Superselection sectors in quantum spin systems*

Pieter Naaijkens
Leibniz Universität Hannover

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Abstract

In certain quantum mechanical systems one can build superpositions of states whose relative phase is not observable. This is related to superselection sectors: the algebra of observables in such a situation acts as a direct sum of irreducible representations on a Hilbert space. Physically, this implies that there are certain global quantities that one cannot change with local operations, for example the total charge of the system.

Here I will discuss how superselection sectors arise in quantum spin systems, and how one can deal with them mathematically. As an example we apply some of these ideas to Kitaev's toric code model, to show how the analysis of the superselection sectors can be used to get a complete understanding of the "excitations" or "charges" in this model. In particular one can show that these excitations are so-called anyons.

These notes introduce the concept of superselection sectors of quantum spin systems, and discuss an application to the analysis of charges in Kitaev's toric code. The aim is to communicate the main ideas behind these topics: most proofs will be omitted or only sketched. The interested reader can find the technical details in the references. A basic knowledge of the mathematical theory of quantum spin systems is assumed, such as given in the lectures by Bruno Nachtergaele during this conference. Other references are, among others, [3, 4, 15, 12]. Parts of these lecture notes are largely taken from [12].

1 Superselection sectors

We will say that two representations π and ρ of a C^*-algebra are *unitary equivalent* (or simply equivalent) if there is a unitary operator $U : \mathcal{H}_\pi \to \mathcal{H}_\rho$ between the corresponding Hilbert spaces and in addition we have that $U\pi(A)U^* = \rho(A)$ for all $A \in \mathfrak{A}$. If this is the case, we also write $\pi \cong \rho$. In general, a C^*-algebra has many inequivalent representations. Here we discuss some of the consequences for quantum mechanical systems of the existence of inequivalent representations. In the next section we will consider an example of a system with inequivalent representations. This happens only for systems with infinitely many degrees of freedom. For finite systems the situation is different. For example, von Neumann [16] showed that there is only one irreducible representation of the canonical commutation relations $[P, Q] = i\hbar$ (up to unitary equivalence). The same is

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true for a spin-1/2 system. If we consider the algebra generated by S_x, S_y, S_z, satisfying $[S_i, S_j] = i\varepsilon_{ijk}S_k$ and $S_x^2 + S_y^2 + S_z^2 = \frac{3}{4}I$, each irreducible representation of these relations is unitary equivalent to the representation generated by the Pauli matrices [18]. A similar results is true for a finite number of copies of such systems. We will see that the existence of inequivalent representations has consequences for the superposition principle.

If $\pi : \mathfrak{A} \to \mathfrak{B}(\mathcal{H})$ is a representation of a C^*-algebra, there is an easy way to obtain different states on $\mathfrak{A}$. Take any vector $\psi \in \mathcal{H}$ of norm one. Then the assignment $A \mapsto \langle \psi, \pi(A)\psi \rangle$ defines a state. Such states are called *vector states* for the representation π. Note that by the GNS construction it is clear that *any* state can be realised as a vector state in some representation. Consider two states ω_1 and ω_2 that are both vector states for the same representation π. Hence there are vectors ψ_1 and ψ_2 such that $\omega_i(A) = \langle \psi_i, \pi(A)\psi_i \rangle$ for all $A \in \mathfrak{A}$. Consider now $\psi = \alpha\psi_1 + \beta\psi_2$ with $\alpha, \beta \in \mathbb{C}$ such that $|\alpha|^2 + |\beta|^2 = 1$ and both α and β are non-zero. Then $\omega(A) = \langle \psi, \pi(A)\psi \rangle$ again is a state. However, it may be the case that the resulting state is not pure (even if the ω_i are) and we have a mixture

$$\omega(A) = |\alpha|^2\omega_1(A) + |\beta|^2\omega_2(A). \tag{1.1}$$

If this is the case for any representation π in which ω_1 and ω_2 are vector states, we say that the two states are not *superposable* or not *coherent*. This situation was first analysed by Wick, Wightman and Wigner [17].

Theorem 1.1 ([2, Thm 6.1])**.** *Let ω_1 and ω_2 be pure states. Then they are not superposable if and only if the corresponding GNS representations π_{ω_1} and π_{ω_2} are inequivalent.*

Proof. Consider a representation π such that ω_1 and ω_2 are vector states in this representation. Write $\psi_i \in \mathcal{H}$ for the corresponding vectors. Then we can consider the subspaces $\mathcal{H}_i$ of $\mathcal{H}$, defined as the closure of $\pi(\mathfrak{A})\psi_i$. The projections on these subspaces will be denoted by P_i.

