{n=-\infty}^{\infty}2^{-|n|}=3<\infty$$

Now since $f(x)>0$ for all $x\in\mathbb{R}$, then any simple function $\phi\ge f$ must also be positive *everywhere*, so that in the representation

$$\phi=\sum_{i=1}^{n}\alpha_i\chi_{A_i}$$

all the values α_i are positive: $\alpha_i > 0$ for all $i = 1, \dots, n$. At the same time at least one subset A_{i_0} must have infinite measure, i.e., $m(A_{i_0}) = \infty$. Therefore

$$\int \phi \, dm = \sum_{i=1}^{n} \alpha_i m(A_i) \geq \alpha_{i_0} m(A_{i_0}) = \infty$$

for every simple function $\phi \geq f$. Thus $\inf \int_{\mathbb{R}} \phi \, dm = \infty$, while $\int_{\mathbb{R}} f \, dm < \infty$.

4 Lebesgue integral: general

JPE, May 2011. Prove that a measurable function $f(x)$ belongs to $L^1(0,1)$ if and only if

$$\sum_{n=1}^{\infty} 2^n \cdot m\{x \in [0,1] \colon |f(x)| \geq 2^n\} < \infty.$$

JPE, May 2006. Let $\mu(X) < \infty$. Prove that a non-negative measurable function $f(x)$ belongs to $L^1(X, \mu)$ if and only if

$$\sum_{n=1}^{\infty} 2^n \cdot \mu\{x \in X \colon f(x) \geq 2^n\} < \infty.$$

The above two problems are almost identical, we only solve the first one. Let us partition $[0,1]$ into subsets

$$\begin{aligned}
E_0 &= \{x \colon |f(x)| < 2\} & g(x) &= 1 \\
E_1 &= \{x \colon 2 \leq |f(x)| < 2^2\} & g(x) &= 2 \\
&\dots \\
E_n &= \{x \colon 2^n \leq |f(x)| < 2^{n+1}\} & g(x) &= 2^n \\
&\dots
\end{aligned}$$

and a new function g by $g(x) = 2^n$ for all $x \in E_n$, as shown above. Note that

$$g(x) - 1 \leq |f(x)| \leq 2g(x) \qquad \Rightarrow \qquad \int g - 1 \leq \int |f| \leq 2 \int g,$$

thus $f \in L^1(0,1)$ if and only if $g \in L^1(0,1)$.

Denote

$$S \colon = \sum_{n=1}^{\infty} 2^n \cdot m\{x \in [0,1] \colon |f(x)| \geq 2^n\}$$

and observe that

$$m\{x \colon |f(x)| \geq 2^n\} = \sum_{k=n}^{\infty} m(E_k).$$

Now on the one hand,

$$\int_{[0,1]} g\,dm = \sum_{n=0}^{\infty} 2^n m(E_n) \le 2m(E_0) + \sum_{n=1}^{\infty} 2^n m\{x\colon |f(x)| \ge 2^n\} \le 2 + S.$$

On the other hand,

$$S = \sum_{n=1}^{\infty} 2^n \sum_{k=n}^{\infty} m(E_k) = \sum_{k=1}^{\infty} m(E_k) \sum_{n=1}^{k} 2^n \le \sum_{k=1}^{\infty} 2^{k+1} m(E_k) \le 2 \int_{[0,1]} g\,dm$$

thus

$$\frac{S}{2} \le \int_{[0,1]} g\,dm \le S + 2.$$

This implies $g \in L^1(0,1)$ if and only if $S < \infty$.

JPE, May 2010. Is the following true or false?
Let $\{f_k\}$ be a sequence of non-negative measurable functions on $\mathbb{R}$ such that $f_k \to f$ a.e. in $\mathbb{R}$. Then $\lim_{k\to\infty} \int_{\mathbb{R}} f_k\,dm$ exists and

$$\int_{\mathbb{R}} f\,dm \le \lim_{k\to\infty} \int_{\mathbb{R}} f_k\,dm.$$

False. The limit need not exist. For example, let

$$f \equiv 0 \qquad \text{and} \qquad f_k = k\big(1 + (-1)^k\big)\chi_{[0,k^{-1}]}$$

Then $\int_{\mathbb{R}} f_k\,dm$ alternatively takes values 2 (for all even k) and 0 (for all odd k), so it does not have a limit.

We note, however, that whenever $\lim_{k\to\infty} \int_{\mathbb{R}} f_k\,dm$ does exist, the claimed indeed immediately follows from Fatou's Lemma.

JPE, May 2010. If $f \in L^1(0,1)$, find

$$\lim_{k\to\infty} \int_0^1 k \ln\Big(1 + \frac{|f(x)|^2}{k^2}\Big)\,dx.$$

The limit is zero. The integrand can be written (and bounded above) as

$$\frac{1}{k}\,\ln\Big(1 + \frac{|f(x)|^2}{k^2}\Big)^{k^2} \le \frac{1}{k}\,\ln e^{|f(x)|^2} = \frac{1}{k}\,|f(x)|^2,$$

thus it converges to zero pointwise a.e. (more precisely, for every $x \in [0,1]$ such that $|f(x)| < \infty$). However, the Lebesgue Dominated Convergence does not apply (yet), because $|f^2|$ is not necessarily in $L^1(0,1)$, i.e., we may have $\int_0^1 |f(x)|^2\,dx = \infty$ (example: $f(x) = 1/\sqrt{x}$ is in $L^1(0,1)$, but $\int_0^1 |f(x)|^2\,dx = \infty$).

To get a better upper bound we can use an elementary inequality $\ln(1+t) \le 2\sqrt{t}$, which is true for all $t \ge 0$; see a proof below. This inequality gives

$$k \ln\Big(1 + \frac{|f(x)|^2}{k^2}\Big) \le 2k\,\frac{|f(x)|}{k} = 2|f(x)|,$$

which is integrable. Now the Lebesgue Dominated Convergence applies and finishes the job. Lastly, here is the proof of the elementary inequality:

$$\ln(1+t) \le 2\sqrt{t} \quad \Leftrightarrow \quad 1+t \le e^{2\sqrt{t}} = 1 + 2\sqrt{t} + \frac{4t}{2} + \cdots$$

and the latter is obvious.

JPE, Sept 2009 and Sept 2004. Assume that $\{f_n\}, \{g_n\}, f, g$ are in $L^1(\mathbb{R}^n)$, $f_n \to f$ pointwise a.e., $g_n \to g$ pointwise a.e., $|f_n| \le g_n$ a.e., and $\int_{\mathbb{R}^n} g_n\, dm \to \int_{\mathbb{R}^n} g\, dm$. Show that $\int_{\mathbb{R}^n} f_n\, dm \to \int_{\mathbb{R}^n} f\, dm$.

By the triangle inequality $|f_n - f| \le g_n + |f|$, hence

$$g_n + |f| - |f_n - f| \ge 0.$$

Now by Fatou's Lemma (as in the proof of Lebesgue Dominated Convergence)

$$\begin{aligned}
\int g + \int |f| &= \int \liminf (g_n + |f| - |f_n - f|) \\
&\le \liminf \int (g_n + |f| - |f_n - f|) \\
&= \liminf \Big(\int g_n + \int |f| - \int |f_n - f| \Big) \\
&= \int g + \int |f| - \limsup \int |f_n - f|
\end{aligned}$$

therefore $\int |f_n - f| \to 0$. Lastly, by the integral triangle inequality

$$\Big| \int f_n - \int f \Big| = \Big| \int (f_n - f) \Big| \le \int |f_n - f| \to 0,$$

therefore $\int f_n \to \int f$.

JPE, Sept 2009 and Oct 1990 Assume that $\{f_k\}$ and f are in $L^1(\mathbb{R}^n)$ and $f_k \to f$ pointwise a.e. and $\int_{\mathbb{R}^n} |f_k| \to \int_{\mathbb{R}^n} |f|$. Show that for any measurable set $E \subset \mathbb{R}^n$

$$\int_E f_k\, dm \to \int_E f\, dm$$

By the triangle inequality

$$\big| |f_k| - |f_k - f| \big| \le |f|.$$

Since $f \in L^1(\mathbb{R}^n)$, the Lebesgue Dominated Convergence gives

$$\int_{\mathbb{R}^n} |f_k| - \int_{\mathbb{R}^n} |f_k - f| = \int_{\mathbb{R}^n} \big(|f_k| - |f_k - f|\big) \xrightarrow[k\to\infty]{} \int_{\mathbb{R}^n} |f|$$

therefore $\int_{\mathbb{R}^n} |f_k - f| \to 0$. Lastly by the integral triangle inequality

$$\left|\int_E f_k - \int_E f\right| = \left|\int_E (f_k - f)\right| \le \int_E |f_k - f| \le \int_{\mathbb{R}^n} |f_k - f| \to 0.$$

JPE, May 2009. Let $f \in L^1([0,1])$ be real-valued. Prove the following statements:
(a) $x^k f(x) \in L^1([0,1])$ for all $k \in \mathbb{N}$.
(b) $\lim_{k\to\infty} \int_0^1 x^k f(x)\,dx = 0$.
(c) If $\lim_{x\uparrow 1} f(x) = a$ for some real number a, then

$$\lim_{k\to\infty} k \int_0^1 x^k f(x)\,dx = a.$$

(a) $|x^k f(x)| \le |f(x)|$, hence $x^k f(x) \in L^1([0,1])$.

(b) We have $x^k f(x) \to 0$ as $k \to \infty$ for all $x \in [0,1)$. Then the claim follows from Part (a) and the Lebesgue Dominated Convergence.

(c) An elegant solution exists when f is continuously differentiable. Since $\lim \frac{k+1}{k} = 1$, we can replace the factor k with $k+1$ and then integrate by parts:

$$(k+1)\int_0^1 x^k f(x)\,dx = \int_0^1 f(x)\,dx^{k+1} = x^{k+1} f(x)\Big|_0^1 - \int_0^1 x^{k+1} f'(x)\,dx.$$

The first terms is

$$x^{k+1} f(x)\Big|_0^1 = 1^{k+1} f(1) - 0^{k+1} f(0) = a,$$

and the second converges to zero, due to Part (b).

Next we outline a solution for an arbitrary $f \in L^1([0,1])$.
(i) Choose small $\varepsilon > 0$ and $\delta > 0$ such that $|f(x) - a| < \varepsilon$ for all $x \in (1-\delta, 1]$.

(ii) Show that $\lim_{k\to\infty} k \int_0^{1-\delta} x^k f(x)\,dx = 0$ by using the Lebesgue Dominated Convergence. Note that

$$\sup_{k\ge 1} \sup_{x\in[0,1-\delta)} kx^k = \sup_{k\ge 1} k(1-\delta)^k < \infty,$$

which provides an integrable upper bound. Now it is enough to prove that

$$\lim_{k\to\infty} k \int_{1-\delta}^1 x^k f(x)\,dx = a.$$

(iii) Note that $a - \varepsilon < f(x) < a + \varepsilon$ on the interval $[1-\delta, 1]$, thus

$$(a-\varepsilon)k \int_{1-\delta}^{1} x^k \, dx \le k \int_{1-\delta}^{1} x^k f(x) \, dx \le (a+\varepsilon)k \int_{1-\delta}^{1} x^k \, dx.$$

Computing the integral

$$\int_{1-\delta}^{1} x^k \, dx = \frac{1-(1-\delta)^{k+1}}{k+1}$$

gives

$$\frac{(a-\varepsilon)k}{k+1}\left[1-(1-\delta)^{k+1}\right] \le k \int_{1-\delta}^{1} x^k f(x) \, dx \le \frac{(a+\varepsilon)k}{k+1}\left[1-(1-\delta)^{k+1}\right]$$

Taking the limit $k \to \infty$ shows that the middle integral will be eventually "squeezed" between $a - \varepsilon$ and $a + \varepsilon$. Since $\varepsilon > 0$ is arbitrary, Part (c) follows.

JPE, May 2009. Suppose that f_n is a sequence of non-negative Lebesgue measurable functions on $(0, 10)$ such that $f_n(x) \to f(x)$ for almost all $x \in (0, 10)$. Let $F(x) = \int_0^x f \, dm$ and $F_n(x) = \int_0^x f_n \, dm$. Prove that

$$\int_0^{10} (f + F) \, dm \le \liminf_{n\to\infty} \int_0^{10} (f_n + F_n) \, dm.$$

We apply Fatou's Lemma twice. First,

$$F(x) = \int_0^x f \, dm = \int_0^x \liminf f_n \, dm \le \liminf \int_0^x f_n \, dm = \liminf F_n(x).$$

Second,

$$\begin{aligned}
\int_0^{10} (f + F) \, dm &\le \int_0^{10} (f + \liminf F_n) \, dm \\
&= \int_0^{10} \liminf (f_n + F_n) \, dm \\
&\le \liminf \int_0^{10} (f_n + F_n) \, dm.
\end{aligned}$$

JPE, Sept 2008. Let $f \in L^1(0, \infty)$. Prove that there is a sequence $x_n \to \infty$ such that $\lim_{n\to\infty} x_n f(x_n) = 0$.

