PART III & QUASI-ADIABATIC EVOLUTION

(Developed by M. B. Hastings in 2004)

Question: Assume that we are given a family of grouped Hamiltonians $H(s)$, that are sums of (quasi-)local interactions. Can we construct a corresponding family of unitaries $U(s)$, such that (i) $\partial_s U(s) = i \cdot D(s) \cdot U(s)$, with $U(0) = 1$ and $D(s)$ a sum of quasi-local interactions,

(ii) For $P_0(s)$ the groundstate projector of $H(s)$, we can write $P_0(s) = U(s) \cdot P_0(0) \cdot U^+(s)$?

Answer: Yes! The idea is to create an operator $D(s)$ (Hermitian) that simulates the generator of the true adiabatic evolution on the groundstate subspace, but in a way that preserves locality of interactions as they change the Hamiltonian $H(s)$. 
STEP 1: Assume \( H(s) \) is family of gapped Hamiltonians with gap \( E_1(s) - E_0(s) = \delta(s) \geq \delta > 0 \), and \( H(s) \) is differentiable.

Differentiating \( (H(s) - E_0(s)) P_0(s) = 0 \), w.r.t. \( s \), we get:

\[
\{ \frac{\partial}{\partial s} (H(s) - E_0(s)) \} P_0(s) = - (H(s) - E_0(s)) \frac{\partial}{\partial s} P_0(s).
\]

Since \( P_0^2(s) = P_0(s) \), we have \( \frac{\partial}{\partial s} P_0^2(s) = \frac{\partial}{\partial s} P_0(s) = (\partial_s P_0(s)) P_0(s) + P_0(s) (\partial_s P_0(s)) \)

\[\Rightarrow P_0(s) \frac{\partial}{\partial s} P_0(s) P_0(s) = 0 \quad \text{and} \quad (1 - P_0(s)) \frac{\partial}{\partial s} P_0(s) (1 - P_0(s)) = 0.\]

This implies that \( \frac{\partial}{\partial s} P_0(s) = (1 - P_0(s)) (\partial_s P_0(s)) P_0(s) + P_0(s) (\partial_s P_0(s)) (1 - P_0(s)) \)

Combined with \( \circ \), we get:

\[
\frac{\partial}{\partial s} P_0(s) = \frac{1}{H(s) - E_0(s)} (H(s) - E_0(s)) \frac{\partial}{\partial s} P_0(s).
\]

\[
\Rightarrow \frac{\partial}{\partial s} P_0(s) = - (1 - P_0(s)) \frac{\partial}{\partial s} H(s) P_0(s) + P_0(s) \frac{\partial}{\partial s} H(s) (1 - P_0(s)) \frac{1}{H(s) - E_0(s)}
\]

where the inverse \((H(s) - E_0(s))^{-1}\) is well-defined in the \((1 - P_0(s))\) subspace. Denoting \( \dot{G}(s) = i (H(s) - E_0(s))^{-1} \frac{\partial}{\partial s} (H(s) - E_0(s)) \),

\[
\frac{\partial}{\partial s} P_0(s) = i \left[ \dot{G}(s), P_0(s) \right].
\]

Let \( \hat{G}(s) = G(s) + G^+(s) = i G(s) P_0(s) - i P_0(s) G^-(s) \)

\[
\hat{G}(s) = G(s) + G^+(s) = i G(s) P_0(s) - i P_0(s) G^-(s)
\]
STEP 2: Recalling that \( G(s) = i (1-P_0(s)) (H(s)-E_0(s))^{-1} (1-P_0(s)) \),
we wish to find an operator \( D(s) = D^+(s) \), such that
\[
G(s) P_0(s) = D(s) \cdot P_0(s)
\]
and \( D(s) \) is sum of quasi-local terms.

Recalling that \( P_0(s) G(s) P_0(s) = 0 \), we want \( P_0(s) D(s) P_0(s) = 0 \)
and \( (1-P_0(s)) G(s) P_0(s) = (1-P_0(s)) D(s) P_0(s) \), or equivalently,
\[
\langle \psi_n(s) | G(s) | \psi_0(s) \rangle = \langle \psi_n(s) | D(s) | \psi_0(s) \rangle,
\]
where \( | \psi_0(s) \rangle \) is an eigenstate of \( H(s) \) with energy \( E_n(s) \).

Now, recall that \( E_n(s) - E_0(s) \geq \delta(s) \geq \delta \), for \( n \geq 1 \).

