On the Random XY Spin Chain

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The Isotropic XY-Spin Chain

Fix a real-valued sequence \( \{\nu_j\}_{j \geq 1} \) and for each integer \( n \geq 1 \), set

\[
H_n = \sum_{j=1}^{n-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_{j=1}^n \nu_j \sigma_j^z,
\]

acting on

\[
\mathcal{H}_n = \bigotimes_{j=1}^n \mathbb{C}^2
\]

Here, the Pauli matrices are

\[
\sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

and for \( \alpha \in \{x, y, z\} \) and \( 1 \leq j \leq n \) we embed these into \( B(\mathcal{H}_n) \):

\[
\sigma_j^\alpha = \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \sigma^\alpha \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I} \text{ with } \sigma^\alpha \text{ in the } j\text{-th factor.}
\]
A Locality Review

Let us denote the Heisenberg dynamics associated to $H_n$ by

$$\tau_t^n(A) = e^{itH_n} A e^{-itH_n} \quad \text{for all} \quad A \in \mathcal{B}(\mathcal{H}_n)$$

For simplicity, take $A_k \subset \mathcal{B}(\mathcal{H}_n)$ to be all those observables supported at a single site $1 \leq k \leq n$.

For bounded sequences $\{\nu_j\}$, it is clear that Lieb-Robinson bounds apply and for any $\mu > 0$,

$$\|[\tau_t^n(A), B]\| \leq C \|A\| \|B\| e^{-\mu(|k-k'|-\nu|t|)}$$

for any $A \in A_k$, $B \in A_{k'}$, and $t \in \mathbb{R}$ with some $\nu > 0$ depending on $\mu$ and the sequence $\{\nu_j\}$.

Can one prove a stronger statement if the $\{\nu_j\}$ are random?
A Strong Form of Dynamical Localization

Assume \( \{\nu_j\} \) is an i.i.d. random sequence with compactly supported bounded density \( \rho \).

**Theorem (Hamza-S-Stolz ’11)**

*There are positive numbers \( C \) and \( \eta \) for which, given any integer \( n \geq 1 \) the bound*

\[
E \left( \sup_{t \in \mathbb{R}} \left\| [\tau^n_t(A), B] \right\| \right) \leq C \|A\| \|B\| e^{-\eta |k-k'|}
\]

*holds for any \( A \in \mathcal{A}_k \) and \( B \in \mathcal{A}_{k'} \) with \( 1 \leq k < k' \leq n \).*

One can think of this as a Lieb-Robinson bound with zero velocity.

After reviewing some basic properties of this model, the main goal of this talk is to prove this result.
Diagonalizing the Hamiltonian

As Gunter discussed, *diagonalizing* this Hamiltonian goes back to LSM ’61:

First, one introduces *raising* and *lowering* operators:

\[ a^* = \frac{1}{2}(\sigma^x + i\sigma^y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \]

\[ a = \frac{1}{2}(\sigma^x - i\sigma^y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \]

It will also be useful to observe that:

\[ a^*a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad aa^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \]

i.e., these four operators form a basis for \( \mathcal{B}(\mathbb{C}^2) = \mathbb{C}^{2 \times 2} \).
Jordan-Wigner Transform

Then, one introduces the non-local Jordan-Wigner transform:

\[ c_1 = a_1 \quad \text{and} \quad c_j = \sigma^z_1 \cdots \sigma^z_{j-1} a_j \]

for each \( j \geq 2 \).