Note that ψ_i is, by definition, cyclic for the representation $\pi(\mathfrak{A})$ restricted to $\mathcal{H}_i$. Let us write π_i for these restricted representations. But since the vectors ψ_i implement the state, it follows that the representation π_i must be (unitary equivalent to) the GNS representations π_{ω_i}. Let $U : \mathcal{H}_1 \to \mathcal{H}_2$ be a bounded linear map such that $U\pi_1(A) = \pi_2(A)U$ for all $A \in \mathfrak{A}$. By first taking adjoints, we then see that $U^*U\pi_1(A) = \pi_1(A)U^*U$. By irreducibility of π_1 it follows that $U^*U = \lambda I$ for some $\lambda \in \mathbb{C}$. In fact, λ must be real since U^*U is self-adjoint. A similar argument holds for UU^*. Hence by rescaling we can choose U to be unitary, unless $U^*U = 0$. Hence non-zero maps U intertwining the representations only exist if π_1 and π_2 are unitarily equivalent.

To get back to the original setting, let $T \in \pi(\mathfrak{A})'$. Then P_2TP_1 can be identified with a map $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $U\pi_1(A) = \pi_2(A)U$. Conversely, any such map can be extended to an operator in $P_2\pi(\mathfrak{A})'P_1$. Hence

$$P_2\pi(\mathfrak{A})'P_1 = \{0\}$$

if and only if π_1 and π_2 are not unitarily equivalent.

Since $I \in \pi(\mathfrak{A})'$ it is clear that $P_1P_2 = P_2P_1 = 0$ if π_1 and π_2 are not equivalent. This implies that $\mathcal{H}_1$ and $\mathcal{H}_2$ are orthogonal subspaces of $\mathcal{H}$. Hence

$$\langle \psi_2, \pi(A)\psi_1 \rangle = 0 = \langle \psi_1, \pi(A) \rangle \psi_2 \rangle.$$

Consequently, if $\psi = \alpha\psi_1 + \beta\psi_2$ with $|\alpha|^2 + |\beta|^2 = 1$, then

$$\omega(A) := \langle \psi, \pi(A)\psi \rangle = |\alpha|^2 \omega_1(A) + |\beta|^2 \omega_2(A).$$

Hence ω_1 and ω_2 are not superposable.

Conversely, suppose that π_{ω_1} and π_{ω_2} are unitarily equivalent. Then there must be some unitary U in $\pi(\mathfrak{A})'$ such that $P_2 U P_1 \neq 0$. This is only possible if there are vectors $\varphi_i \in \mathcal{H}_i$ such that $\langle \varphi_2, U\varphi_1 \rangle \neq 0$. Since $\pi(\mathfrak{A})\psi_1$ is dense in $\mathcal{H}_1$ (and similarly for $\mathcal{H}_2$), there must be $A_1, A_2 \in \mathfrak{A}$ such that

$$\langle \pi(A_1)\psi_2, U\pi(A_2)\psi_1 \rangle \neq 0. \tag{1.2}$$

Set $\varphi = U\psi_1$. Since U commutes with $\pi(A)$ for every A, it follows that

$$\langle \varphi, \pi(A)\varphi \rangle = \omega_1(A).$$

Now consider the vector $\psi = \alpha\varphi + \beta\psi_2$. This induces a state ω, and we find

$$\omega(A) - |\alpha|^2\omega_1(A) - |\beta|^2\omega_2(A) = \overline{\alpha}\beta\langle U\psi_1, \pi(A)\psi_2\rangle + \alpha\overline{\beta}\langle \psi_2, \pi(A)U\psi_1\rangle. \tag{1.3}$$

Consider then $A = A_2^* A_1$, where A_1 and A_2 are as above. Then the right hand side of equation (1.3) becomes

$$2\,\mathrm{Re}(\alpha\overline{\beta}\langle \pi(A_1)\psi_2, U\pi(A_2)\psi_1\rangle).$$

By choosing a suitable multiple if A we can make the right hand side non-zero, because of equation (1.2). It follows that equation (1.1) does not hold. □

This result shows that as soon as a C^*-algebra has inequivalent representations, there are states that are not coherent. That is, there are pure states ω_1 and ω_2 such that a superposition of those states is never pure. The proof also makes clear that if we have vector states corresponding to inequivalent representations, there can never be a transition from one state to the other, not even by applying any operation available in $\mathfrak{A}$, because $\langle \psi_1, \pi(A)\psi_2 \rangle$ is zero. Such a rule that forbids such transitions is called a *superselection rule*. There are many different (but strongly related) notions of a superselection rule around, see for example [8] for a discussion.