Denote $c = \liminf_{x\to\infty} x|f(x)|$. If $c = 0$, then a sequence as above exists. If $c > 0$, then there exists $A > 0$ such that $x|f(x)| > c/2$ for all $x > A$. Then

$$\int_{(0,\infty)} |f| \, dm \ge \int_{(A,\infty)} |f| \, dm \ge \int_A^{\infty} \frac{c}{2x} \, dx = \infty,$$

which contradicts the assumption $f \in L^1(0,\infty)$.

JPE, May 2008 and Sept 2009. Is the following true or false?
There exists a sequence $\{f_n\}$ of functions in $L^1(0,\infty)$ such that $|f_n(x)| \leq 1$ for all x and for all n, $\lim_{n\to\infty} f_n(x) = 0$ for all x, and

$$\lim_{n\to\infty} \int_{(0,\infty)} f_n \, dm = 1.$$

True. An example: $f_n = n^{-1}\chi_{(0,n)}$.

JPE, May 2003. Is the following true or false?
There exists a sequence $\{f_n\}$ of functions in $L^1(0,1)$ such that $f_n \to 0$ pointwise and yet $\int_{[0,1]} f_n \, dm \to \infty$.

True. An example: $f_n = n^2\chi_{(0,n^{-1})}$.

JPE, May 2008 and Oct 1991. Is the following true or false?
There exists a sequence $\{g_n\}$ of functions on $[0,1]$ such that

$$\lim_{n\to\infty} \int_{[0,1]} g_n \, dm = 0$$

but $g_n(x)$ converges for no $x \in [0,1]$.

True. See "Amazing shrinking sliding rectangles" in the class notes. Note: in the 1991 version, the functions g_n must be continuous. This requires a slight modification of the "sliding rectangles" example.

JPE, Sept 2004. Is the following true or false?
There are measurable functions f_n, $n = 1, 2, \ldots$, and f on $[0,1]$ such that $f_n(x) \to f(x)$ for every $x \in [0,1]$, but $\int_{[0,1]} f_n \, dm \not\to \int_{[0,1]} f \, dm$.

True. Example: $f_n = n\chi_{(0,\frac{1}{n})}$ and $f = 0$.

JPE, May 2004. Let $f \geq 0$ on $[0,1]$ be measurable.
(a) Show that $\int_{[0,1]} f^n\, dm$ converges to a limit in $[0,\infty]$ as $n \to \infty$.
(b) If $\int_{[0,1]} f^n\, dm = C < \infty$ for all $n = 1, 2, \ldots$, then prove the existence of a measurable subset B of $[0,1]$ such that $f(x) = \chi_B(x)$ for almost every x.

(Question (b) was given, as a separate problem, in **JPE, Jan 1989.**)

JPE, May 1995. Let $f \geq 0$ on $[0,1]$ be measurable. Suppose that $\limsup_n \int_{[0,1]} f^n\, dm < \infty$. Prove that $f(x) \leq 1$ a.e. on $[0,1]$.

JPE, Sept 1993. Let $f \geq 0$ on $[0,1]$ be measurable. Prove that $\lim_n \int_{[0,1]} f^n\, dm$ exists (as a finite number) if and only if $m(\{x \in [0,1] \colon f(x) > 1\}) = 0$.

See also a much harder version of this problem in Section on the L^p spaces.

We solve the 2004 problem. The 1995 and 1993 problems will follow as a side result.

Part (a): Let us partition $[0,1]$ into four subsets:

$$\begin{aligned} G &= \{x \in [0,1] \colon f(x) > 1\} \\ E &= \{x \in [0,1] \colon f(x) = 1\} \\ L &= \{x \in [0,1] \colon 0 < f(x) < 1\} \\ Z &= \{x \in [0,1] \colon f(x) = 0\}. \end{aligned}$$

Note that

$$\int_E f^n\, dm = m(E) \qquad \text{and} \qquad \int_Z f^n\, dm = 0$$

for all $n = 1, 2, \ldots$. Next, we have

$$\begin{aligned} & f(x) < f^2(x) < f^3(x) < \ldots \to \infty && \text{for all } x \in G \\ & f(x) > f^2(x) > f^3(x) > \ldots \to 0 && \text{for all } x \in L. \end{aligned}$$

On the subset $G \subset [0,1]$ we apply the Lebesgue Monotone Convergence:

$$\lim_{n\to\infty} \int_G f^n\, dm = \int_G \infty\, dm = \infty \cdot m(G).$$

On the subset $L \subset [0,1]$, we have $f^n(x) < 1$ for all n and all $x \in L$, hence we can apply the Lebesgue Dominated Convergence:

$$\lim_{n\to\infty} \int_L f^n\, dm = \int_L 0\, dm = 0.$$

Putting it all together gives

$$\lim_{n\to\infty} \int_{[0,1]} f^n\, dm = \infty \cdot m(G) + m(E).$$

solving Part (a). Note that

$$\infty \cdot m(G) = \begin{cases} \infty & \text{if } \ m(G) > 0 \\ 0 & \text{if } \ m(G) = 0 \end{cases}$$

which solves the 1995 problem, too.
Part (b): The assumption $\int_{[0,1]} f^n \, dm = C < \infty$ implies that

$$\lim_{n\to\infty} \int_{[0,1]} f^n \, dm = C < \infty$$

thus we must have $m(G) = 0$. In addition, we must have $m(L) = 0$. Indeed, if $m(L) > 0$, then we would have

$$f(x) > f^2(x) \ (\forall x \in L) \quad \Rightarrow \quad \int_L f \, dm > \int_L f^2 \, dm \quad \Rightarrow \quad \int_{[0,1]} f \, dm > \int_{[0,1]} f^2 \, dm$$

As a result, $f = \chi_E$ a.e.

JPE, May 2001. Let f and f_n, $n = 1, 2, \ldots$, be non-negative measurable functions on $[0, 1]$ such that f_n converges pointwise to f. Under each of the following additional assumptions, either prove that $\int_0^1 f \, dm \to \int_0^1 f \, dm$ or show that this is not generally true.
(a) $f_n \geq f$ and $f_n \in L^1([0, 1])$ for all n.
(b) $f_n \geq f_{n+1}$ for all n.
(c) $f_n \leq f$ for all n.
(Question (b) was also given in **JPE, Sept 1993 and Jan 1989**.)
(Question (c) was also given in **JPE, Sept 1995**.)

(a) False: $f_n = n\chi_{(0,1/n)}$ and $f = 0$.

(b) False: $f_n = 1/(nx)$ and $f = 0$. (Note that $\int_0^1 f_n \, dm = \infty$ for all n.)

(c) True. First note that we cannot apply Lebesgue Dominated Convergence, because f may not be integrable. We proceed as follows. On the one hand,

$$f_n \leq f \quad \Rightarrow \quad \int_0^1 f_n \, dm \leq \int_0^1 f \, dm \quad \Rightarrow \quad \limsup \int_0^1 f_n \, dm \leq \int_0^1 f \, dm$$

On the other hand, by Fatou's Lemma

$$\int_0^1 f \, dm = \int_0^1 \liminf f_n \, dm \leq \liminf \int_0^1 f_n \, dm,$$

thus $\lim \int_0^1 f_n \, dm$ exists and is equal to $\int_0^1 f \, dm$.

JPE, May 1999. Let $f \in L^1(\mathbb{R})$ and $E_1 \subset E_2 \subset \cdots$ be measurable subsets of $\mathbb{R}$. Prove that

$$\lim_{n\to\infty} \int_{E_n} f\, dm$$

exists.

The above limit equals $\int_E f\, dm$, where $E = \cup_{n\geq 1} E_n$. Indeed, we have

$$\left| \int_E f\, dm - \int_{E_n} f\, dm \right| = \left| \int_{E\setminus E_n} f\, dm \right| \leq \int_{E\setminus E_n} |f|\, dm.$$

We know that $\mu(A) = \int_A |f|\, dm$ is a measure on $\mathbb{R}$, and $\mu(\mathbb{R}) = \int_{\mathbb{R}} |f|\, dm < \infty$. Thus by the continuity

$$\int_{E\setminus E_n} |f|\, dm = \mu(E \setminus E_n) \to \mu(E \setminus \cup_{n=1}^{\infty} E_n) = \mu(\emptyset) = 0.$$

JPE, Sept 1995, May 1995 and May 1989. Let $\alpha > 2$ be a real number. Define

$$E = \left\{ x \in [0,1] \colon \left| x - \frac{p}{q} \right| < q^{-\alpha} \text{ for infinitely many } p, q \in \mathbb{N}^2 \right\}.$$

Prove that $m(E) = 0$.

In the May 1995 and May 1989 versions, α was set to 3.

Denote

$$E_{p,q} = \left\{ x \in [0,1] \colon \left| x - \frac{p}{q} \right| < \frac{1}{q^{\alpha}} \right\}.$$

Note that $m(E_{p,q}) = 2/q^{\alpha}$, and $0 \leq p \leq q$. Thus

$$\sum_{p,q} m(E_{p,q}) \leq \sum_{q=1}^{\infty} \frac{2(q+1)}{q^{\alpha}} \leq \sum_{q=1}^{\infty} \frac{4}{q^{\alpha-1}} < \infty,$$

so the claim follows from the Borel-Cantelli lemma.

JPE, May 1995. Assume that f and f_n are measurable functions on $[0,1]$ and that $f_n \geq 0$ a.e. on $[0,1]$. Prove that

$$\int_{[0,1]} f_n e^{-f_n}\, dm \to \int_{[0,1]} f e^{-f}\, dm.$$

Note that it is not necessarily true that $\int_{[0,1]} f_n\, dm \to \int_{[0,1]} f\, dm$ under the given conditions. For example, let $f \equiv 0$ and $f_n = n\chi_{(0,\frac{1}{n})}$. Then we have $f_n(x) \to f(x)$ for all $x \in [0,1]$ but $\int_{[0,1]} f_n\, dm = 1 \neq 0 = \int_{[0,1]} f\, dm$.

It is then really amazing that $\int_{[0,1]} f_n e^{-f_n}\, dm \to \int_{[0,1]} f e^{-f}\, dm$.

The reason is that $f_n(x)e^{-f_n(x)} \to f(x)e^{-f(x)}$ a.e. (quite obviously), and the integrands here are bounded by one integrable function:

$$0 \le f_n(x)e^{-f_n(x)} \le g(x) = 1.$$

In fact, the bound can be tightened: the function te^{-t} for $t \ge 0$ reaches its maximum at $t = 1$ and its maximum value is $e^{-1} < 1$.

Now the Lebesgue Dominated Convergence applies and completes the solution.

JPE, Sept 1994. Let $(X, \mathfrak{M}, \mu)$ be a measure space with a positive measure μ. Let f be a non-negative function on X such that for any $n = 1, 2, \ldots$ there are two measurable functions g_n and h_n on X such that $g_n(x) \le f(x) \le h_n(x)$ for all $x \in X$, and $\int_X (h_n - g_n)\, d\mu < \frac{1}{n}$. Prove that f is measurable and

$$\int_X f\, d\mu = \lim_{n\to\infty} \int_X g_n\, d\mu.$$

Note that $h_n - g_n \ge 0$, hence $\int_X (h_n - g_n)\, d\mu = \|h_n - g_n\|_1$, so we have $\|h_n - g_n\|_1 < \frac{1}{n}$, i.e., $\|h_n - g_n\|_1 \to 0$. By a corollary proved in class, there is a subsequence n_k such that $h_{n_k} - g_{n_k} \to 0$ a.e.

A side note: we can arrive at the same conclusion without using the corollary or the 1-norm. We can first prove that $\int_X (h_n - g_n)\, d\mu \to 0$ implies $h_n - g_n \to 0$ in measure, and then refer to a homework exercise were we proved that convergence in measure implies the existence of a subsequence converging a.e.

Now since $h_{n_k}(x) - g_{n_k}(x) \to 0$ a.e. and $g_{n_k}(x) \le f(x) \le h_{n_k}(x)$, then $g_{n_k}(x) \to f(x)$ a.e. In other words, there is a full measure set $E \subset X$ such that $g_{n_k}(x) \to f(x)$ for all $x \in E$. Thus the restriction $f|_E$ of f to E is a limit of measurable functions, hence $f|_E$ is measurable. The values of f on the null set $X \setminus E$ do not affect the measurability of f, hence f is measurable on the whole space X.