These operators are particularly useful because they satisfy the canonical anti-commutation relations (CAR):

\[
\{ c_j, c^*_k \} = \delta_{jk} \mathbb{I} \quad \text{and} \quad \{ c_j, c_k \} = \{ c^*_j, c^*_k \} = 0 .
\]

We often collect them as vectors:

\[
c = \begin{pmatrix}
    c_1 \\
    c_2 \\
    \vdots \\
    c_n 
\end{pmatrix}
\quad \text{or} \quad
\begin{pmatrix}
    c^*_1 \\
    c^*_2 \\
    \vdots \\
    c^*_n 
\end{pmatrix}
\]
Re-writing the Hamiltonian

A short calculation shows that

\[ H_n = \sum_{j=1}^{n-1} (\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) + \sum_{j=1}^{n} \nu_j \sigma_j^z \]

\[ = 2 \sum_{j=1}^{n-1} (a_j^* a_{j+1} + a_{j+1}^* a_j) + \sum_{j=1}^{n} \nu_j (2a_j^* a_j - \mathbb{1}) \]

\[ = -2 \sum_{j=1}^{n-1} (c_j^* c_{j+1} + c_{j+1}^* c_j) + \sum_{j=1}^{n} \nu_j (2c_j^* c_j - \mathbb{1}) \]

\[ = 2c^* M_n c - E \mathbb{1} \]

where

\[ M_n = \begin{pmatrix} \nu_1 & -1 & & & \\ -1 & \nu_2 & \ddots & & \\ & \ddots & \ddots & \ddots & -1 \\ & & \ddots & \ddots & -1 \\ -1 & & & & \nu_n \end{pmatrix} \quad \text{and} \quad E = \sum_{j=1}^{n} \nu_j \]
The Anderson model:

In the case that the \( \{ \nu_j \} \) are random, \( M_n \) corresponds to the well-studied Anderson model corresponding to a single quantum particle in a random environment.

With our assumptions on the \( \{ \nu_j \} \), one can prove

**Theorem (Dynamical Localization)**

*There exist positive numbers \( C' \) and \( \eta' \) such that for all integers \( n \geq 1 \) and any \( k, k' \in \{1, \ldots, n\} \),

\[
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \left| (e^{-itM_n})_{kk'} \right| \right) \leq C' e^{-\eta'|k-k'|}.
\]

**For the purposes of this talk, we will assume this is well-known; see, for example, Kunz-Souillard or Aizenman-Molchanov.**
Summary of findings so far:

We have seen that this many-body XY Hamiltonian can be expressed in terms of a single-particle Hamiltonian, i.e.,

\[ H_n = 2c^* M_n c - E \mathbb{1} \]

In the case that the \( \{ \nu_j \} \) are nice random variables, we also know that the single-particle dynamics, i.e. \( e^{itM_n} \), is dynamically localized.

We now show that this implies the desired result for the many-body dynamics, i.e. \( \tau^n_t(\cdot) \).
Bogoliubov Transformations

As we saw, $M_n$ is real symmetric. In this case, there is a real orthogonal $U_n$ such that

$$U_n^t M_n U_n = \Lambda = \text{diag}(\lambda_k)$$

Take

$$\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix} = U_n^t \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix} \quad \text{i.e.} \quad b = U_n^t c$$

Observe that these $b$-operators also satisfy the CAR and moreover

$$H_n = 2c^* M_n c - E \mathbb{1} = 2 \sum_{k=1}^n \lambda_k b_k^* b_k - E \mathbb{1}$$

Thus, this last transform expresses $H_n$ as a system of free Fermions.
Calculating the Many-Body Dynamics

As Gunter also discussed, a simple calculation shows that

\[ \tau_n^t(b_k) = e^{-2it\lambda_k} b_k \quad \text{and} \quad \tau_n^t(b_k^*) = e^{2it\lambda_k} b_k^* \]

or

\[ \tau_n^t(b) = e^{-2it\Lambda} b \]

A further calculation, using that \( c = U_n b \), shows that

\[ \tau_n^t(c) = e^{-2itM_n} c \quad \text{or} \quad \tau_n^t(c_j) = \sum_k \left( e^{-2itM_n} \right)_{jk} c_k \]

and so the many-body dynamics (of the \( c \)-operators) can be expressed explicitly in terms of the single-particle dynamics.