As an example of irreducible representations we consider a spin-1/2 chain on the line, that is, $\Gamma = \mathbb{Z}$. We define a vector $|\{s_n\}\rangle$ by specifying the spin in the z-direction for each site n. In particular, we set ψ^+ to be the state where each $s_n = +1$, and ψ^-1 the state with each $s_n = -1$. That is, the states are those with all spins in the up direction, and all in the down direction. This induces states $\omega_\pm$ on the quasilocal algebra $\mathfrak{A}$.

Using the GNS representation we can then get representations $(\pi^\pm, \mathcal{H}^\pm, \psi^\pm)$. The representation works in the way one would expect, by acting with the Pauli matrices on the individual sites. It is however a good idea to look a bit more closely at what the Hilbert spaces are. The cyclic vectors are just the states $\psi^\pm$ defined above. According to the GNS construction, the Hilbert space is then generated by acting with *local* observables on this state. But a local observable can only flip a *finite* number of spins. This implies that the Hilbert space $\mathcal{H}^+$ is spanned by vectors $|\{s_n\}\rangle$ with only *finitely* many $s_n = -1$. Note that that this is a countable set, unlike the set of all possible sequences of ± 1. In a similar way the Hilbert space $\mathcal{H}^-$ is spanned by vectors with only finitely many spins in the up direction.

One can show that the representations $\pi^\pm$ are irreducible. Here we want to show that they are, however, *not* unitarily equivalent. To see this we consider the *polarization* operators, defined by

$$S_N = \frac{1}{2N+1} \sum_{n=-N}^{N} \sigma_n^z.$$

That is, it measures the average spin in the z-direction. Since in the Hilbert space $\mathcal{H}^+$ most spins are in the up direction, it follows that

$$\lim_{N\to\infty} \pi^+(S_N)\psi = \psi,$$

for all $\psi \in \mathcal{H}^+$. On the other hand, $\lim_{N\to\infty} \pi^-(S_N)\psi = -\psi$ for all $\psi \in \mathcal{H}^-$. It follows that

$$\lim_{N\to\infty} \langle \varphi_\pm, \pi^\pm(S_N)\psi_\pm \rangle = \pm\langle \varphi_\pm, \psi_\pm \rangle$$

for all $\varphi_\pm, \psi_\pm \in \mathcal{H}^\pm$.

To conclude the example, assume that π^+ and π^- are unitarily equivalent. Then there is a unitary $U : \mathcal{H}^+ \to \mathcal{H}^-$ such that $U\pi^-(A)U^* = \pi^+(A)$ for all $A \in \mathfrak{A}$. It follows that

$$\begin{aligned}\langle \varphi, \psi \rangle = \lim_{n\to\infty} \langle \varphi, \pi^+(S_N)\psi \rangle &= \lim_{n\to\infty} \langle U^*\varphi, \pi^-(S_N)U^*\psi \rangle \\ &= -\langle U^*\varphi, U^*\psi \rangle = -\langle \varphi, \psi \rangle.\end{aligned}$$

This is a contradiction.

2 The toric code

We now apply the results and techniques developed in this section to an important example in quantum information theory, the *toric code*. This model was first introduced by Kitaev [11]. The reason that it is called the toric code is two-fold: the model is often considered on a torus (i.e., as a finite system with periodic boundary conditions in the x and y direction) and it is an example of a *quantum code*. Quantum codes are used to store quantum information and correct errors. We will only make some brief comments later on this aspect.

Instead of on a torus, we will consider the model on an infinite plane. That is, consider the lattice $\mathbb{Z}^2$. The set Γ of sites is defined to be the *edges* between nearest-neighbour points in the lattice (see Figure 1). At each of these edges there is a spin-$1/2$ degree of freedom, with corresponding observable algebra $M_2(\mathbb{C})$. We can then define the quasi-local algebra $\mathfrak{A}(\Gamma)$ as before. Note that there is a natural action of the translation group, so that it makes sense to talk about translation invariant interactions or states.

There are two special subsets of sites that we will consider. For any vertex v there are in total four edges that begin or end in that vertex (see the picture). Such a set will be called a *star*. Similarly, one can define a plaquette, as the edges around a vertex in the dual lattice (see the solid lines in the picture). To a star s and a plaquette p we associate the following operators:

$$A_s = \bigotimes_{j\in s} \sigma_j^x, \quad B_p = \bigotimes_{j\in p} \sigma_j^z.$$

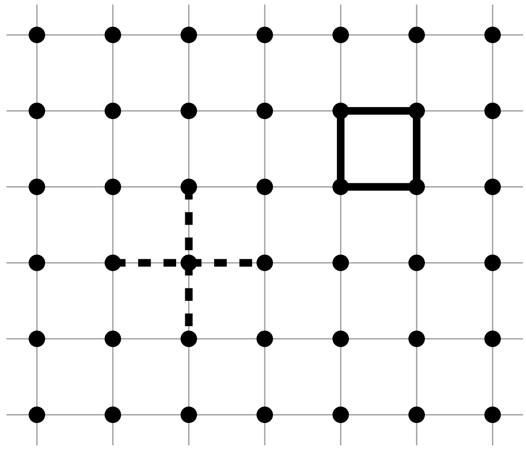

Figure 1: Lattice on which Kitaev's toric code is defined. The spin degrees of freedom live on the *edges* between the solid dots. Also indicated are a star (dashed lines) and a plaquette (solid lines). Picture from [13].