Next we need to prove that $\int_X g_n\, d\mu$ exists in a certain sense. The function g_n is not necessarily non-negative, so the existence of $\int_X g_n\, d\mu$ is not a trivial issue. Since $h_n(x) \ge f(x) \ge 0$, we have

$$0 \le g_n^-(x) = \max\{0, -g_n(x)\} \le \max\{0, h_n(x) - g_n(x)\} = h_n(x) - g_n(x),$$

hence

$$\int_X g_n^-\, d\mu \le \int_X (h_n - g_n)\, d\mu < \infty.$$

This implies that $\int_X g_n\, d\mu$ exists, at least in the extended sense (defined in class); though note that its value may be $+\infty$. Also note that $f \ge 0$ and $f - g_n \ge 0$, so the existence of $\int_X f\, d\mu$ and $\int_X (f - g_n)\, d\mu$ need no justification. Now we have

$$f = g_n + (f - g_n) \qquad \Rightarrow \qquad \int_X f\, d\mu = \int_X g_n\, d\mu + \int_X (f - g_n)\, d\mu.$$

Note that

$$0 \le \int_X (f - g_n)\, d\mu \le \int_X (h_n - g_n)\, d\mu \le \frac{1}{n}.$$

Thus taking the limit $n \to \infty$ proves the last claim of the problem.

JPE, May 1990. Does there exist a sequence of functions $f_n \in L^1(\mathbb{R})$ that converges uniformly to zero on every compact set, but $\int_{\mathbb{R}} f_n\, dm = 1$ for all n?

Yes, Example: $f_n = \chi_{(n,n+1)}$.

JPE, May 1990. Let f be a non-negative function defined on $\mathbb{R}$. Assume that for all $n \ge 1$

$$\int_{\mathbb{R}} \frac{n^2}{n^2 + x^2} f(x)\, dm \le 1.$$

Show that $f \in L^1(\mathbb{R})$ and $\|f\|_1 \le 1$.

Note that for each $x \in \mathbb{R}$, the sequence $\frac{n^2}{n^2+x^2}$ monotonically increases and converges to 1, as $n \to \infty$. Thus the integrand $\frac{n^2}{n^2+x^2} f(x)$ monotonically increases (in n) and converges to $f(x)$ as $n \to \infty$. By the Lebesgue Monotone Convergence

$$\int_{\mathbb{R}} \frac{n^2}{n^2 + x^2} f(x)\, dm \to \int_{\mathbb{R}} f(x)\, dm = \|f\|_1.$$

JPE, Sept 1989. Let f_n be a sequence of continuous functions Lebesgue integrable on $[0, \infty)$ which converges uniformly to a function f Lebesgue integrable on $[0, \infty)$. Is it true that

$$\lim_{n \to \infty} \int_0^{\infty} |f(x) - f_n(x)|\, dx = 0.$$

False. Example: we set $f(x) \equiv 0$ and define f_n by $f_n(x) = \frac{1}{n} - \frac{x}{n^2}$ for $x \in (0, n)$ and $f(x) = 0$ for $x \ge n$. Then $\int_0^{\infty} |f(x) - f_n(x)|\, dx = \frac{1}{2}$ for all n.

5 Lebesgue integral: "equipartitions"

Note: in all the problems of this section the function f must be real-valued, even though this assumption is NOT made explicitly in any of them, for some reason.

JPE, May 2011. Let $f \in L^1(-\infty, \infty)$. Let $\{a_n\}$ be a sequence of strictly positive numbers, i.e., $a_n > 0$ for any n, such that $\sum_{n=1}^{\infty} a_n = 1$. Prove that there exists a partition of $\mathbb{R}$ into measurable sets $\{E_n\}_{n=1}^{\infty}$ such that $\int_{E_n} f\, dm = a_m \int_{\mathbb{R}} f\, dm$ for all n.

Consider the function $F(x) = \int_{(-\infty,x)} f\,dm$. It is a continuous function on $\mathbb{R}$. Indeed,

$$F(x+\delta) - F(x) = \int_{[x,x+\delta)} f\,dm$$

By the triangle inequality

$$|F(x+\delta) - F(x)| \le \int_{[x,x+\delta)} |f|\,dm$$

We know that $\mu(E) = \int_E |f|\,dm$ is a measure in $\mathbb{R}$, and it is a finite measure because $\mu(\mathbb{R}) = \int_{\mathbb{R}} |f|\,dm < \infty$. Hence by the continuity we have

$$\int_{[x,x+\delta)} |f|\,dm = \mu\big([x,x+\delta)\big) \xrightarrow[\delta\to 0]{} \mu(\{x\}) = \int_{\{x\}} |f|\,dm = 0,$$

because $\{x\}$ is a singleton whose Lebesgue measure is zero. (Note: the continuity of $F(x)$ was a separate problem in **JPE, Sept 1995 and May 1992.**)

Next, we have the following limits:

$$\lim_{x\to-\infty} F(x) = 0, \qquad \lim_{x\to\infty} F(x) = I: \; = \int_{\mathbb{R}} f\,dm.$$

Indeed,

$$|F(x)| \le \int_{(\infty,x)} |f|\,dm$$

and by the continuity we have

$$\lim_{x\to-\infty} \int_{(-\infty,x)} |f|\,dm = \int_{\emptyset} |f|\,dm = 0,$$

Similarly,

$$|I - F(x)| \le \int_{[x,\infty)} |f|\,dm$$

and by the continuity we have

$$\lim_{x\to\infty} \int_{[x,\infty)} |f|\,dm = \int_{\emptyset} |f|\,dm = 0.$$

Now suppose first that $I \neq 0$. By the intermediate value theorem there are points $-\infty < x_1 < x_2 < \cdots < \infty$ such that

$$F(x_n) = (a_1 + \cdots + a_n)I \qquad \text{for all } \; n \ge 1.$$

Let $x_* = \lim_{n\to\infty} x_n$. If $x_* = \infty$, we choose $E_1 = (-\infty, x_1]$ and $E_n = (x_{n-1}, x_n]$ for all $n \ge 2$. If $x_* < \infty$, then $\int_{[x_*,\infty)} f\,dm = 0$, and we choose $E_1 = (-\infty, x_1] \cup [x_*, \infty)$ and $E_n = (x_{n-1}, x_n]$ for all $n \ge 2$.

In the case $I = 0$, i.e., $\int_{\mathbb{R}} f\,dm = 0$, we can just choose $E_2, E_3, \dots$ to be arbitrary disjoint null sets, and define $E_1 = \mathbb{R} \setminus \cup_{n=2}^{\infty} E_n$.

Note: in the case $\int_{\mathbb{R}} f\,dm = 0$ we can also choose sets E_n differently, so that each of them has positive measure. To this end we need to apply the previous argument to f^+ and f^- separately.

JPE, May 2003. Let f be integrable on $[0,1]$. Prove that there exists $c \in [0,1]$ such that $\int_{[0,c]} f\,dm = \int_{[c,1]} f\,dm$.

Similarly to the previous problem, let $F(c) = \int_{[0,c]} f\,dm - \int_{[c,1]} f\,dm$. It is a continuous function on $[0,1]$ and $F(0) = -F(1)$. If $F(0) \neq 0$, the existence of c such that $F(c) = 0$ follows from the intermediate value theorem. If $F(0) = 0$, we can choose $c = 0$ (or $c = 1$).

Note: in the case $F(0) = 0$ the choices $c = 0$ and $c = 1$ may be the only possible. Example: $f(x) = \sin(\pi x)$. We have $\int_0^1 f(x) = 0$, but for any $c \in (0,1)$ we have $\int_{[0,c]} f(x) > 0$ and $\int_{[c,1]} f(x) < 0$.

JPE, Sept 2007 and May 2001. Let $f \in L^1(\mathbb{R}^2)$. Prove that there exists a subset $E \subset \mathbb{R}^2$ such that

$$\int_E f\,dm = \int_{\mathbb{R}^2 \setminus E} f\,dm.$$

Similarly to the previous two problems, let

$$F(r) = \int_{D_r} f\,dm - \int_{\mathbb{R}^2 \setminus D_r} f\,dm,$$

where $D_r = \{(x,y)\colon x^2 + y^2 < r^2\}$ is the disk of radius r centered at the origin. Let us prove that $F(r)$ is a continuous function of r. Indeed,

$$F(r+\delta) - F(r) = 2 \int_{R_{r,r+\delta}} f\,dm$$

where $R_{r,r+\delta} = \{r^2 \le x^2 + y^2 < (r+\delta)^2\}$ is the ring with the inner radius r and the outer radius $r + \delta$. By the triangle inequality

$$|F(r+\delta) - F(r)| \le 2 \int_{R_{r,r+\delta}} |f|\,dm$$

We know that $\mu(E) = \int_E |f|\,dm$ is a measure in $\mathbb{R}^2$, and it is a finite measure because $\mu(\mathbb{R}^2) = \int_{\mathbb{R}^2} |f|\,dm < \infty$. Hence by the continuity we have

$$\lim_{\delta \to 0} \int_{R_{r,r+\delta}} |f|\,dm = \int_{R_{r,r}} |f|\,dm = 0,$$

because $R_{r,r} = \{x^2 + y^2 = r^2\}$ is a circle (a "ring" with zero width) whose Lebesgue measure is zero. Next, similarly to the previous problems,

$$F(0) = -\int_{\mathbb{R}^2} f\,dm \quad \text{and} \quad \lim_{r \to \infty} F(r) = \int_{\mathbb{R}^2} f\,dm = -F(0).$$

If $F(0) \neq 0$, the existence of r such that $F(r) = 0$ follows from the intermediate value theorem, and we can choose $E = D_r$. If $F(0) = 0$, we can choose E to be any null set (or make $\mathbb{R}^2 \setminus E$ a null set).

Note: in the case $\int_{\mathbb{R}^2} f\, dm = 0$ we can choose a more "interesting" set E so that $m(E) > 0$ and $m(E^c) > 0$. To this end we need to apply the above argument to f^+ and f^- separately.

6 Limits of integrals of specific functions

Preliminary Note. In many problems one has to use the following upper bound:

$$\left(1 + \frac{x}{n}\right)^n \le e^x \tag{1}$$

which holds for all

$$n \ge 1 \qquad \text{and} \qquad x > -n \tag{2}$$

This bound needs to be proved! You cannot just refer to Calculus or Advanced Calculus where this fact may have been mentioned.

To prove it, take logarithm of both sides in (1) and convert it to

$$\ln \frac{x+n}{n} \le \frac{x}{n} = \frac{x+n}{n} - 1$$

Since $t = \frac{x+n}{n} > 0$ due to (2), the last inequality can be written as

$$\ln t \le t - 1 \qquad\qquad \forall\, t > 0 \tag{3}$$

which is a standard fact which I believe can be used.

With a little more effort we can show that the sequence

$$a_n = \left(1 + \frac{x}{n}\right)^n$$

is monotonically increasing in n for all n satisfying (2). To prove this, treat n as a continuous variable and take the derivative:

$$\frac{da_n}{dn} = a_n \left[\ln\left(1 + \frac{x}{n}\right) - \frac{x}{x+n}\right]$$

Since $a_n > 0$, we have $\frac{da_n}{dn} \ge 0$ if and only if

$$\ln \frac{x+n}{n} \ge \frac{x}{x+n}$$

or equivalently

$$\ln \frac{n}{x+n} \le -\frac{x}{x+n} = \frac{n}{x+n} - 1$$

Since $t = \frac{n}{x+n}$ due to (2), we again reduce our inequality to the standard fact (3).

JPE, May 2012. Find

$$\lim_{n\to\infty} \int_{[0,n]} \left(1+\frac{x}{n}\right)^n e^{-\pi x}\, dx$$

First, we can replace the above limit with

$$\lim_{n\to\infty} \int_{[0,\infty)} \left(1+\frac{x}{n}\right)^n e^{-\pi x} \chi_{[0,n]}\, dx$$

Due to Preliminary Note, $\left(1+\frac{x}{n}\right)^n$ is a monotonically increasing sequence converging to e^x. Thus the integrand monotonically converges to $e^{x-\pi x}$. Then we can apply the Lebesgue Monotone Convergence Theorem and get

$$\lim_{n\to\infty} \int_{[0,n]} \left(1+\frac{x}{n}\right)^n e^{-\pi x}\, dx = \int_{[0,\infty)} e^{-(\pi-1)x} dx = \frac{1}{\pi-1}.$$

JPE, May 2011. Find $\lim_{n\to\infty} \int_0^\infty f_n\, dm$, where

$$f_n(x) = \frac{1}{1+x^{\frac{\sqrt{n}}{\ln(n+2011)}}}, \quad x \geq 0.$$

JPE, May 2006. Find

$$\lim_{n\to\infty} \int_0^\infty \frac{1}{1+x^{\frac{n}{\ln n + 2006}}}\, dx.$$

The above two problems are almost identical, we only solve the first one.