As in PART II (ENERGY FILTERING), we will make use of the "filter" function \( \omega_\delta(t) \), which satisfies:

(i) \( \omega_\delta(t) \geq 0 \) and \( \hat{\omega}_\delta(x) = 0 \), for \( |x| \geq \delta \)

(ii) \( \int_{-\infty}^{\infty} \omega_\delta(t) = 1 \) (normalized \( \hat{\omega}_\delta(0) \)) and \( \omega_\delta(t) = \omega_\delta(-t) \),

(iii) \( \omega_\delta(t) \leq C \left( \exp \frac{\delta}{\delta^2} \frac{\delta |t|}{\log^2 (\delta |t|)} \right) \)
up order of, up to polynomial factors in \( \delta |t| \).
Step 3: We compute \[ \langle \psi_n(s) | G(s) | \psi_0(s) \rangle = \frac{i}{E_n(s) - E_0(s)} \langle \psi_n(s) | \partial_s H(s) | \psi_0(s) \rangle \]

Define \[ D(s) = \int_{-\infty}^{\infty} dt \omega_\xi(t) \int_0^t du e^{iuH(s)} \partial_s H(s) e^{-iuH(s)} \]

Calculating the term \[ \langle \psi_n(s) | D(s) | \psi_0(s) \rangle \], we get:

\[ \langle \psi_n(s) | D(s) | \psi_0(s) \rangle = \int_{-\infty}^{\infty} dt \omega_\xi(t) \int_0^t du e^{iu(H(s)-E_0(s))} \cdot \langle \psi_n(s) | \partial_s H(s) | \psi_0(s) \rangle \]

\[ = \int_{-\infty}^{\infty} dt \omega_\xi(t) \left( \frac{e^{it(E_n(s)-E_0(s))}}{i(E_n(s)-E_0(s))} - 1 \right) \cdot \langle \psi_n(s) | \partial_s H(s) | \psi_0(s) \rangle \]

\[ = \left[ \frac{\hat{\omega}_\xi(E_n(s)-E_0(s))}{i(E_n(s)-E_0(s))} + \frac{i}{E_n(s)-E_0(s)} \right] \langle \psi_n(s) | \partial_s H(s) | \psi_0(s) \rangle \]

\[ = \langle \psi_n(s) | G(s) | \psi_0(s) \rangle \], since \[ \hat{\omega}_\xi(E_n(s)-E_0(s)) = 0 \]

Finally, since \[ \omega_\xi(t) \] is an even function, we have:

\[ \langle \psi_0(s) | D(s) | \psi_0(s) \rangle = \left( \int_{-\infty}^{\infty} dt \omega_\xi(t) \cdot t \right) \cdot \langle \psi_0(s) | \partial_s H(s) | \psi_0(s) \rangle \]

\[ = 0 \], as desired.

At this point, we have shown that \( \Theta \Theta \) in page 3 is valid, which implies that \( \partial_s \rho_0(s) = i [D(s), \rho_0(s)] \) and hence, \( \rho_0(s) = U(s) \cdot \rho_0(0) \cdot U^*(s) \), with \( \partial_s U(s) = i [D(s), U(s)] \), \( U(0) = 1 \).
STEP 4: It remains to show that $D(s)$ is a sum of quasi-local terms. Recall that $H(s)$ is a sum of (quasi-)local terms such that $H(s) = \sum_{\mathbf{z} \in \Lambda} H_2(s)$, where $\Lambda$ may be taken as a subset of sites within a ball of radius $R_0$ of site $\mathbf{z} \in \Lambda \subset \mathbb{Z}^d$.

Then, $\partial_s H(s) = \sum_{\mathbf{z} \in \Lambda} \partial_s H_2(s)$, which is also local, with $\partial_s H_2(s)$ supported on sites that $H_2(s)$ acted on, non-trivially.

Since we will be making use of Lieb-Robinson bounds (presented in PART I), we assume appropriate bounds on the locality and strength of $H_2(s)$ and $\partial_s H_2(s)$.

Now, we may write $D(s) = \sum_{\mathbf{z} \in \Lambda}^{\mathcal{I}} \int_{-\infty}^{\infty} dt \, \omega(t) \int_{0}^{t} du \, \text{T}_u H(s)(\partial_s H_2(s))$

where $\text{T}_u H(s)(A) = e^{iuH(s)} A e^{-iuH(s)}$.

Setting $D_2(s) = \int_{-\infty}^{\infty} dt \, \omega(t) \int_{0}^{t} du \, \text{T}_u H(s)(\partial_s H_2(s))$, we have $D(s) = \sum_{\mathbf{z} \in \Lambda}^{\mathcal{I}} D_2(s)$. It remains to show that each $D_2(s)$ can be decomposed into a sum of terms with $\uparrow$ support and $\downarrow$ strength.
STEP 5: (Locality of $D_Z(s)$)

Back in PART II of the series, we proved a Lemma that used Lieb-Robinson Bounds to calculate the worst error we could make by approximating the evolution $T_t^{H(s)}(A_w)$ by a local version $T_t^{H_u(s,r)}(A_w)$, where $H_u(s,r)$ is the Hamiltonian $H(s)$ with interactions restricted within a ball of radius $r$, around the support of the operator $A_w$.

Let us now recall that Lemma and the statement of Lieb-Robinson bounds for $H(s) = \sum_{z \in \Lambda} H_Z(s)$, where the support of each $H_Z(s)$ is assumed to be $B_Z(z)$, the ball of radius 1, centered on $z \in \Lambda \subset \mathbb{Z}^d$, (see p. 3 of PART II).