An important property is that $[A_s, B_p] = 0$ for any star s and plaquette p. This can be seen because a star and a plaquette always have an even number of edges in common. Commuting the operators at each edge give a minus sign, because of the anti-commutation of Pauli matrices. Since the number of minus signs is even the claim follows. Another property is that $A_s^2 = B_p^2 = I$.

The star and plaquette operators will be used to define the interactions of the model. Namely, for $\Lambda \in \mathcal{P}_f(\Gamma)$ we set

$$\Phi(\Lambda) = \begin{cases} -A_s & \Lambda = s \text{ for some star } s \\ -B_p & \Lambda = p \text{ for some plaquette } p \\ 0 & \text{else} \end{cases}.$$

Note that the interactions is of finite range and bounded. Moreover, it is translation invariant. It is then easy to show that this generates a one-parameter group of dynamics $t \mapsto \alpha_t$, and one can then talk about ground states. In the present situation these are those states ω_0 such that $-i\omega_0(X^*\delta(X)) \geq 0$ for all local operators X. Here δ is defined as

$$\delta(X) = \lim_{\Lambda \to \infty} [H_\Lambda, X]$$

for X local. Since the interaction is of finite range it is easy to see that this is well defined. The toric code model is very simple: the Hamiltonian consists of sums of mutually commuting terms and a ground state minimizes the expectation value of each of these terms individually.[1] This is one of the reasons why it is possible to describe the ground states explicitly, and one can show that in fact there is only one ground state.

Theorem 2.1 ([1])**.** *The toric code on the plane has a unique ground state ω_0. It is the unique state that satisfies $\omega_0(A_s) = \omega_0(B_p) = 1$ for all stars s and plaquettes p.*

[1]One says that the model is *frustration free* in this case.

The proof of this result is not difficult, but somewhat tedious, and we won't comment on it here. It should be remarked that we consider here a particular example of the toric code. The model is usually defined on a (compact) orientable surface, by drawing an oriented graph on it and assigning to each edge a qubit [11]. The ground space degeneracy is then 4^g, where g is the genus of the surface. Note that this is consistent with the theorem above, since the plane has genus zero. As another generalisation one can consider so-called quantum double models, introduced by Kitaev in the same paper [11]. One recovers the toric code if one takes the Hopf algebra $H = \mathbb{C}[\mathbb{Z}_2]$. Many of the results we discuss below can be generalised to such models, at least if it comes from the group algebra of a finite abelian group [9].

3 Superselection sectors in the toric code

Now consider the ground state ω_0 and its GNS representation $(\pi_0, \Omega, \mathcal{H})$. The representation π_0 is automatically faithful, so we can identify $\pi_0(A)$ with A, which we will do from now on.

It is well known that the toric code model has so-called anyonic excitations. Unlike bosons or fermions, such an excitation acquires a non-trivial phase as one is interchanged with the other. The goal is here to understand this in the context of superselection sectors. To this end it is illustrative how such excitations can be obtained from the ground state, which we understand well by the results mentioned in the previous section. It turns out that excitations always come in pairs in the toric code. To obtain such a pair, choose a path ξ on the lattice. The path is nothing but a collection of edges, and we write $j \in \xi$ for such an edge. One can also consider the dual lattice, and choose a path $\widehat{\xi}$ there. This is the same as a path connecting the plaquettes in the original lattices. We then identify $\widehat{\xi}$ with all those edges of the original lattice that this path crosses. To such paths we associate the following operators:

$$F_\xi = \bigotimes_{j \in \xi} \sigma_j^z, \quad F_{\widehat{\xi}} = \bigotimes_{j \in \widehat{\xi}} \sigma_x^j. \tag{3.1}$$

There is one important property that is easy to verify: these "path operators" commute with all star and plaquette operators, *except* those at the endpoints of the path. In the path on the lattice case, it *anti*-commutes with the star operators at the end. Similarly, in the dual path case there is anti-commutation with the plaquette operators. This can be seen by nothing that a path always has an even number of edges in common with a star, except at the endpoints), and then use the anti-commutation property of the Pauli matrices.