Note that

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\ln n + 2011} = \infty,$$

hence we have pointwise convergence $f_n(x) \to f(x)$, where

$$f(x) = \begin{cases} 1 & \text{for } \ 0 < x < 1 \\ \frac{1}{2} & \text{for } \ x = 1 \\ 0 & \text{for } \ x > 1 \end{cases}$$

Also we have $f_n(x) \leq 1$ for all $0 < x \leq 1$ and $n \geq 1$. Next,

$$\frac{\sqrt{n}}{\ln n + 2011} \geq 2$$

for all $n \geq n_0$ for some $n_0 \geq 1$, hence

$$f_n(x) \leq \frac{1}{1+x^2}$$

for all $x > 1$ and $n \geq n_0$. Thus we have a dominating integrable function:

$$f_n(x) \leq g(x) = \begin{cases} 1 & \text{for } \ 0 < x \leq 1 \\ \frac{1}{1+x^2} & \text{for } \ x > 1 \end{cases}$$

Thus by the Lebesgue Dominated Convergence, we have

$$\int_0^\infty f_n \, dm \to \int_0^\infty f \, dm = 1.$$

JPE, Sept 2010. Find

$$\lim_{k\to\infty} \int_0^\infty \frac{dx}{\left(1+\frac{x}{k}\right)^k \sqrt[k]{x}}.$$

We have, for all, $x > 0$

$$\lim_{k\to\infty} \frac{1}{\left(1+\frac{x}{k}\right)^k \sqrt[k]{x}} = \frac{1}{e^x \cdot 1} = e^{-x}$$

pointwise. Due to Preliminary Note, $\left(1+\frac{x}{k}\right)^k$ is a monotonically increasing sequence (converging to e^x). Thus for all $k \geq 2$ the integrand is bounded by

$$\frac{1}{\left(1+\frac{x}{k}\right)^k \sqrt[k]{x}} \leq \frac{1}{\left(1+\frac{x}{2}\right)^2}$$

which is an integrable function (it is in $L^1(0,\infty)$). Thus by Lebesgue Dominated Convergence our integrals converge to $\int_0^\infty e^{-x}\, dx = 1$.

JPE, May 2010 and May 1990. Is the following true or false?

$$\lim_{n\to\infty} \int_0^1 e^{x^2/n}\, dx = \int_0^1 \lim_{n\to\infty} e^{x^2/n}\, dx.$$

True. The integrand converges pointwise to $f(x) \equiv 1$, and is bounded by $g(x) = e^{x^2}$, which is a bounded, hence integrable function on $(0,1)$. Thus Lebesgue Dominated Convergence applies.

JPE, Sept 2009 and May 2008. Find

$$\lim_{n\to\infty} \int_0^{\pi/2} \sqrt{n \sin \tfrac{x}{n}}\, dx.$$

You can use the fact $0 \leq \sin\theta \leq \theta$ for $\theta \in [0, \pi/2]$.

The pointwise limit of the integrand is $\sqrt{x}$, which is also its upper bound. Hence by Lebesgue Dominated Convergence the limit is $\int_0^{\pi/2} \sqrt{x}\, dx = \frac{2}{3}(\frac{\pi}{2})^{3/2}$.

JPE, May 2007. Find

$$\lim_{n\to\infty} \int_{[0,n]} \left(\frac{\sin x}{x}\right)^n dm.$$

You can use the fact that $|\sin x| < x$ for all $x > 0$.

JPE, Sept 2006. Find

$$\lim_{n\to\infty} \int_0^\infty \left(\frac{\sin x}{x}\right)^n dx.$$

For all $x > 0$ we have $\left|\frac{\sin x}{x}\right| < 1$ hence

$$\left(\frac{\sin x}{x}\right)^n \xrightarrow[n\to\infty]{} 0 \qquad \text{pointwise.}$$

Next we find a good upper bound. First, the integrand is bounded by $g_1(x) = 1$. Second, for all $n \geq 2$ the integrand is bounded by $g_2(x) = 1/x^2$. Thus we have

$$\left|\left(\frac{\sin x}{x}\right)^n\right| \leq \min\{g_1(x), g_2(x)\} = \min\{1, 1/x^2\}$$

and this upper bound is integrable (it belongs to $L^1(0,\infty)$). Thus the Lebesgue Dominated Convergence applies. Therefore the above limit is zero.

JPE, Sept 2005. Find the limit and justify your answer

$$\lim_{n\to\infty} \int_1^\infty \frac{\ln(nx)}{x + x^2 \ln n}\, dx.$$

The integrand converges to $f(x) = \frac{1}{x^2}$ pointwise. Its upper bound is

$$\frac{\ln n + \ln x}{x + x^2 \ln n} \leq \frac{1}{x^2} + \frac{\ln x}{x^2},$$

which is an integrable function (it is in $L^1(1,\infty)$). Thus the Lebesgue Dominated Convergence applies and gives the limit $\int_1^\infty \frac{1}{x^2}\, dx = 1$.

JPE, May 2005. Find the limit and justify your answer

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(nx)}{1 + x^2}\, dx.$$

This problem is unusual, because the limit of the integral here is NOT equal to the integral of the pointwise limit function. In fact, the pointwise limit of the integrand does not even exist. We need to integrate by parts first:

$$\begin{aligned}\int_0^\infty \frac{\sin(nx)}{1+x^2}\,dx &= -\frac{1}{n}\int_0^\infty \frac{d\cos(nx)}{1+x^2}\,dx \\ &= -\frac{1}{n}\frac{\cos(nx)}{1+x^2}\Big|_0^\infty - \frac{1}{n}\int_0^\infty \frac{2x\cos(nx)}{(1+x^2)^2}\,dx \\ &= \frac{1}{n} - \frac{1}{n}\int_0^\infty \frac{2x\cos(nx)}{(1+x^2)^2}\,dx.\end{aligned}$$

For the last integral we have

$$\left|\int_0^\infty \frac{2x\cos(nx)}{(1+x^2)^2}\,dx\right| \le \int_0^\infty \frac{|2x\cos(nx)|}{(1+x^2)^2}\,dx \le \int_0^\infty \frac{2xdx}{(1+x^2)^2} = 1,$$

Hence our original integral converges to zero.

JPE, May 2003 and Sept 1999 Let $E = [0,\infty)$. Prove that $\lim_{n\to\infty}\int_E \frac{x}{1+x^n}\,dx$ exists and find its value.

We have $\lim_{n\to\infty}\frac{x}{1+x^n} = f(x)$ pointwise, where $f(x) = x$ for $x \in [0,1)$, $f(x) = 0$ for $x \in (1,\infty)$, and $f(1) = 0.5$. For all $n \ge 3$ our integrand is bounded by the function g such that $g(x) = x$ for $x \in [0,1]$ and $g(x) = \frac{x}{1+x^3}$ for $x \in (1,\infty)$. We can easily see that $g \in L^1(E)$, thus Lebesgue Dominated Convergence applies, and our integral converges to $\int_E f\,dm = 0.5$.

JPE, Sept 2002 and May 1994. Find the limit and justify your steps

$$\lim_{n\to\infty}\int_0^1 \frac{(nx)^2}{(1+x^2)^n}\,dx.$$

This problem is unusual, because the limit of the integral here is NOT equal to the integral of the pointwise limit function. In fact, the integrand converges to zero pointwise while the integral converges to infinity.

A quick inspection shows that the integrand takes high values for $x \sim 1/\sqrt{n}$. So let us estimate the integral from below by

$$\int_0^1 \frac{(nx)^2}{(1+x^2)^n}\,dx \ge \int_0^{1/\sqrt{n}} \frac{(nx)^2}{(1+x^2)^n}\,dx$$

Now for all $x \in [0, \frac{1}{\sqrt{n}}]$ we have

$$\frac{(nx)^2}{(1+x^2)^n} \ge \frac{(nx)^2}{(1+1/n)^n} \ge \frac{(nx)^2}{e}$$

because $(1+1/n)^n \le e$. Therefore

$$\int_0^1 \frac{(nx)^2}{(1+x^2)^n}\,dx \ge \frac{1}{e}\int_0^{1/\sqrt{n}} (nx)^2\,dx = \frac{\sqrt{n}}{3e} \to \infty.$$

(a) **JPE, Sept 2001.** Compute

$$\lim_{k\to\infty}\int_0^k \left(1-\frac{x}{k}\right)^k e^{x/3}\,dx.$$

(b) **JPE, May 2000.** Compute

$$\lim_{k\to\infty}\int_0^k \left(1-\frac{x}{k}\right)^k e^{x/2}\,dx.$$

(a) Due to Preliminary Note, $\left(1-\frac{x}{n}\right)^n$ is a monotonically increasing sequence converging to e^{-x}. Thus the integrand monotonically converges to $f(x)=e^{-2x/3}$ pointwise. By Lebesgue Monotone Convergence, our integral converges to $\int_0^\infty f\,dx=\frac{3}{2}$.

(b) The solution is almost identical to Part (a). The answer is 2.

JPE, May 2001 and May 1991. Compute

$$\lim_{n\to\infty}\int_0^1 \frac{nx^{n-1}}{2+x}\,dx.$$

(in **JPE, May 1991**, the value of the limit $(=\frac{1}{3})$ was given.)

This problem is unusual, because the limit of the integral here is NOT equal to the integral of the pointwise limit function. In fact, the integrand converges to zero pointwise while the integral converges to a positive number.

We can integrate by parts first:

$$\int_0^1 \frac{nx^{n-1}}{2+x}\,dx=\int_0^1\frac{dx^n}{2+x}=\left.\frac{x^n}{2+x}\right|_0^1+\int_0^1\frac{x^n}{(2+x)^2}\,dx$$

The first term on the right hand side is $\frac{1}{3}$. The last integral converges to zero, because the integrand converges to zero pointwise for all $x\in[0,1)$ and is bounded by an integrable function (in fact, it is bounded by a constant function: $\frac{x^n}{(2+x)^2}\le\frac{1}{4}$).

Alternatively, we can change variable $y=x^n$, which gives us

$$\lim_{n\to\infty}\int_0^1\frac{dy}{2+y^{1/n}}.$$

Now the integrand is bounded by $\frac{1}{2}$ and its pointwise limit is $\frac{1}{3}$, so the integral converges to $\frac{1}{3}$.

JPE, Sept 1999. Find the limit

$$\lim_{n\to\infty}\int_{[0,1]}\cos(x^n)\,dx.$$

For all $x \in [0,1)$ we have

$$x^n \to 0 \qquad \Rightarrow \qquad \cos(x^n) \to \cos 0 = 1,$$

and for $x = 1$ we have $\cos(x^n) = \cos 1$. Note that $|\cos(x^n)| \le 1$, hence the Lebesgue Dominated Convergence applies and gives

$$\lim_{n\to\infty} \int_{[0,1]} \cos(x^n)\, dm = \int_{[0,1]} 1\, dm = 1.$$

JPE, Sept 1998 and Sept 1993. Find the limit

$$\lim_{n\to\infty} \int_1^\infty \frac{\ln(1+nx)}{1+x^2\ln n}\, dx.$$

Note that

$$x \ge 1,\ n \ge 1 \qquad \Rightarrow \qquad nx < 1+nx \le 2nx,$$

hence

$$\frac{\ln n + \ln x}{1+x^2\ln n} \le \frac{\ln(1+nx)}{1+x^2\ln n} \le \frac{\ln n + \ln x + \ln 2}{1+x^2\ln n}.$$

By the squeeze theorem, for each $x \ge 1$

$$\lim_{n\to\infty} \frac{\ln(1+nx)}{1+x^2\ln n} = \frac{1}{x^2}.$$

We also have the following upper bound:

$$\frac{\ln(1+nx)}{1+x^2\ln n} \le \frac{1}{x^2} + \frac{\ln x}{x^2} + \frac{\ln 2}{x^2}$$

which is an integrable function on $(1,\infty)$. Thus the Lebesgue Dominated Convergence applies and gives

$$\lim_{n\to\infty} \int_1^\infty \frac{\ln(1+nx)}{1+x^2\ln n}\, dx = \int_1^\infty \frac{1}{x^2}\, dx = 1.$$

JPE, May 1998 and Sept 1995. Find the limit

$$\lim_{n\to\infty} \int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\, dx.$$

We have

$$\lim_{n\to\infty} \frac{\sin(x/n)}{(x/n)} = 1 \qquad \Rightarrow \qquad \lim_{n\to\infty} \frac{n\sin(x/n)}{x(1+x^2)} = \frac{1}{1+x^2}$$

for all $x > 0$, pointwise. Also note that $\sin\theta \le \theta$ for all $\theta \ge 0$, thus

$$\frac{n\sin(x/n)}{x(1+x^2)} \le \frac{1}{1+x^2},$$

which is an integrable function on $(0,\infty)$. Thus the Lebesgue Dominated Convergence applies and gives

$$\lim_{n\to\infty}\int_0^\infty \frac{n\sin(x/n)}{x(1+x^2)}\,dx. = \int_0^\infty \frac{1}{1+x^2}\,dx = \tan^{-1}x\Big|_0^\infty = \pi/2.$$

JPE, Sept 1997. Show that the hyperelliptic integral

$$\int_2^\infty \frac{x\,dx}{\sqrt{(x^2-\varepsilon^2)(x^2-1)(x-2)}}$$

converges to the elliptic integral

$$\int_2^\infty \frac{dx}{\sqrt{(x^2-1)(x-2)}}$$

as ε tends to zero.