But, first, we write $D_Z(s) = \sum_{r \geq 0} D_Z(s, r) + \partial_S H_Z(s)$, where

$$D_Z(s, r) = \int_{-\infty}^{\infty} dt \omega_y(t) \int_0^t du \left[ H_y(s, r) (H_u(s, r) - H_u(s, 0)) \right]$$

with $H_u(s, r) = \sum_{z \in \mathbb{Z}^d} H_Z(s)$ and $H_u(s, 0) = 0$.

$B_Z(z) \subset B_u(r)$
(1) (Lieb-Robinson Bounds): Given a Hamiltonian $H = \sum_{z} H_z$, and operators $A_x$ and $B_y$, supported on subsets $X, Y$ of the lattice $\mathbb{Z}^d$, the following bound holds:

$$\| \left[ T_t^H(A_x), B_y \right] \| \leq 2 \min \{ |x|, |y| \} \| A_x \| \cdot \| B_y \| e^{\frac{v_0 |t|}{\mu_0}} e^{-\mu_0 d(x,y)}$$

where $T_t^H(A_x) = e^{itH} A_x e^{-itH}$, $d(x,y) = \min_{x \in X, y \in Y} d(x,y)$, $\mu_0 > 0$, and $v_0 = e^{\mu_0} (2 J B_1(d))$, where $J = \max_{\mathbb{Z} \in A} \| H_z \|$ and $B_1(d) = $ volume of unit ball in $\mathbb{Z}^d$.

(2) ("Localizing evolution" Lemma): Let $H_u(r) = \sum_{z \in B_1(r)} H_z$ and $T_t^H_u(r)(A_u)$, as above. Then, the following bound holds:

$$\| T_t^H_u(r+t)(A_u) - T_t^H_u(r)(A_u) \| \leq \| A_u \| \cdot (c_0 \cdot r^{d-1}) e^{-\mu_0 r} e^{v_0 |t|}$$

where $c_0 = \text{surface area}$ of the unit ball in $\mathbb{Z}^d$.

**NOTE:** The proofs of (1) and (2) are given in PART I and PART II, respectively. In particular, (2) follows from an application of (1).
STEP 5 (continued...)

Recall that \( D_z(s) = \sum_{r > 0} D_z(s, r) + \varphi_s H_z(s) \).

It remains to show that \( D_z(s, r) \) has support on \( B_u(r+1) \) and strength \( \| D_z(s, r) \| \leq \| \varphi_s H_z(s) \| \cdot f_0(r) \), where \( f_0 \) is a rapidly decaying function.

(i) By definition of \( D_z(s, r) \) and \( \frac{H_u(s, r+1)}{\varphi_s H_z(s)} \), we see that each interaction term is supported non-trivially on \( B_u(r+1) \), the support of \( H_u(s, r+1) \).

(ii) Using a similar argument as the bound on p. 6 of PART II, we get:

\[
\| D_z(s, r) \| \leq \int_{-T}^{T} dt \, \omega_\delta(t) \left( \| \varphi_s H_z(s) \| \cdot a_t \cdot e^{-\mu r} \right) \left[ \int_0^{\| \varphi_s H_z(s) \|} e^{-\lambda u} du \right] \\
+ 2 \| \varphi_s H_z(s) \| \int_{|t| > T} dt \, |t| \cdot \omega_\delta(t) \\
\leq \| \varphi_s H_z(s) \| \cdot f_0(r),
\]

where \( f_0(r) = 2 \left[ c_0 \cdot r d^{-1} \cdot e^{-\left( \frac{\mu r}{2} \right)} \right] \left[ \int_{|t| > T} dt \, |t| \cdot \omega_\delta(t) \right] \), where \( T = \frac{\mu r}{\omega_\delta} \).
Conclusions: Recalling the almost-exponential decay of
\[ \omega_g(t) \sim \exp\left( -\frac{\gamma}{\beta} \frac{r}{\log^2(r)} \right) \], we immediately see that
the function \( f(r) \) has decay of the order \( \exp\left( -\frac{\gamma}{\beta} \frac{r}{\log^2(r)} \right) \)
for \( c_1 = \frac{\gamma}{\beta} \left( \frac{\sqrt{\nu_0}}{\sqrt{\nu_0}} \right) \), which is super-polynomial in \( r \). This implies
that the quasi-adiabatic evolution \( U(t) \) is generated by a
quasi-local Hermitian operator, exactly what we need for
Lieb-Robinson-type bounds on evolutions based on \( U(t) \).
These Lieb-Robinson bounds are proven using techniques very
similar to those in Part I and have applications in proving
rigorous results about the properties of low-energy states
of interacting Hamiltonians with a gap to the excited sector.
For some good examples, see recent results on the arXiv
by Hastings, Nachtergaele, Ogata and Sims (some
with co-authors and others individually.)