Now let Λ be a finite subset of the lattice and ξ a path inside Λ. Then we have that

$$H_\Lambda F_\xi \Omega = \left(-\sum_{s \subset \Lambda} A_s - \sum_{p \subset \Lambda} B_p \right) F_\xi \Omega = F_\xi (4 + H_\Lambda) \Omega,$$

where we used that the string operators anti-commute with the star operators at its endpoint. This calculation shows that the energy of $F_\xi \Omega$ is four higher than the ground state. This is precisely because of the anti-commutation with the star operators at the endpoint. Another way to think of it is that the constraint $A_s\Omega = \Omega$, which is easy to verify, is violated at the endpoints of the string. Hence it makes sense to interpret this as a pair of excitations, sitting at the end of ξ. For the dual path $\widehat{\xi}$ a similar situation is true, with plaquettes replacing the star operators. Finally, an important property of the toric code model is that the state $F_\xi\Omega$ does *not* depend on the path, but only on its endpoint. That is, if ξ' is another path with the same endpoints, then $F_\xi\Omega = F_{\xi'}\Omega$.

We can get a better understanding of these excitations using the theory of superselection sectors. The first step is to find examples of different sectors. We do this by first finding certain "charged" states, and then construct a corresponding representation using the GNS construction. We recall the following Lemma, which turns out the be very useful.

Lemma 3.1. *Let $\mathfrak{A} := \mathfrak{A}(\Gamma)$ be the quasi-local observable algebra of some spin system and suppose that ω_1 and ω_2 are pure states on $\mathfrak{A}$. Then the following criteria are equivalent:*

1. *The corresponding GNS representations π_1 and π_2 are equivalent.*
2. *For each $\varepsilon > 0$, there is a $\Lambda_\varepsilon \in \mathcal{P}_f(\Gamma)$ such that*
$$|\omega_1(A) - \omega_2(A)| < \varepsilon\|A\|,$$
for all $A \in \mathfrak{A}(\Lambda)$ with $\Lambda \in \mathcal{P}_f(\Lambda_\varepsilon^c)$.

Here Λ_ε^c is the complement of Λ_ε in Γ.

The Lemma as stated here is a specialisation of a more lemma to the case that we need here. The more general statement and a proof can be found in [3, Corollary 2.6.11].

We now have the necessary tools to construct different superselection sectors. As mentioned before, the path operators F_ξ create a *pair* of excitations. These excitations are conjugate charges, and their total charge is zero. Another way to say this is that the state of these two excitations is in the neutral charge sector. To create a *single* excitation, the idea is to move one of the ends of the paths all the way to infinity. In this way we are left with a single charged state.

To do this, choose a semi-infinite path ξ. The first n parts of the path will be denoted by ξ_n. Let A be a local observable. We can then define the following map:
$$\alpha(A) = \lim_{n\to\infty} F_{\xi_n} A F_{\xi_n}^*.$$
Because A is local, it follows that the right hand side becomes constant for n large enough, hence the limit converges. This defines a continuous map, and hence can be extended to all of $\mathfrak{A}$. In fact, it is not so difficult to show that it is an automorphism. We will also write α^z here, since it is obtained by acting with Pauli-z matrices. In a similar manner one can define automorphisms α^x by choosing a path on the *dual* lattice, and using the same construction. Finally, one can choose a path and a dual path at the same time (for convenience we will assume that they do not intersect), and conjugate with operators $F_\xi F_{\widehat{\xi}}$. That is, a combination of the previous cases. We write α^y for the result. For convenience α^0 will be the trivial automorphism. Note that $\omega_0 \circ \alpha^k$ for $k = x, y, z$ can be interpreted as a state where there is a *single* excitation, at the end of the path.

Theorem 3.2. *Let ρ^k be any of the automorphisms constructed above. Then the states $\omega_k := \omega_0 \circ \rho^k$ are all mutually inequivalent. Moreover, two states of the same type k, but defined with respect to different paths, are equivalent.*

Proof (sketch). Consider two states ω_k and ω_l with $k \neq l$. Choose a closed loop on the lattice (or possible one on the dual lattice), enclosing both endpoints of the semi-finite paths. Let W be the operator obtained by the tensor product of Pauli matrices σ_x around the loop (or σ_z for dual loops).[2] Then an easy calculation shows that we have $\rho_1(W) = \pm W = \mp\rho_2(W)$. Moreover, one can show that $\omega_0(W) = 1$. Since we can make the loop as big as we like, the Lemma above implies that the two states must be inequivalent. The operator W can be understood as measuring the charge in the region it encloses.