For each $x > 2$, we have pointwise convergence, as $\varepsilon \to 0$,

$$\frac{x}{\sqrt{(x^2-\varepsilon^2)(x^2-1)(x-2)}} \to \frac{x}{\sqrt{x^2(x^2-1)(x-2)}} = \frac{1}{\sqrt{(x^2-1)(x-2)}}.$$

An upper bound for the integrand is

$$\frac{x}{\sqrt{(x^2-\varepsilon^2)(x^2-1)(x-2)}} \le \frac{x}{\sqrt{(x^2-1)(x^2-1)(x-2)}} = \frac{x}{(x^2-1)\sqrt{(x-2)}}.$$

This is an integrable function on $(2,\infty)$. Indeed, its tail, as $x\to\infty$, has asymptotic $\sim x^{-3/2}$, which is integrable. It grows to infinity near $x=2$ at rate $\sim \mathcal{O}(1/\sqrt{x-2})$, which is also integrable. Thus the Lebesgue Dominated Convergence applies and gives the result.

JPE, May 1994. Show that

$$\lim_{n\to\infty}\sum_{m=1}^\infty \frac{(-1)^m n}{n+nm^2+1} = \sum_{m=1}^\infty \frac{(-1)^m}{m^2+1}$$

Let $X=\mathbb{N}$ and μ be the counting measure. Then the above claim may be stated in terms of Lebesgue integrals:

$$\lim_{n\to\infty}\int_X \frac{(-1)^x n}{n+nx^2+1}\,d\mu = \int_X \frac{(-1)^x}{x^2+1}\,d\mu.$$

We have pointwise convergence

$$\frac{(-1)^x n}{n+nx^2+1} \to \frac{(-1)^x}{x^2+1} \qquad \text{as} \quad n \to \infty.$$

And for all $n \geq 1$ our integrand has an upper bound

$$\left|\frac{(-1)^x n}{n+nx^2+1}\right| \leq \frac{1}{x^2+1},$$

which is an integrable function on X, because $\sum_{m=1}^{\infty} \frac{1}{m^2+1} < \infty$. Thus the Lebesgue Dominated Convergence applies and gives the result.

JPE, May 1993. Evaluate

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{e^{-x^2/n}}{1+x^2}\, dx$$

For every $x \in \mathbb{R}$ the sequence $\{e^{-x^2/n}\}$ monotonically increases and converges to $e^0 = 1$. Thus by the Lebesgue Monotone Convergence

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \frac{e^{-x^2/n}}{1+x^2}\, dx = \int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \tan^{-1} x \Big|_{-\infty}^{\infty} = \pi.$$

JPE, Oct 1991. (1) Prove that $f(t) = \frac{t}{1+t^2}$ is a bounded function on $(-\infty, \infty)$.
(2) Let

$$f_n(x) = \frac{nx}{1+n^2x^2} \qquad (x \in [0,1],\ n = 1, 2, \ldots)$$

Then prove that $f_n(x)$ does not converge uniformly on $[0,1]$.
(3) Find $\lim_{n\to\infty} \int_0^1 f_n(x)\, dx$, if it exists.

(1) This a calculus exercise. Let us just denote the bound by $C = \max \frac{t}{1+t^2}$.

(2) For $x = 1/n$ we have $f_n(x) = 1/2$, thus there is no uniform convergence.

(3) We have pointwise convergence $f_n(x) \to 0$ for all $x \in [0,1]$ and a common upper bound $f_n(x) \leq C$. Thus Lebesgue Dominated Convergence applies.

JPE, Oct 1991. Let

$$f_n = n^\alpha \chi_{[\frac{1}{n+1}, \frac{1}{n}]}$$

where α is a constant, $1 \leq \alpha < 2$.
(i) Is there an integrable function Φ on $[0,1]$ such that $0 \leq f_n(x) \leq \Phi(x)$ for all n and $x \in [0,1]$?
(ii) Find $\lim_{n\to\infty} \int_{[0,1]} f_n\, dm$, if it exists.

(i) No. Even if we set

$$\Phi(x) = \sup_n f_n(x) = \sum_{n=1}^{\infty} n^\alpha \chi_{[\frac{1}{n+1},\frac{1}{n}]}$$

then

$$\int_{[0,1]} \Phi\, dm = \sum_{n=1}^{\infty} n^\alpha \big|[\tfrac{1}{n+1},\tfrac{1}{n}]\big| = \sum_{n=1}^{\infty} \frac{n^{\alpha-1}}{n+1} \geq \sum_{n=1}^{\infty} \frac{1}{n+1} = \infty.$$

(ii) The limit is zero:

$$\int_{[0,1]} f_n\, dm = \frac{n^{\alpha-1}}{n+1} \to 0.$$

JPE, Oct 1991. (i) Let f be a step function on a bounded interval $[a, b]$. Then prove that

$$(*) \qquad \lim_{n\to\infty} \int_a^b f(x)\sin(nx)\, dx.$$

(ii) Let f be a bounded measurable function on $[a, b]$. Then prove (*).
(iii) Let f be an integrable function on $(-\infty, \infty)$. Then prove (*).

This problem is almost a research project... We only sketch the solution. First, let $f(x) = \alpha\chi_{(c,d)}$ be a constant function on an interval (c, d). Then

$$\int_c^d f(x)\sin(nx)\, dx = -\tfrac{\alpha}{n}\, \cos(nx)\Big|_c^d = -\tfrac{\alpha}{n}\,(\cos(nd) - \cos(nc)),$$

hence

$$\left|\int_c^d f(x)\sin(nx)\, dx\right| \leq \tfrac{2|\alpha|}{n} \to 0.$$

Next, if f is a step function, then $f = \sum_{i=1}^N \alpha_i\chi_{(c_i,d_i)}$, and the result follows by additivity.

(ii) If f is a bounded measurable function on $[a, b]$, then $f \in L^1([a, b])$. We can approximate f by step functions (in the L^1 norm) and apply Part (i).

(iii) The same strategy: approximate f by step functions (in the L^1 norm) and apply Part (i).

JPE, May 1991. Let

$$f_n(x) = \frac{n^{1/4}e^{-x^2 n}}{1 + x^2}.$$

(a) Prove that $f_n \in L^1(0, \infty)$.
(b) Find $\lim_{n\to\infty} \int_{(0,\infty)} f_n\, dm$.

(a) Since $e^{-x^2 n} \le 1$, we have

$$\int_{(0,\infty)} f_n\,dm \le \int_{(0,\infty)} \frac{n^{1/4}}{1+x^2}\,dm = n^{1/4}\tan^{-1}x\Big|_0^\infty = n^{1/4}\pi/2,$$

hence $f_n \in L^1(0,\infty)$.

(b) We will need a common upper bound for all f_n's. To this end we combine $n^{1/4}$ with $e^{-x^2 n}$ as follows. The function $g(t) = te^{-t^4}$ is bounded on $(0,\infty)$. Indeed, by a standard Calculus-I argument its maximum is achieved at $t = 1/\sqrt{2}$ and its maximum value is $C = \frac{1}{\sqrt{2}}\,e^{-1/4}$. Therefore

$$x^{1/2}n^{1/4}e^{-x^2 n} \le C \qquad \Rightarrow \qquad n^{1/4}e^{-x^2 n} \le C/\sqrt{x}.$$

So we have

$$\int_{(0,\infty)} f_n\,dm \le \int_{(0,\infty)} \frac{C/\sqrt{x}}{1+x^2}\,dm < \infty.$$

The last integral is finite, because the integrand has tail $\mathcal{O}(x^{-5/2})$, as $x \to \infty$, which is integrable. And it grows to infinity near $x = 0$ at a rate $\sim \mathcal{O}(1/\sqrt{x})$, which is also integrable. Thus $f_n \in L^1(0,\infty)$ and f_n are bounded by one integrable function. It is easy to see that $\lim_{n\to\infty} f_n(x) = 0$ for each $x > 0$, thus by the Lebesgue Dominated Convergence

$$\lim_{n\to\infty}\int_{(0,\infty)} f_n\,dm = \int_{(0,\infty)} 0\,dm = 0.$$

JPE, May 1990. Let

$$f_n(x) = \sqrt{n}xe^{-nx^3}.$$

(i) Find the maximum value assumed by f_n in the interval $[0,1]$.
(ii) Find $\lim_{n\to\infty}\int_{[0,1]} f_n\,dm$.

(i) This is a Calculus-I problem, never mind. And we do not need it for Part (ii).

(ii) We need a common upper bound on all f_n's. Just like in the previous problem, the function $g(t) = te^{-t^2}$ is bounded on $(0,\infty)$. Indeed, by a standard Calculus-I argument its maximum is achieved at $t = 1/\sqrt{2}$ and its maximum value is $C = \frac{1}{\sqrt{2}}\,e^{-1/2}$. Therefore

$$n^{1/2}x^{3/2}e^{-nx^3} \le C \qquad \Rightarrow \qquad \sqrt{n}xe^{-nx^3} \le C/\sqrt{x}.$$

So we have

$$\int_{[0,1]} f_n\,dm \le \int_{[0,1]} \frac{C}{\sqrt{x}}\,dm = 2C.$$

It is easy to see that $\lim_{n\to\infty} f_n(x) = 0$ for each $x > 0$, thus by the Lebesgue Dominated Convergence

$$\lim_{n\to\infty}\int_{[0,1]} f_n\,dm = \int_{[0,1]} 0\,dm = 0.$$

7 Series of non-negative functions

JPE, Sept 2010. For all $n \in \mathbb{N}$ and $k \in \mathbb{N}$ let $f_{n,k}$ be a non-negative and measurable on $\mathbb{R}$ and assume that $\int_{\mathbb{R}} f_{n,k}\, dm \le \frac{1}{n^2}$. Show that

$$f\colon = \sum_{n=1}^{\infty} \liminf_{k\to\infty} f_{n,k} \in L^1(\mathbb{R}).$$

Here the series and lim inf are defined pointwise.

Since the integrands are non-negative functions, the summation and integration "commute". Next we use Fatou's Lemma:

$$\begin{aligned}\int_{\mathbb{R}} f\, dm &= \sum_{n=1}^{\infty} \int_{\mathbb{R}} \liminf_{k\to\infty} f_{n,k}\, dm \\ &\le \sum_{n=1}^{\infty} \liminf_{k\to\infty} \int_{\mathbb{R}} f_{n,k}\, dm \\ &\le \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty,\end{aligned}$$

thus $f \in L^1(\mathbb{R})$.

JPE, Sept 2008 and Sept 2007. Let f_n, $n = 1, 2, \ldots$, be a sequence of non-negative continuous functions on $\mathbb{R}$ such that $\int_{\mathbb{R}} f_n\, dm < \frac{1}{n^3}$. Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Prove that $f(x)$ is integrable on $\mathbb{R}$.

For non-negative functions, the summation and integration "commute", hence

$$\int_{\mathbb{R}} f\, dm = \sum_{n=1}^{\infty} \int_{\mathbb{R}} f_n\, dm < \sum_{n=1}^{\infty} \frac{1}{n^3} < \infty.$$

Note: the continuity of f_n's is not necessary, it is enough to assume their measurability. The bound $\frac{1}{n^3}$ can be relaxed to $\frac{1}{n^2}$ or $\frac{1}{n^{1+a}}$ for any $a > 0$.

JPE, May 2006. Let f_n, $n = 1, 2, \ldots$, be a sequence of non-negative Lebesgue measurable functions on $\mathbb{R}$ such that $\int f_n\, dm < \frac{1}{2^n}$. Let $f(x) = \sum_{n=1}^{\infty} f_n(x)$. Show that $f \in L^1(0, \infty)$.

For non-negative functions, the summation and integration "commute", hence

$$\int_{(0,\infty)} f\, dm = \sum_{n=1}^{\infty} \int_{(0,\infty)} f_n\, dm < \sum_{n=1}^{\infty} \frac{1}{2^n} < \infty.$$

Note: strangely, the domain of integration is not specified in the first formula of the problem. One may naturally assume that it is $\mathbb{R}$. But then it is not clear why they ask for $f \in L^1(0,\infty)$, instead of $f \in L^1(\mathbb{R})$.

JPE, Sept 1997. Evaluate

$$\sum_{n=0}^{\infty} \int_{1/2}^{\infty} (1-e^{-t})^n e^{-t^2}\, dt.$$

Since the integrands are non-negative functions, the summation and integration "commute", hence

$$\begin{aligned}
\sum_{n=0}^{\infty} \int_{1/2}^{\infty} (1-e^{-t})^n e^{-t^2}\, dt &= \int_{1/2}^{\infty} \sum_{n=0}^{\infty} (1-e^{-t})^n e^{-t^2}\, dt \\
&= \int_{1/2}^{\infty} e^{-t^2} \sum_{n=0}^{\infty} (1-e^{-t})^n\, dt \\
&= \int_{1/2}^{\infty} e^{-t^2} \frac{1}{1-(1-e^{-t})}\, dt \\
&= \int_{1/2}^{\infty} e^{t-t^2}\, dt
\end{aligned}$$

The last integral is computed by "completing the square" and changing variable:

$$\int_{1/2}^{\infty} e^{t-t^2}\, dt = e^{1/4} \int_{1/2}^{\infty} e^{-(t-1/2)^2}\, dt = e^{1/4} \int_{0}^{\infty} e^{-s^2}\, ds = e^{1/4}\sqrt{\pi}/2.$$

Note: apparently it is assumed here that the students know the value of the integral $\int_0^\infty e^{-s^2}\, ds$, or can quickly compute it.