We can now begin the proof of the main result.
Recall the Main Result:

Theorem

Assume \( \{\nu_j\} \) is an i.i.d. random sequence with compactly supported bounded density \( \rho \). There are positive numbers \( C \) and \( \eta \) for which, given any integer \( n \geq 1 \) the bound

\[
\mathbb{E} \left( \sup_{t \in \mathbb{R}} \| [\tau^n_t(A), B] \| \right) \leq C \| A \| \| B \| e^{-\eta|k-k'|}
\]

holds for any \( A \in \mathcal{A}_k \) and \( B \in \mathcal{A}_{k'} \) with \( 1 \leq k < k' \leq n \).
Proof of Main Result:

Fix $1 \leq k < k' \leq n$ as indicated and take $B \in \mathcal{A}_{k'}$. Consider first the non-local $A = c_k$, i.e.:

$$[\tau_t^n(c_k), B] = \sum_{j=1}^{n} \left( e^{-2itM_n} \right)_{kj} [c_j, B]$$

$$= \sum_{j=k'}^{n} \left( e^{-2itM_n} \right)_{kj} [c_j, B]$$

Using the single-particle dynamical localization result, we find that

$$\mathbb{E} \left( \sup_t \| [\tau_t^n(c_k), B] \| \right) \leq 2C' \| B \| \sum_{j=k'}^{n} e^{-\eta'(j-k)} \leq \frac{2C' \| B \|}{1 - e^{-\eta'}} e^{-\eta'(k' - k)}$$

This is an estimate of the type desired; excepting that it is for the non-local observable $c_k$. By taking adjoints, it is clear that a similar result holds for $c_k^\ast$. 
Proof of Main Result (cont.):

Now take $A = a_k$. Recall $a_k = \sigma_1^z \cdots \sigma_{k-1}^z c_k$. Observe that

$$[	au^n_t(a_k), B] = \tau^n_t(\sigma_1^z) \cdots \tau^n_t(\sigma_{k-1}^z)[\tau^n_t(c_k), B] +$$
$$+[	au^n_t(\sigma_1^z) \cdots \tau^n_t(\sigma_{k-1}^z), B]\tau^n_t(c_k)$$

where we have used the automorphism property of $\tau^n_t(\cdot)$, i.e.

$$\tau^n_t(AB) = \tau^n_t(A)\tau^n_t(B)$$

and the Leibnitz rule:


It is clear then that

$$\|\tau^n_t(a_k), B\| \leq \|\tau^n_t(c_k), B\| + \|\tau^n_t(\sigma_1^z) \cdots \tau^n_t(\sigma_{k-1}^z), B\|$$
Proof of Main Result (cont.):
Now, for any \( j \), the quantity appearing above satisfies
\[
\| [\tau^n_t(\sigma^z_j) \cdots \tau^n_t(\sigma^z_1), B] \| \leq \| [\tau^n_t(\sigma^z_j), B] \| + \| [\tau^n_t(\sigma^z_{j-1}) \cdots \tau^n_t(\sigma^z_1), B] \|
\]
again by Leibnitz. Moreover,
\[
\sigma^z_j = 2a^*_ja_j - \mathbb{I} = 2c^*_jc_j - \mathbb{I}
\]
and so
\[
[\tau^n_t(\sigma^z_j), B] = 2[\tau^n_t(c^*_j), B]\tau^n_t(c_j) + 2\tau^n_t(c^*_j)[\tau^n_t(c_j), B]
\]
and, in fact:
\[
\| [\tau^n_t(\sigma^z_j), B] \| \leq 2\| [\tau^n_t(c^*_j), B] \| + 2\| [\tau^n_t(c_j), B] \|
\]
Consequently,
\[
\| [\tau^n_t(a_k), B] \| \leq 2 \sum_{j=1}^{k} (\| [\tau^n_t(c^*_j), B] \| + \| [\tau^n_t(c_j), B] \|)
\]
Proof of Main Result (cont.):

Using our previous result, it is clear that

\[ \mathbb{E} \left( \sup_t \left\| [\tau^n_t(a_k), B] \right\| \right) \leq \frac{8 C' \|B\|}{1 - e^{-\eta'}} \sum_{j=1}^{k} e^{-\eta'(k' - j)} \]

\[ \leq \frac{8 C' \|B\|}{(1 - e^{-\eta'})^2} e^{-\eta'(k' - k)} \]

which is the result for \( A = a_k \).