To show the remaining claim, one first shows that the state ω_k only depends on the *endpoint* of the path used to define ρ_k. This readily follows from the important property that $F^k_\xi \Omega = F^k_{\xi'}\Omega$ for any pair of paths ξ and ξ' having the same endpoints. Finally, consider two automorphisms ρ^k_i of the same type, and choose a path ξ from the endpoint of the first path to the endpoint of the second. Then by the independence of the path one sees that $\omega_0 \circ \rho_1$ and $\omega_0(F^k_\xi \rho_2(\cdot)(F^k_\xi)^*)$ are actually equal, and it follows that the GNS representations of $\omega_0 \circ \rho^k_i$ are equivalent. □

So there are at least four inequivalent classes of states (and hence superselection sectors), counting the ground state. Each of these states give the expectation values of the quasi-local observables in the presence of a *single* background charge, where we also consider the ground state as a separate charge (or more precisely, the absence of a charge).

Let ω be any of such states. To get back to the "usual" quantum mechanics picture, we construct the GNS representation $(\pi, \Omega, \mathcal{H})$. The vector states in $\mathcal{H}$ can be understood as all "configurations" with total charge given by the charge of ω. By acting with local operators $\pi(A)$ one can only create pairs of conjugate excitations (hence not changing the total charge), or for example move around the excitation ω. However, completely getting rid of the charge ω is not possible with such local operations. Indeed, from the constructing it is clear that this would require a non-local operation. The only thing that one can do with local operations is move the charge around, but one cannot get rid of it.

4 Analysis of the sectors

The excitations in the toric code are quite special. By the discussion above we can interpret the as quasi-particles. But they are not just ordinary particles, but rather special: they are anyons. Unlike bosons and fermions, states of anyons do not just acquire a sign, but possibly a phase or even some non-abelian operation. The anyons can be identified with different superselection sectors, by the construction above. But there is a lot more structure than just the labelling of different anyons, for example how they behave under interchange or under "fusion". That is, what happens if we bring two charges closely together? It turns out that a more careful of the superselection sectors reveals all this additional structure. The tools for such an analysis were developed by Doplicher,

[2]The operator W is essentially a so-called Wilson loop.

Haag and Roberts in the context of algebraic quantum field theory [6, 7], but the main ideas can be applied to spin systems as well. To illustrate the main ideas we apply them to the toric code model [13].

A C^*-algebra usually has very many inequivalent representations, most of which are not meaningful for physics. Hence it is natural to look at what other properties the physically relevant representations have. The representations constructed above have such an important property: they are *localizable* in a suitable sense. To get an understanding for this, consider the GNS representation $(\pi_0, \mathcal{H}, \Omega)$ of the ground state representation. Choose a cone-like region of the lattice (the shape is not so important, we just want to choose a path to infinity in the cone, that does not "spiral" around), and a path ξ to infinity inside the cone. This gives an automorphism α_ξ^k as before. Then, essentially by construction, $(\pi_0 \circ \alpha_\xi^k, \mathcal{H}, \Omega)$ is a GNS triple for the state ω_k. This representation is localized in Λ, in the sense that for *any* observable outside Λ, that is $A \in \mathfrak{A}(\Lambda^c)$, we have $\pi_0 \circ \alpha_\xi^k(A) = \pi_0(A)$. This follows from locality and the definition of the automorphism.

Now comes a crucial point. Suppose that we had chosen another cone Λ' and another path ξ' in this cone, in such a way that ξ and ξ' have the same endpoint. From the independence of the states $F_\rho\Omega$ for finite paths ρ, it follows that the states $\omega_0 \circ \alpha_\xi^k$ and $\omega_0 \circ \alpha_{\xi'}^k$ are in fact equal. By the uniqueness of the GNS representation, it then follows that there is a unitary U such that $U\pi_0 \circ \alpha_\xi(A)U^* = \pi_0 \circ \alpha_{\xi'}(A)$ for all $A \in \mathfrak{A}$. By conjugating with path operators it is not so difficult to see that even if the paths do not end at the same site, the corresponding representations are still unitarily equivalent. So to summarise, for each of the superselection sectors we constructed it is true that when one restricts to observables localised *outside* an *arbitrary* cone-like region Λ, the charged representations and the ground state representations are unitarily equivalent:

Theorem 4.1. *For each representation π in the equivalence class of one of the representations constructed above, it holds for* any *cone Λ that*

$$\pi_0 \restriction \mathfrak{A}(\Lambda^c) \cong \pi \restriction \mathfrak{A}(\Lambda^c).$$

Here Λ^c is the complement of Λ in Γ.