JPE, Sept 1995 and Oct 1991. Evaluate

$$\sum_{n=0}^{\infty} \int_{0}^{\pi/2} (1-\sqrt{\sin x})^n \cos x\, dx.$$

Since the integrands are non-negative functions, the summation and integration "commute", hence

$$\sum_{n=0}^{\infty}\int_0^{\pi/2}(1-\sqrt{\sin x})^n\cos x\,dx = \int_0^{\pi/2}\sum_{n=0}^{\infty}(1-\sqrt{\sin x})^n\cos x\,dx$$
$$= \int_0^{\pi/2}\cos x\sum_{n=0}^{\infty}(1-\sqrt{\sin x})^n\,dx$$
$$= \int_0^{\pi/2}\cos x\frac{1}{1-(1-\sqrt{\sin x})}\,dx$$
$$= \int_0^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}\,dx$$

The last integral is rather elementary:

$$\int_0^{\pi/2}\frac{\cos x}{\sqrt{\sin x}}\,dx = \int_0^{\pi/2}\frac{d\sin x}{\sqrt{\sin x}} = 2\sqrt{\sin x}\Big|_0^{\pi/2} = 2.$$

8 Riemann integral vs Lebesgue integral

JPE, Sept 2011. Is it true that the characteristic function of the Cantor set is Lebesgue integrable in $[0,1]$ but not Riemann integrable?

False. The characteristic function of the Cantor set is continuous on the complement to the Cantor set (that complement consists of open intervals on which the function is identically zero). Thus the set of discontinuity points is exactly the Cantor set, which measure zero. This implies Riemann integrability.

JPE, Sept 2004. Let $f\colon [0,1]\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } \ x \ \text{ is irrational} \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that f is measurable.
(ii) Is f Lebesgue integrable? If yes, find its Lebesgue integral.
(iii) Is f Riemann integrable? If yes, find its Riemann integral.

(i) just like in some homework exercises.

(ii) yes, because f is measurable and bounded. Changing f on a set of measure zero will not affect its Lebesgue integral, so we can replace f with $g(x)=\sqrt{x}$ for all $x\in[0,1]$. Now

$$\int_{[0,1]} f\,dm = \int_{[0,1]} g\,dm = \int_0^1\sqrt{x}\,dx = \frac{2}{3}.$$

(iii) no, because f is discontinuous at every $x > 0$, hence the set of the discontinuity points has positive Lebesgue measure.

JPE, Sept 2001. Let $f\colon [0,1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \sqrt{x} & \text{if } \ x \ \text{ is rational} \\ 0 & \text{otherwise} \end{cases}$$

(i) Show that f is measurable.
(ii) Prove or disprove that f is of bounded variation on $[0,1]$.
(iii) Is f Lebesgue integrable but not Riemann integrable?

(i) just like in some homework exercises.

(ii) see some other section...

(iii) yes. It is Lebesgue integrable because it is measurable and bounded. It is not Riemann integrable because f is discontinuous at every $x > 0$, hence the set of the discontinuity points has positive Lebesgue measure.

9 L^p spaces: general

JPE, May 2012. Let $p \in [1,\infty)$. Suppose that $f_n \in L^p([0,1])$ converges a.e. to $f \in L^p([0,1])$. Show that f_n converges to f in $L^p([0,1])$ if and only if $\|f_n\|_p \to \|f\|_p$.

Note: this is the most popular JPE problem. Its variants were given in the following exams: **JPE, May 2005, Sept 1998, and Jan 1989** (for $p = 2$), and **JPE, May 2003, Sept 2002, Sept 2001, Sept 1999, May 1995, Sept 1994, and Sept 1993** (for $p = 1$). Sometimes $[0,1]$ is replaced with $\mathbb{R}$ or with an unspecified measurable subset $E \subset \mathbb{R}$. See also a clever "extended" version of this problem given in JPE, May 1994, in this section.

The "only if" part is simple: just use the triangle inequality

$$\|f_n - f\|_p \geq \big|\|f_n\|_p - \|f\|_p\big|,$$

hence $\|f_n - f\|_p \to 0$ implies $\|f_n\|_p \to \|f\|_p$.

We now turn to the difficult "if" part. Consider first the simpler case $p = 1$. There is a relatively fast solution that goes as follows. By the triangle inequality

$$\big||f_n| - |f_n - f|\big| \leq |f|.$$

Since $f \in L^1([0,1])$, the Lebesgue Dominated Convergence gives

$$\int |f_n| - \int |f_n - f| = \int \big(|f_n| - |f_n - f|\big) \xrightarrow[n\to\infty]{} \int |f|$$

therefore $\int |f_n - f| \to 0$, i.e., $f_n \to f$ in L^1. However, this solution does not seem to generalize to the $p > 1$ case. Hence we present another solution below.

By the triangle inequality $|f_n - f| \le |f_n| + |f|$, hence

$$|f_n| + |f| - |f_n - f| \ge 0.$$

Now by Fatou's Lemma (as in the proof of Lebesgue Dominated Convergence)

$$\begin{aligned} 2\int |f| &= \int \liminf(|f_n| + |f| - |f_n - f|) \\ &\le \liminf \int (|f_n| + |f| - |f_n - f|) \\ &= \liminf \int |f_n| + \int |f| - \int |f_n - f| \\ &= 2\int |f| - \limsup \int |f_n - f|. \end{aligned}$$

Therefore $\limsup \int |f_n - f| \le 0$, which implies $\int |f_n - f| \to 0$, i.e., $f_n \to f$ in L^1.

In the case $p > 1$ we use the convexity of the function $\varphi(x) = |x|^p$ on the entire real line $-\infty < x < \infty$ and Jensen's inequality to get

$$\left|\frac{f_n - f}{2}\right|^p = \left|\frac{f_n + (-f)}{2}\right|^p \le \frac{|f_n|^p + |-f|^p}{2} = \frac{|f_n|^p + |f|^p}{2}$$

hence

$$2^{p-1}|f_n|^p + 2^{p-1}|f|^p - |f_n - f|^p \ge 0;$$

after that the previous solution for $p = 1$ is repeated almost verbatim.

JPE, May 2012 and Sept 2004. Are the following true of false?
(a) If $f \in L^p([0,1])$ for all $p \in (1,\infty)$, then $f \in L^\infty([0,1])$.
(b) If $1 \le p < q < \infty$, then $L^q([1,\infty)) \subset L^p([1,\infty))$.

(a) False. Counterexample: $f = \ln x$.

(b) False. Counterexample: $f = x^{-r}$ for any $\frac{1}{q} < r < \frac{1}{p}$.

JPE, Sept 2010 and Sept 1988. Is the following true of false?
Suppose $f_n \in L^1(0,1)$ for all $n \in \mathbb{N}$, that $f_n(x) \to g(x)$ for almost every $x \in [0,1]$ and that $f_n \to f$ in $L^1(0,1)$. Then $f(x) = g(x)$ for almost every $x \in [0,1]$.

True. By a corollary proved in class: If $f_n \to f$ in the L^p norm ($p \in [1,\infty)$), then there is a subsequence $\{f_{n_k}\}$ such that $f_{n_k} \to f$ a.e. So the subsequence f_{n_k} converges a.e. to f and to g. This implies $f = g$ a.e.

JPE, Sept 2009. Is the following true of false?
If $\{f_k\} \in L^p(\mathbb{R}^n) \cap L^r(\mathbb{R}^n)$ for some $p, r \in [1,\infty)$, $f_k \to g$ in $L^p(\mathbb{R}^n)$ and $f_k \to h$ in $L^r(\mathbb{R}^n)$. Then $g(x) = h(x)$ a.e. in $\mathbb{R}^n$.

True. By a corollary proved in class: If $f_k \to g$ in the L^p norm ($p \in [1, \infty)$), then there is a subsequence $\{f_{k_m}\}$ such that $f_{k_m} \to g$ a.e. Now that subsequence converges to h in the L^r norm, so by the same corollary there is a subsubsequence $\{f_{k_{m_s}}\}$ such that $f_{k_{m_s}} \to h$ a.e. Hence the same sequence of functions, $\{f_{k_{m_s}}\}$, converges to both g and h a.e. This implies $g = h$ a.e.

JPE, Sept 2010. Let $f_n \in L^1(0,1) \cap L^2(0,1)$ for all $n \in \mathbb{N}$. Prove or disprove:
(a) If $\|f_n\|_1 \to 0$, then $\|f_n\|_2 \to 0$.
(b) If $\|f_n\|_2 \to 0$, then $\|f_n\|_1 \to 0$.

(a) False. Example: $f_n = n\chi_{[0,n^{-2}]}$. Here $\|f\|_1 = \frac{1}{n} \to 0$, but $\|f_n\|_2 = 1$ ($\forall n \in \mathbb{N}$).

(b) True. By the Schwarz inequality,

$$\|f_n\|_1 = \int |f_n| \cdot 1 \le \left[\int |f_n|^2\right]^{1/2} \left[\int 1^2\right]^{1/2} = \|f_n\|_2$$

hence $\|f_n\|_2 \to 0$ implies $\|f_n\|_1 \to 0$.

JPE, Sept 2010. Let f be a positive measurable function defined on a measurable set E with $m(E) < \infty$. Prove that

$$\Big(\int_E f\,dm\Big)\Big(\int_E \frac{1}{f}\,dm\Big) \ge m^2(E).$$

By the Schwarz inequality,

$$m(E) = \int_E 1\,dm = \int_E \sqrt{f} \cdot \frac{1}{\sqrt{f}}\,dm \le \Big(\int_E f\,dm\Big)^{1/2} \Big(\int_E \frac{1}{f}\,dm\Big)^{1/2}.$$

JPE, May 2010 and May 2004. (a) Let f be measurable on $[0,1]$. For $1 \le p < \infty$ define

$$g(p) = \Big(\int_0^1 |f(x)|^p\,dx\Big)^{1/p}.$$

Show that g is non-decreasing on $[1, \infty)$.
(b) Assume in addition that $f \notin L^\infty(0,1)$. Show that $\lim_{p\to\infty} g(p) = \infty$.

(a) Let $p < q$. By the Hölder inequality

$$\int_0^1 |f(x)|^p \cdot 1\,dx \le \Big(\int_0^1 \big[|f(x)|^p\big]^{q/p}\,dx\Big)^{p/q} \Big(\int_0^1 1^{q'}\,dx\Big)^{1/q'}$$
$$= \Big(\int_0^1 |f(x)|^q\,dx\Big)^{p/q}.$$

where $q' = q/(q-p)$ is just the exponent conjugate to q/p. Now raising both sides to the power $1/p$ proves that $g(p) \le g(q)$.

(b) For any $A > 0$ we have $m_A\colon = m\{x\colon |f(x)| > A\} > 0$, hence

$$g(p) \ge \left(\int_{\{x\colon |f(x)|>A\}} |f(x)|^p\,dx\right)^{1/p} \ge [A^p m_A]^{1/p} = A m_A^{1/p}.$$

Taking the limit $p \to \infty$ gives $\liminf_{p\to\infty} g(p) \ge A$. Since A is arbitrary, we have $\lim_{p\to\infty} g(p) = \infty$.

JPE, May 2009. Let $(X, \mathcal{M}, \mu)$ be a measure space with a positive, finite measure μ. Consider a function $f \in L^\infty(\mu)$ such that $\|f\|_\infty > 0$.
(a) Show that, for every positive ε, the set $\{x\colon |f(x)| > \|f\|_\infty - \varepsilon\}$ has positive measure.
(b) Show that

$$\lim_{n\to\infty} \|f\|_n = \lim_{n\to\infty} \frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} = \|f\|_\infty.$$

(The convergence $\|f\|_n \to \|f\|_\infty$ was also given, as a separate problem, in **JPE, May 1998, Sept 1995, May 1994, and May 1990.**)

(a) Trivial. Let us just denote $m_\varepsilon\colon = \{x\colon |f(x)| > \|f\|_\infty - \varepsilon\} > 0$.

(b) First,

$$\|f\|_n \le \left[\|f\|_\infty^n \mu(X)\right]^{1/n} = \|f\|_\infty \mu(X)^{1/n}.$$

Taking the limit $n \to \infty$ gives $\limsup \|f\|_n \le \|f\|_\infty$. On the other hand, for any $\varepsilon > 0$ we have

$$\|f\|_n \ge \left[(\|f\|_\infty - \varepsilon)^n m_\varepsilon\right]^{1/n} = (\|f\|_\infty - \varepsilon) m_\varepsilon^{1/n}.$$

Taking the limit $n \to \infty$ gives $\liminf \|f\|_n \ge \|f\|_\infty - \varepsilon$. Since $\varepsilon > 0$ is arbitrary, we have $\liminf \|f\|_n \ge \|f\|_\infty$. Thus $\lim \|f\|_n = \|f\|_\infty$.

