Introduction to the Mathematics of the XY Spin Chain

Essentially: Lieb-Schultz-Mattis 1961

The isotropic XY spin chain in transversal field:

$$H = -\sum_{j=1}^{n-1} \mu_j (\sigma_j^X \sigma_{j+1}^X + \sigma_j^Y \sigma_{j+1}^Y) - \sum_{j=1}^{n} \nu_j \sigma_j^Z$$

$$\mathcal{H} = \bigotimes_{j=1}^{n} \mathbb{C}^2, \quad \dim \mathcal{H} = 2^n$$

$$\sigma^X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Variable coefficients: $\nu_j \in \mathbb{R}, \quad \mu_j \in \mathbb{R} \setminus \{0\}$

Spin lowering and spin raising operators:

$$a := \frac{1}{2}(\sigma^X - i\sigma^Y) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

$$a^* := \frac{1}{2}(\sigma^X + i\sigma^Y) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

$$a^* a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad aa^* = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Commutation relations:

$$[a_j, a_k^*] = [a_j, a_k] = 0, \quad j \neq k$$

$$\{a_j, a_j^*\} = I, \quad a_j^2 = (a_j^*)^2 = 0$$

Particle number preservation:

$$H = -2\sum_{j=1}^{n-1} \mu_j (a_j^* a_{j+1} + a_{j+1}^* a_j) - \sum_{j=1}^{n} \nu_j (2a_j^* a_j - I)$$

Thus H preserves "particle number" (number of up-spins and down-spins, respectively):

$$e_0 := \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad e_1 := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Product basis: $e_\alpha := e_{\alpha_1} \otimes \ldots \otimes e_{\alpha_n}, \; \alpha \in \{0,1\}^n$

$$\mathcal{H}_k := \operatorname{span}\{e_\alpha : \#\{j : \alpha_j = 1\} = k\}$$

$\implies \mathcal{H}_k$ invariant under H for all $0 \le k \le n$

Jordan-Wigner transform:

$$c_1 := a_1, \quad c_j := \sigma_1^Z \dots \sigma_{j-1}^Z a_j, \quad j = 2, \dots, n$$

Canonical anti-commutation relations (CAR):

$$\{c_j, c_k^*\} = \delta_{jk} I, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0$$

- Good: “Fermionic creation and annihilation operators”
- Not so good: c_j non-local

Fermionic represenation of H:

$$
\begin{aligned}
H &= 2\sum_{j=1}^{n-1} \mu_j (c_j^* c_{j+1} + c_{j+1}^* c_j) - \sum_{j=1}^{n} \nu_j (2c_j^* c_j - I) \\
&= 2\,(c_1^*, \ldots, c_n^*)\; M \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} + E_0 I
\end{aligned}
$$

$$
E_0 := \sum_j \nu_j, \quad M := \begin{pmatrix} -\nu_1 & \mu_1 & & \\ \mu_1 & -\nu_2 & \ddots & \\ & \ddots & \ddots & \mu_{n-1} \\ & & \mu_{n-1} & -\nu_n \end{pmatrix}
$$

M effective one-particle Hamiltonian, acting on n-dimensional space.

Bogolubov transformation:

M real symmetric, so there exists orthogonal U such that

$$UMU^t = \Lambda = \mathrm{diag}\,(\lambda_j)$$

Let

$$\begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} := U \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}$$

$\Longrightarrow \{b_j\}_{j=1}^n$ satisfy CAR and

$$H = 2\sum_{j=1}^{n} \lambda_j b_j^* b_j + E_0 I$$

Free Fermion system!

Properties of the Fermionic operators $\{b_j\}_{j=1}^n$:

- $b_j^* b_j$, $j = 1, \ldots, n$, pairwise commuting orthogonal projections
- $N = \bigcap_{j=1}^n \ker(b_j^* b_j) = \bigcap_{j=1}^n \ker(b_j)$ one-dimensional. Pick normalized $\Omega \in N$ ("vacuum state").
- $\psi_\alpha := (b_1^*)^{\alpha_1} \ldots (b_n^*)^{\alpha_n}\Omega$, $\alpha \in \{0,1\}^n$ form ONB of $\mathcal{H}$.
- All ψ_α are eigenvectors of each $b_j^* b_j$:

$$b_j^* b_j \psi_\alpha = \begin{cases} 0 & \text{if } \alpha_j = 0 \\ 1 & \text{if } \alpha_j = 1 \end{cases}$$

Eigenvectors and eigenvalues of H:

All ψ_α are also eigenvectors of H:

$$H\psi_\alpha = \left(2 \sum_{j:\alpha_j=1} \lambda_j + E_0 \right) \psi_\alpha$$

$$\sigma(H) = \left\{ 2 \sum_{j:\alpha_j=1} \lambda_j + E_0 : \ \alpha \in \{0,1\}^n \right\}$$

Ground state energy:

$$2 \sum_{j=1}^{n} \min\{0, \lambda_j\} + E_0$$

$$\text{non-degenerate} \quad \Longleftrightarrow \quad \lambda_j \neq 0 \text{ for all } j$$

Finding eigenvalues and eigenvectors of H has been reduced to finding eigenvalues and eigenvectors of

$$M := \begin{pmatrix} -\nu_1 & \mu_1 & & \\ \mu_1 & -\nu_2 & \ddots & \\ & \ddots & \ddots & \mu_{n-1} \\ & & \mu_{n-1} & -\nu_n \end{pmatrix}$$

Lieb, Schultz, Mattis:

$$\nu_j = \nu, \quad \mu_j = \mu$$

Exactly solvable!

In general: Dimension of Hilbert space reduced from 2^n to n.

Heisenberg dynamics under H:

$$\tau_t(A) := e^{itH} A e^{-itH} \quad \text{for } A \in B(\mathcal{H})$$

Lemma

$$\tau_t(b_k) = e^{-2it\lambda_k} b_k, \quad \tau_t(b_k^*) = e^{2it\lambda_k} b_k^*$$

Proof.

$$\begin{aligned}
\frac{d}{dt}\tau_t(b_k) &= -i\tau_t([b_k, H]) \\
&= -2i \sum_j \lambda_j \tau_t([b_k, b_\ell^* b_\ell]) \\
&= -2i\lambda_k \tau_t([b_k, b_k^* b_k]) = -2i\lambda_k \tau_t(b_k)
\end{aligned}$$

Also: $\tau_0(b_k) = b_k$. Unique solution: $\tau_t(b_k) = e^{-2it\lambda_k} b_k$

□

Heisenberg dynamics under H:

See notes for: $\begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} := U^t \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} \Longrightarrow$

$$\tau_t(c_j) = \sum_{\ell} \left(e^{-2iMt}\right)_{j\ell} c_\ell$$

Thus: One-particle dynamics e^{-2iMt} determines many-body dynamics $\tau_t(c_j)$.

Problem: How to "undo" Jordan-Wigner to get dynamics of local operators such as a_j, a_j^*?

See Online Notes: Extension to *anisotropic* XY chain