This can be interpreted as follows. The single charged states look very much like the ground state, as long as one restricts to observables *outside* any given cone. Indeed, the only way that one can detect the charge is by doing a Wilson-loop measurement (as we have seen in the proof above). Such measurements are precisely excluded in the description above.

To proceed it is convenient to not work with the representations itself, but rather directly with the automorphisms α_ξ^k that we constructed above. To ease the notation we identify $\pi_0 \circ \alpha_\xi^k$ with α_ξ^k. We then make the following definition:

Definition 4.2. *An automorphism (or generally, an endomorphism) α of $\mathfrak{A}$ is said to be* localized *if there is some cone Λ such that $\alpha(A) = A$ for all $A \in \mathfrak{A}(\Lambda^c)$. It is called* transportable *if in addition for any other cone $\widehat{\Lambda}$, there is an automorphism $\widehat{\alpha}$ localized in $\widehat{\Lambda}$ that is unitary equivalent to α.*

It follows from the previous discussion that the automorphisms α_ξ^k are localized and transportable. It should be noted that the unitary setting up the equivalence in the definition of transportability need not be in $\mathfrak{A}$ (and in general, it isn't). Rather, it can be seen as a unitary in $\mathfrak{B}(\mathcal{H})$, with $V\pi_0 \circ \alpha = \pi_0 \circ \widehat{\alpha}V$. Using locality one can get better control over the algebra in which V lives, but we will not go into that here. Such a unitary V is called a *charge transporter*, and one can indeed think of it as moving the charge from one cone to the other.

A natural question is to ask what happens if we add *two* (possibly distinct) charges in the system. By the discussion before, the automorphisms α^k describe how the observables change in the presence of a charge. Hence if we have two such automorphisms α and β, $(\alpha \otimes \beta)(A) := \alpha \circ \beta(A)$ describes the effect of first adding a charge β, and then a charge α. The tensor product symbol here is just the standard notation, it has nothing to do with the tensor product of vector spaces in this case.

The combination of two charges can often be decomposed in simpler ("irreducible") parts again. This is very much analogous to the decomposition of the tensor product of two group representations in a direct sum of irreducibles. In the present case these decompositions are particularly easy. They can be obtained by fixing paths to infinity and explicitly calculating the composed automorphisms. This gives for example the following "fusion rules", where strictly speaking the equality is up to unitary equivalence:

$$\alpha^x \otimes \alpha^z = \alpha^y, \quad \alpha^k \otimes \alpha^k = \iota.$$

Fusion with ι is always trivial, the other combinations can be obtained straightforwardly. The second part is nothing but saying that all charges in the model are self-dual: if we add two of the same charges together, we end up with the trivial charge again.

It is easy to see that if α and β are localized, then $\alpha \otimes \beta$ is localized in any region that contains the localization regions of α and β. Now suppose that S and T are intertwiners, that is $S\alpha(A) = \widehat{\alpha}(A)T$ for localized endomorphisms α and $\widehat{\alpha}$, and similarly for T. Define $S \otimes T := S\alpha(T)$, then we have[3]

$$(S \otimes T)\alpha \otimes \beta(A) = S\alpha(T\beta(A)) = \widehat{\alpha} \otimes \widehat{\beta}(A)(S \otimes T).$$

Hence from intertwiners (and in particular charge transporters) one can obtain intertwiners for the fused charges.

Remark 4.3. In the language of tensor categories, we can consider the category with as objects localized and transportable endomorphisms, and as morphisms the intertwiners between such maps. By following the constructions above, we can make it into a so-called fusion category. A much more detailed description can be found in, among others, [10].

The final part of the construction is to find a so-called *braiding*. It is here that the alluded anyonic nature of the excitations will come in. As we mentioned this is related to what happens if we interchange two excitations. In the present context, this is nothing but relating $\alpha \otimes \beta$ to $\beta \otimes \alpha$. This can be done roughly in the following way. Let Λ_1 and Λ_2 be the localisation regions. Now choose a third cone $\widehat{\Lambda}$ to the left of the both cones.[4] By transportability, there is a

[3] One actually has to be a bit more careful here, since the intertwiners T are in general not in $\mathfrak{A}$, so that $\alpha(T)$ is not defined. However one can show that one can extend the map α to a larger algebra that contains T. Details can be found in [13].