Next, we have

$$\|f\|_{n+1}^{n+1} = \int_X |f|^{n+1}\,d\mu \le \int_X \|f\|_\infty |f|^n\,d\mu \le \|f\|_\infty \|f\|_n^n,$$

thus

$$\limsup \frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} \le \|f\|_\infty.$$

On the other hand, by the Hölder inequality

$$\|f\|_n^n = \int_X |f|^n \cdot 1\,d\mu \le \left[\int_X |f|^{n+1}\,d\mu\right]^{\frac{n}{n+1}} \cdot \left[\int_X 1^{n+1}\,d\mu\right]^{\frac{1}{n+1}} = \|f\|_{n+1}^n \mu(X)^{\frac{1}{n+1}}$$

hence

$$\frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} \ge \frac{\|f\|_{n+1}}{\mu(X)^{\frac{1}{n+1}}}$$

and taking the limit $n \to \infty$ gives

$$\liminf \frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} \ge \lim \frac{\|f\|_{n+1}}{\mu(X)^{\frac{1}{n+1}}} = \|f\|_\infty$$

(because $\|f\|_{n+1} \to \|f\|_\infty$ and $\mu(X)^{\frac{1}{n+1}} \to 1$). Thus $\lim \frac{\|f\|_{n+1}^{n+1}}{\|f\|_n^n} = \|f\|_\infty$.

JPE, May 2001. Show by example that there exist two functions $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$ such that $f+g$ is neither in $L^1(\mathbb{R})$ nor in $L^2(\mathbb{R})$.

Here: $f = \chi_{(0,1)}/\sqrt{x}$ and $g = \chi_{(1,\infty)}/x$.

JPE, May 2000. Find a function $f \in L^1(\mathbb{R})$ which does not belong to any $L^p(\mathbb{R})$ with $p > 1$.

Here: $f = \chi_{(0,1)}/(x \ln^2 x)$.

JPE, Sept 1997. (i) Prove that convergence in L^1 implies convergence in measure. (ii) Is the converse true?

(i) By way of contradiction, if f_n converges to f in L^1 but not in measure, then $\exists \varepsilon > 0$ such that for all $N \ge 1$ there is $n > N$ such that

$$\mu\{x \colon |f_n(x) - f(x)| > \varepsilon\} > \varepsilon.$$

In that case

$$\int_X |f_n - f| \, d\mu \ge \varepsilon \mu\{x \colon |f_n(x) - f(x)| > \varepsilon\} \ge \varepsilon^2 > 0,$$

thus f_n cannot converge to f in L^1, a contradiction.

(ii) The converse is false. Example: $X = [0,1]$, μ is the Lebesgue measure, and $f_n = n\chi_{(0,1/n)}$. Then $f_n \to f \equiv 0$ in measure, but not in L^1, because $\int_{[0,1]} |f_n - f| \, d\mu = 1$ for all n.

JPE, May 1997. Find a function f on $(0,\infty)$ such that $f \in L^p(0,\infty)$ if and only if $1 < p < 2$.

Here:

$$f(x) = \begin{cases} 1/\sqrt{x} & \text{if } \ x \in (0,1] \\ 1/x & \text{if } \ x \in (1,\infty) \end{cases}$$

JPE, May 1995. Suppose that $f_n \to f$ in $L^1([0,1])$ and that $h\colon \mathbb{R} \to \mathbb{R}$ is a continuous function. Prove or disprove: $h \circ f_n \to h \circ f$ in $L^1([0,1])$.

This is false. Example: $f \equiv 0$, $f_n = \sqrt{n}\,\chi_{[0,\frac{1}{n}]}$, and $h(x) = x^2$.

JPE, May 1995. Let $\mathcal{M}$ be a subspace of $L^2([0,1])$ with the following property: there is a constant C so that if $f \in \mathcal{M}$, then $|f(x)| \le C\|f\|_2$ for a.e. $x \in [0,1]$. Let $f_1, \dots, f_n$ be an orthonormal set in $\mathcal{M}$.
(a) Show that $\sum_{j=1}^n |f_j(x)|^2 \le C^2$ for a.e. $x \in [0,1]$.
(b) Show that the dimension of $\mathcal{M}$ is bounded above by C^2.

(a) An orthonormal set $f_1, \dots, f_n$ satisfies $\int_{[0,1]} f_i \overline{f_j}\, dm = \delta_{ij}$, which is the Kronecker delta symbol (the bar here means complex conjugate). Now let $c_1, \dots, c_n$ be any complex numbers. Then $c_1 f_1 + \cdots + c_n f_n \in \mathcal{M}$, hence

$$|c_1 f_1(x) + \cdots + c_n f_n(x)| \le C\|c_1 f_1 + \cdots + c_n f_n\|_2 = C\left[|c_1|^2 + \cdots + |c_n|^2\right]^{1/2}$$

for a.e. $x \in \mathcal{M}$. Setting $c_1 = \overline{f_1(x)}, \dots, c_n = \overline{f_n(x)}$ and squaring gives

$$\Big(\sum_{j=1}^n |f_j(x)|^2\Big)^2 \le C^2 \sum_{j=1}^n |f_j(x)|^2,$$

which completes the proof of (a).

(b) Integrating the inequality in (a) gives $n \le C^2$, thus the cardinality of any orthonormal set does not exceed C^2.

Comment: the subspace $\mathcal{M}$ cannot contain all functions with the given property (i.e., there is a constant C so that $|f(x)| \le C\|f\|_2$ for a.e. $x \in [0,1]$). For example, let $C = 10$ and consider two functions: $f = 1 + 9\chi_{(0,\varepsilon)}$ and $g = -1 + 9\chi_{(0,\varepsilon)}$. Note that $\|f\|_2 > 1$ and $\|g\|_2 > 1$, and at the same time $|f(x)| \le 10$ and $|g(x)| \le 10$ for all $x \in [0,1]$; hence f and g have the above property. Now $f + g = 18\chi_{(0,\varepsilon)}$, so we have $\|f + g\|_2 = 18\sqrt{\varepsilon}$ and ess-sup$(|f + g|) = 18$, thus $f + g$ does NOT have the above property (provided ε is sufficiently small).

JPE, May 1994. Let $f_n \in L^1(\mathbb{R})$ for $n = 1, 2, \dots$ and $f_n \to f$ a.e. as $n \to \infty$. Suppose $\|f_n\|_1 \to A$ as $n \to \infty$.
(a) Show that f is integrable on $\mathbb{R}$ and $\|f - f_n\|_1 \to A - \|f\|_1$.
(b) Must $\|f\|_1 = A$? Give a proof or a counterexample.

Note: this problem resembles the most popular JPE problem (see May 2012 in this section), but it goes a little further: it allows the limit 1-norm of f_n to exceed that of f, so that some "mass" of f_n "falls through cracks" and "does not reach" f.

(a) The convergence $f_n \to f$ a.e. implies $|f_n| \to |f|$ a.e., so by Fatou's Lemma

$$\int |f| = \int \liminf |f_n| \le \liminf \int |f_n| = A < \infty$$

therefore $f \in L^1(\mathbb{R})$.

Next, by the triangle inequality

$$\big||f_n| - |f_n - f|\big| \le |f|.$$

Since $f \in L^1(\mathbb{R})$, the Lebesgue Dominated Convergence gives

$$\int |f_n| - \int |f_n - f| = \int \big(|f_n| - |f_n - f|\big) \xrightarrow[n\to\infty]{} \int |f|$$

therefore

$$\|f_n - f\|_1 = \int |f_n - f| \to A - \|f\|_1.$$

(b) Not necessarily true. Example: $f_n = n\chi_{(0,1/n)}$. We have $f_n \to f \equiv 0$ pointwise and $A = \lim \|f_n\|_1 = 1$, but $\|f\|_1 = 0 \ne 1 = A$.

JPE, Sept 1993. Prove or disprove the following statement:
If $(X, \mathfrak{M}, \mu)$ is a measure space, φ is a convex function on (a, b) and $f\colon X \to (a, b)$ is an integrable function, i.e. $f \in L^1_\mu(X)$, then

$$\varphi\Big(\int_X f\,d\mu\Big) \le \int_X (\varphi \circ f)\,d\mu.$$

This looks very much like Jensen's inequality, but a crucial assumption is missing: $\mu(X) = 1$. Without this assumption, the statement is false. Example: $\varphi = x^2$, $X = [0, 10]$, μ the Lebesgue measure, and $f \equiv 1$. Then $\varphi\Big(\int_X f\,d\mu\Big) = 10^2 = 100$ and $\int_X (\varphi \circ f)\,d\mu = 10$.

JPE, May 1993. Let $f \in L^2(\mathbb{R})$. Prove that

$$\lim_{n\to\infty} \int_n^{n+1} f\,dm = 0.$$

First, by the triangle inequality

$$\left|\int_n^{n+1} f\,dm\right| \le \int_n^{n+1} |f|\,dm.$$

Now by the Schwarz inequality

$$\int_n^{n+1} |f| \cdot 1\,dm \le \Big(\int_n^{n+1} |f|^2\,dm\Big)^{1/2} \Big(\int_n^{n+1} 1^2\,dm\Big)^{1/2} = \Big(\int_n^{n+1} |f|^2\,dm\Big)^{1/2}.$$

Next we show that the last integral converges to zero, as $n \to \infty$. Indeed, recall that $\mu(E) = \int_E |f|^2\,dm$ is a measure on $\mathbb{R}$, and it is finite because $\mu(\mathbb{R}) = \int_{\mathbb{R}} |f|^2\,dm < \infty$. Hence by the continuity

$$\int_n^{n+1} |f|^2\,dm = \mu([n, n+1]) \le \mu([n, \infty)) \to \mu(\cap_{n=1}^{\infty}[n, \infty)) = \mu(\emptyset) = 0.$$

JPE, Oct 1991. Let $f \geq 0$ be Lebesgue measurable on $[0,1]$ and

$$\int_{[0,1]} f^2\, dm = \int_{[0,1]} f^3\, dm = \int_{[0,1]} f^4\, dm < \infty$$

Show that $f = f^2$ a.e.

By the Schwarz inequality

$$\int_{[0,1]} f \cdot f^2\, dm \leq \Big[\int_{[0,1]} f^2\, dm\Big]^{1/2} \cdot \Big[\int_{[0,1]} f^4\, dm\Big]^{1/2} = \int_{[0,1]} f^2\, dm.$$

Thus we have an equality in the Schwarz inequality, which is only possible if the two involved functions (in this case f and f^2) are proportional to each other. From the relation $af = bf^2$ a.e. with some $a, b \in \mathbb{R}$ (at least one of which differs from zero) we can easily show that $f = f^2$ a.e.

Note that $f = f^2$ a.e. implies that f only takes two values, 0 and 1 (a.e.), hence $f = \chi_A$ a.e. for a subset $A \subset [0,1]$.

JPE, Sept 1989. Let $p > 1$ and $f \in L^p\big([-1,1]\big)$, i.e.

$$\int_{[-1,1]} |f|^p\, dm < \infty.$$

(i) Prove that $f \in L^1([-1,1])$.
(ii) Let $I_n = (-\frac{1}{n}, \frac{1}{n})$ and $\gamma = \frac{p-1}{p}$. Then prove

$$\lim_{n\to\infty} n^\gamma \int_{I_n} |f|\, dm = 0.$$

(i) By the Hölder inequality

$$\int_{[-1,1]} |f| \cdot 1\, dm \leq \Big[\int_{[-1,1]} |f|^p\, dm\Big]^{1/p} \Big[\int_{[-1,1]} 1^q\, dm\Big]^{1/q}$$
$$= 2^{1/q} \|f\|_p < \infty,$$

where $q = \frac{p}{p-1}$ is the exponent conjugate to p. Note that $\frac{1}{q} = \gamma$.

(ii) Again by the Hölder inequality

$$\int_{I_n} |f| \cdot 1\, dm \leq \Big[\int_{I_n} |f|^p\, dm\Big]^{1/p} \Big[\int_{I_n} 1^q\, dm\Big]^{1/q} = \Big(\frac{2}{n}\Big)^\gamma \Big[\int_{I_n} |f|^p\, dm\Big]^{1/p}.$$

Thus

$$n^\gamma \int_{I_n} |f|\, dm \leq 2^\gamma \Big[\int_{I_n} |f|^p\, dm\Big]^{1/p}.$$

Next we show that the last integral converges to zero, as $n \to \infty$. Indeed, recall that $\mu(E) = \int_E |f|^p\, dm$ is a measure on $[-1, 1]$, it is finite because $\mu([-1, 1]) = \int_{[-1,1]} |f|^p\, dm < \infty$, hence by the continuity

$$\int_{I_n} |f|^p\, dm = \mu(I_n) \to \mu(\cap_{n=1}^{\infty} I_n) = \mu(\{0\}) = \int_{\{0\}} |f|^p\, dm = 0.$$

10 L^p spaces: estimation of specific integrals

JPE, May 2012. Let $f \in L^\infty([0, 1])$ and $\|f\|_\infty \leq 1$.
(a) Show that

$$\int_{[0,1]} \sqrt{1 - f^2}\, dm \leq \sqrt{1 - \Big(\int_{[0,1]} f\, dm\Big)^2}$$

(b) Describe the class of functions for which the equality takes place.