By taking adjoints, a similar result holds for \( A = a_k^* \).

Using Leibnitz again, it is clear that a similar result holds for both \( A = a_k^* a_k \) and \( A = a_k a_k^* \).

Since these operators form a basis for \( \mathcal{A}_k \), this completes the proof.
A Generalization

The isotropic XY model is not the only spin chain that reduces to a system of free Fermions. Consider e.g. the anisotropic XY Spin Hamiltonian

\[ H_n = \sum_{j=1}^{n-1} \mu_j [(1 + \gamma_j)\sigma^x_j \sigma^x_{j+1} + (1 - \gamma_j)\sigma^y_j \sigma^y_{j+1}] + \sum_{j=1}^{n} \nu_j \sigma^z_j \]

with real parameters given by: interaction strengths \{\mu_j\}, anisotropy \{\gamma_j\}, and field strengths \{\nu_j\}. Introducing the same raising and lowering operators and then the Jordan-Wigner transform, this many-body operator can also be written in terms of an effective single-particle Hamiltonian.
Diagonalizing the Hamiltonian

As is discussed in the notes,

\[ H_n = C^* M_n C \]

where

\[ C = (c_1, c_2, \cdots, c_n, c_1^*, c_2^*, \cdots, c_n^*)^t \]

is a column vector and

\[ C^* = (c_1^*, c_2^*, \cdots, c_n^*, c_1, c_2, \cdots, c_n) \]

In this case, the single particle Hamiltonian is a block-matrix

\[ M_n = \begin{pmatrix} A_n & B_n \\ -B_n & -A_n \end{pmatrix} \]
Diagonalizing the Hamiltonian (cont.)

with

$$A_n = \begin{pmatrix}
\nu_1 & -\mu_1 \\
-\mu_1 & \nu_2 & \ddots \\
\ddots & \ddots & \ddots & -\mu_{n-1} \\
-\mu_{n-1} & \nu_n & \ddots & \ddots \\
\end{pmatrix}$$

and

$$B_n = \begin{pmatrix}
0 & \gamma_1 \mu_1 \\
-\gamma_1 \mu_1 & \ddots & \ddots \\
& \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \gamma_{n-1} \mu_{n-1} \\
-\gamma_{n-1} \mu_{n-1} & \ddots & \ddots & \ddots & 0 \\
\end{pmatrix}$$
The More General Result (at least in words)

In our paper, we prove an analogous result:

If the single particle Hamiltonian $M_n$ is dynamically localized, then the many body Hamiltonian satisfies dynamical localization as well, in the sense that we establish a zero-velocity Lieb-Robinson bound (in average).

What we do not quantify (and what remains an interesting open question) is: Under what conditions is this more general, random single particle system dynamically localized?
For the Experts:

By re-ordering the basis vectors, the block-matrix above is easily seen to be unitarily equivalent to

\[
\tilde{M}_n = \begin{pmatrix}
-\nu_1 \sigma^z & \mu_1 S(\gamma_1) \\
\mu_1 S(\gamma_1)^t & -\nu_2 \sigma^z \\
& \ddots & \ddots \\
& & \ddots & \ddots \\
& & & \mu_{n-1} S(\gamma_{n-1}) \\
& & & \mu_{n-1} S(\gamma_{n-1})^t & -\nu_n \sigma^z
\end{pmatrix}
\]

where

\[
S(\gamma) = \begin{pmatrix}
1 & \gamma \\
-\gamma & -1
\end{pmatrix}
\]

The question now becomes: If some of these coefficients are random, is such a one-dimensional, single-particle model dynamically localized?