[4] To define "left" consistently, one has to choose a reference direction.

unitary V such that $V\beta(A) = \widehat{\beta}(A)V$ with $\widehat{\beta}$ localized in $\widehat{\Lambda}$. Finally define $\varepsilon_{\alpha,\beta} := V^*\alpha(V)$. Then it follows that

$$\varepsilon_{\alpha,\beta}\alpha \otimes \beta(A) = \beta \otimes \alpha(A)\varepsilon_{\alpha,\beta}.$$

This can be checked easily, if one remarks that $\alpha\circ\widehat{\beta}(A) = \widehat{\beta}\circ\alpha(A)$ because of the disjoint localization regions. With the interpretation of the charge transporters, one quite literally can interpret this as exchanging the two charges. Finally, one can then imagine "circling" charge α around charge β. This is described by doing two interchanges, or $\varepsilon_{\alpha,\beta}\varepsilon_{\beta,\alpha}$. In the toric code model, these operators can be calculated explicitly, and take the values $\pm I$. This should be contrasted with bosons and fermions, where a double interchange always is a trivial operation, that is, we always have $+I$ in that case. In the toric code $-I$ also is possible, showing that indeed the model supports anyons. A complete analysis shows that the tensor category one obtains in this way is actually the representation category of a certain Hopf algebra, called the quantum double of $\mathbb{Z}_2$. Hence one can understand the excitations completely by studying the representation theory of some Hopf algebra [13].

Remark 4.4. The charged states that we have constructed lead to examples of cone localizable representations, as we have seen before. An alternative approach is to consider all equivalence classes of cone localizable representations (or just the irreducible ones), and postulate that these representations are the ones of physical interest.[5] This is what is usually done in algebraic quantum field theory. Such a rule selecting the physical representations is is called a *superselection criterion*. An interesting result in this context is that of Buchholz and Fredenhagen [5], who showed that in relativistic quantum field theories, representations with a mass gap always obey such localisation properties (where cones are replaced by so-called spacelike cones). Using a technical property called Haag duality [14], one can then obtain localized endomorphisms from such representations, and build up the theory in complete analogy.

References

[1] R. Alicki, M. Fannes, and M. Horodecki. A statistical mechanics view on Kitaev's proposal for quantum memories. *J. Phys. A*, 40(24):6451–6467, 2007.

[2] H. Araki. *Mathematical theory of quantum fields*, volume 101 of *International Series of Monographs on Physics.* Oxford University Press, Oxford, 2009. Translated from the 1993 Japanese original by Ursula Carow-Watamura.

[3] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 1.* Texts and Monographs in Physics. Springer-Verlag, New York, second edition, 1987.

[5] Of course there is a physical motivation behind this particular choice, for the present purpose it is enough to see that examples of them arise in a straightforward way from from charged states.

[4] O. Bratteli and D. W. Robinson. *Operator algebras and quantum statistical mechanics. 2.* Texts and Monographs in Physics. Springer-Verlag, Berlin, second edition, 1997.

[5] D. Buchholz and K. Fredenhagen. Locality and the structure of particle states. *Comm. Math. Phys.*, 84(1):1–54, 1982.

[6] S. Doplicher, R. Haag, and J. E. Roberts. Local observables and particle statistics. I. *Comm. Math. Phys.*, 23:199–230, 1971.

[7] S. Doplicher, R. Haag, and J. E. Roberts. Local observables and particle statistics. II. *Comm. Math. Phys.*, 35:49–85, 1974.

[8] J. Earman. Superselection rules for philosophers. *Erkenntnis*, 69(3):377–414, 2008.

[9] L. Fiedler and P. Naaijkens. Haag duality for kitaev's quantum double model for abelian groups. Preprint, arXiv:1406.1084.

[10] H. Halvorson and M. Müger. Algebraic quantum field theory. In J. Butterfield and J. Earman, editors, *Philosophy of Physics*, pages 731–922. Elsevier, 2006.

[11] A. Kitaev. Fault-tolerant quantum computation by anyons. *Ann. Physics*, 303(1):2–30, 2003.

[12] P. Naaijkens. Quantum spin systems on infinite lattices. Lecture notes, available as arXiv:1311.2717.

[13] P. Naaijkens. Localized endomorphisms in Kitaev's toric code on the plane. *Rev. Math. Phys.*, 23(4):347–373, 4 2011.

[14] P. Naaijkens. Haag duality and the distal split property for cones in the toric code. *Lett. Math. Phys.*, 101(3):341–354, 2012.

[15] B. Simon. *The statistical mechanics of lattice gases. Vol. I.* Princeton Series in Physics. Princeton University Press, Princeton, NJ, 1993.

[16] J. von Neumann. Die Eindeutigkeit der Schrödingerschen Operatoren. *Mathematische Annalen*, 104(1):570–578, 1931.

[17] G. Wick, A. Wightman, and E. Wigner. The intrinsic parity of elementary particles. *Physical Review*, 88(1):101–105, 1952.

[18] E. P. Wigner. *Group theory: And its application to the quantum mechanics of atomic spectra.* Expanded and improved ed. Pure and Applied Physics. Vol. 5. Academic Press, New York, 1959.