(a) Denote $g = \sqrt{1 - f^2}$. Observe that $f^2 + g^2 = 1$ a.e. Now by the Schwarz inequality, applied twice,

$$\Big[\int f\Big]^2 + \Big[\int g\Big]^2 \leq \int f^2 + \int g^2 = \int (f^2 + g^2) = \int 1 = 1.$$

(b) For strictly convex functions, equality in Jensen's inequality occurs if and only if the integrand is constant a.e., hence f must be constant a.e.

JPE, Sept 2011. Given that $\int_0^\infty e^{-x} \sin x\, dx = \frac{1}{2}$, prove that

$$\int_0^\infty e^{-x} \sqrt{3 + 2\sin x}\, dx \leq 2.$$

First of all, in the given formula $\int_0^\infty e^{-x} \sin x\, dx = \frac{1}{2}$ the integral is a Riemann integral. The corresponding Lebesgue integral $\int_{(0,\infty)} e^{-x} \sin x\, dm$ has the same value, because the function is absolutely integrable:

$$\int_0^\infty \big|e^{-x} \sin x\big|\, dx \leq \int_0^\infty e^{-x}\, dx = 1.$$

Next we apply the Schwarz inequality to $f = e^{-x/2}$ and $g = e^{-x/2}\sqrt{3 + 2\sin x}$:

$$\int_{(0,\infty)} e^{-x}\sqrt{3 + 2\sin x}\, dm \leq \Big[\int_{(0,\infty)} e^{-x}\, dm\Big]^{1/2} \Big[\int_{(0,\infty)} e^{-x}(3 + 2\sin x)\, dm\Big]^{1/2}$$
$$\leq 1^{1/2} \cdot (3 + 2 \cdot \tfrac{1}{2})^{1/2} = 2.$$

JPE, Sept 2009. (i) Show that

$$\int_1^\infty \frac{\sqrt[3]{1+x}}{x^2}\,dx \le \sqrt[3]{6}.$$

(ii) Show that the strict inequality holds in (i).

(i) By the Hölder inequality

$$\int_1^\infty \frac{\sqrt[3]{1+x}}{x} \cdot \frac{1}{x}\,dx \le \left[\int_1^\infty \frac{1+x}{x^3}\,dx\right]^{1/3} \left[\int_1^\infty \frac{1}{x^{3/2}}\,dx\right]^{2/3}$$
$$= \left(\frac{1}{2}+1\right)^{1/3} \cdot 2^{2/3} = \sqrt[3]{6}.$$

(ii) The Hölder inequality does not turn into an equality here, because the functions $\frac{1+x}{x^3}$ and $\frac{1}{x^{3/2}}$ are obviously not proportional to one another.

JPE, Sept 2008. Find all functions $g(x) \in L^3(0,1)$ satisfying the equation

$$\left(\int_{[0,1]} xg(x)\,dx\right)^3 = \frac{4}{25}\int_{[0,1]} g^3(x)\,dx.$$

By the Hölder inequality

$$\int_{[0,1]} xg(x)\,dx \le \left(\int_{[0,1]} x^{3/2}\,dx\right)^{2/3} \left(\int_{[0,1]} g^3(x)\,dx\right)^{1/3}$$
$$= \left(\frac{2}{5}\right)^{2/3} \left(\int_{[0,1]} g^3(x)\,dx\right)^{1/3}.$$

Since the Hölder inequality turns into an equality, the functions g^3 and $x^{3/2}$ must be proportional, i.e., $g(x) = c\sqrt{x}$, where $c \in \mathbb{C}$ is an arbitrary constant.

JPE, May 2008. Show that

$$\left(\int_0^1 \frac{x^3}{(1-x)^{1/5}}\,dx\right)^5 \le \frac{16}{81}.$$

By the Hölder inequality

$$\int_0^1 \frac{x^3}{(1-x)^{1/5}}\,dx \le \left(\int_0^1 (x^3)^5\,dx\right)^{1/5} \left(\int_0^1 \frac{1}{[(1-x)^{1/5}]^{5/4}}\,dx\right)^{4/5}$$
$$= \left(\int_0^1 x^{15}\,dx\right)^{1/5} \left(\int_0^1 \frac{1}{(1-x)^{1/4}}\,dx\right)^{4/5}$$
$$= \left(\frac{1}{16}\right)^{1/5} \left(\frac{4}{3}\right)^{4/5} = \left(\frac{16}{81}\right)^{1/5}.$$

Note: a much better bound can be achieved by

$$\int_0^1 \frac{x^3}{(1-x)^{1/5}}\,dx \le \left(\int_0^1 (x^3)^{5/3}\,dx\right)^{3/5} \left(\int_0^1 \frac{1}{[(1-x)^{1/5}]^{5/2}}\,dx\right)^{2/5}$$
$$= \left(\int_0^1 x^5\,dx\right)^{3/5} \left(\int_0^1 \frac{1}{(1-x)^{1/2}}\,dx\right)^{2/5}$$
$$= \left(\frac{1}{6}\right)^{3/5} \left(\frac{2}{1}\right)^{2/5} = \left(\frac{1}{54}\right)^{1/5}$$

Here are the corresponding numerical values:

$$\left(\frac{16}{81}\right)^{1/5} \approx 0.723, \qquad \left(\frac{1}{54}\right)^{1/5} \approx 0.4503$$

and the exact value of the integral is

$$\int_0^1 \frac{x^3}{(1-x)^{1/5}}\,dx = \frac{5}{4} - \frac{5}{3} + \frac{15}{14} - \frac{5}{19} \approx 0.3916,$$

which can be easily found by change of variable $x = 1-y$. One may wonder why we care about applying Hölder inequality to get a rough upper bound for something that can be computed precisely by any Calculus-II student. Go figure...

JPE, Sept 2004. Prove that

$$\int_0^1 \sqrt{x^4+4x^2+3}\,dx \le \frac{2}{3}\sqrt{10}.$$

By the Schwarz inequality

$$\int_0^1 \sqrt{x^4+4x^2+3}\,dx = \int_0^1 \sqrt{x^2+3}\cdot\sqrt{x^2+1}\,dx$$
$$\le \left(\int_0^1 (x^2+3)\,dx\right)^{1/2} \left(\int_0^1 (x^2+1)\,dx\right)^{1/2}$$
$$= \sqrt{\frac{10}{3}}\cdot\sqrt{\frac{4}{3}} = \frac{2}{3}\sqrt{10}.$$

JPE, May 2001. (a) Let $a_1, \dots, a_n$ be positive numbers. Prove that their harmonic mean is bounded by their arithmetic mean, i.e.

$$\left(\frac{1}{n}\sum_{k=1}^{n}\frac{1}{a_k}\right)^{-1} \le \frac{1}{n}\sum_{k=1}^{n} a_k.$$

(b) Characterize the vectors for which equality holds in (a).

(a) Let $X = \{1, 2, \dots, n\}$ and μ be the counting measure on X. Let $f\colon X \to (0, \infty)$ be defined by $f(k) = \sqrt{a_k}$. Then by the Schwarz inequality

$$n = \int_X 1 \, d\mu = \int_X f \cdot \frac{1}{f} \, d\mu \le \sqrt{\int_X f^2 \, dx} \cdot \sqrt{\int_X \frac{1}{f^2} \, dx} = \sqrt{\sum_{k=1}^n a_k} \cdot \sqrt{\sum_{k=1}^n \frac{1}{a_k}}$$

(b) Equality holds whenever f is proportional to $1/f$, i.e., $a_k = c/a_k$ for some constant c and all $k = 1, \dots, n$. This implies $a_1 = \cdots = a_n$.

JPE, May 1999. (i) Prove that

$$\int_0^{\pi/2} \sqrt{x \sin x} \, dx \le \frac{\pi}{2\sqrt{2}}.$$

(ii) Prove that in fact the inequality is strict.

(i) By the Schwarz inequality

$$\begin{aligned} \int_0^{\pi/2} \sqrt{x \sin x} \, dx &\le \left(\int_0^{\pi/2} x \, dx \right)^{1/2} \left(\int_0^{\pi/2} \sin x \, dx \right)^{1/2} \\ &= \left(\frac{1}{2} \left(\frac{\pi}{2} \right)^2 \right)^{1/2} \cdot 1^{1/2} = \frac{\pi}{2\sqrt{2}}. \end{aligned}$$

(ii) The Schwarz inequality does not turn into an equality here, because the functions x and $\sin x$ are obviously not proportional to one another.

JPE, Sept 1997. Show that

$$\left(\int_0^1 \frac{x^{1/2}}{(1-x)^{1/3}} \, dx \right)^3 \le \frac{8}{5}.$$

By the Hölder inequality

$$\begin{aligned} \int_0^1 \frac{x^{1/2}}{(1-x)^{1/3}} \, dx &\le \left(\int_0^1 (x^{1/2})^3 \, dx \right)^{1/3} \left(\int_0^1 \frac{1}{[(1-x)^{1/3}]^{3/2}} \, dx \right)^{2/3} \\ &= \left(\int_0^1 x^{3/2} \, dx \right)^{1/3} \left(\int_0^1 \frac{1}{(1-x)^{1/2}} \, dx \right)^{2/3} \\ &= \left(\frac{2}{5} \right)^{1/3} \cdot 2^{2/3} = \left(\frac{8}{5} \right)^{1/3}. \end{aligned}$$

JPE, Sept 1994. Prove that

$$\int_1^\infty \frac{\sqrt{1 + x^3}}{x^4} \, dx \le \sqrt{\frac{7}{10}}.$$

By the Schwarz inequality

$$\int_1^\infty \frac{\sqrt{1+x^3}}{x^3}\,\frac{1}{x}\,dx \le \left[\int_1^\infty \frac{1+x^3}{x^6}\,dx\right]^{1/2}\left[\int_1^\infty \frac{1}{x^2}\,dx\right]^{1/2}$$
$$\le \sqrt{\frac{1}{5}+\frac{1}{2}}\cdot\sqrt{1} = \sqrt{\frac{7}{10}}.$$

JPE, Sept 1993. Prove that

$$\int_0^\infty e^{-x}\sqrt{1+\sin x}\,dx \le \sqrt{\frac{3}{2}}.$$

We apply the Schwarz inequality to $f = e^{-x/2}$ and $g = e^{-x/2}\sqrt{1+\sin x}$:

$$\int_0^\infty e^{-x}\sqrt{1+\sin x}\,dx \le \left[\int_0^\infty e^{-x}\,dx\right]^{1/2}\left[\int_0^\infty e^{-x}(1+\sin x)\,dx\right]^{1/2}$$
$$\le 1^{1/2}\cdot(1+\tfrac{1}{2})^{1/2} = \sqrt{\frac{3}{2}}.$$

Note that we needed to compute here the integral $\int_0^\infty e^{-x}\sin x\,dx = \frac{1}{2}$. Even though this is a routine Calculus-II exercise, it certainly takes valuable time during the exam. At a later JPE (Sept 2011) the value of this elementary integral was provided, to save time.

11 ℓ^p spaces

JPE, May 2000. Do there exist two sequences $(a_n) \in \ell^1$ and $(b_n) \in \ell^2$ such that $(a_n + b_n)$ is neither in ℓ^1 nor in ℓ^2?

No. Any sequence in $(a_n) \in \ell^1$ is also in ℓ^2. Indeed, there are only finitely many a_n's such that $|a_n| > 1$, and for all the other terms we have $|a_n|^2 \le |a_n|$. Hence the convergence of $\sum |a_n|$ implies the convergence of $\sum |a_n|^2$. Thus $(a_n + b_n)$ will be in ℓ^2.

JPE, May 1998. Does there exist a sequence (x_n) which is in $\ell^{1+\varepsilon}$ for all $\varepsilon > 0$ but which is not in ℓ^1?

Yes, $x_n = 1/n$.

JPE, May 1997. Suppose $1 \le p \le q \le \infty$. Prove or disprove: $\ell^p \subset \ell^q$.

The inclusion is true. If $p = \infty$, then $q = \infty$ and the inclusion is trivial. If $p < \infty$ and $(a_n) \in \ell^p$, then there are only finitely many a_n's such that $|a_n| > 1$. This implies $(a_n) \in \ell^\infty$. Now suppose $q < \infty$. Then for all a_n such that $|a_n| \leq 1$ we have $|a_n|^q \leq |a_n|^p$. Hence the convergence of $\sum |a_n|^p$ implies the convergence of $\sum |a_n|^q$.

JPE, May 1991. Find a sequence (a_n) which is in ℓ^3 but not in ℓ^2.

Here: $a_n = 1/\sqrt{n}$. (More generally: $a_n = n^{-b}$ with any $\frac{1}{3} < b \leq \frac{1}{2}$.)