A complete criterion for convex-Gaussian states detection

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Outline of the talk

- Introduction
- Fermionic Gaussian states
- Gaussian-symmetric states
- A complete criterion for detection of convex-Gaussian states
- Work in progress

Introduction

Given m Dirac fermions created by the operator a_j^*, $j = 1, .., m$, with CAR

$$\{a_j, a_k^*\} = \delta_{jk} \text{ and } \{a_j, a_k\} = 0,$$

$2m$ **Majorana fermions** are defined as

$$c_{2k-1} = a_k + a_k^* \text{ and } c_{2k} = i(a_k^* - a_k).$$

They satisfy

$$c_k = c_k^*, \text{ and } c_k^2 = 1$$

$$c_k c_j = -c_j c_k.$$

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The number operator

$$n_k = a_k^* a_k$$

has eigenvalues either 0 or 1.
The fermion parity operator

$$P_k = I - 2n_k = (-1)^{n_k} I$$

has eigenvalue $+1$ if the number of fermions is even and -1 if it is odd. In terms of Majorana operators

$$P_k = -ic_{2k-1}c_{2k}.$$

The total fermion parity operator is

$$P_{all} = \prod_k P_k = i^m c_1 c_2 \ldots c_{2m}.$$

Any even pure state $|\psi\rangle \langle\psi|$ is the eigenstate of P_{all}, i.e. $P_{all} |\psi\rangle \langle\psi| = \pm |\psi\rangle \langle\psi|$ the eigenvalues depend on whether the parity of the number of fermions in $|\psi\rangle$ is even or odd.

A Hermitian operator X is called **even** if it has the form

$$X = \alpha_0 I + \sum_{k=1}^{m} (i)^k \sum_{1 \leq j_1 < \ldots < j_{2k} \leq 2m} \alpha_{j_1,\ldots,j_{2k}} c_{j_1} \ldots c_{j_{2k}},$$

where $\alpha_0, \alpha_{j_1,\ldots,j_{2k}}$ are real.
Any even operator commutes with P_{all}, $[X, P_{all}] = 0$.

Fermionic Gaussian states

Fermionic Gaussian state is defined as

$$\rho = \gamma \exp\{-i \sum_{i \neq j} A_{ij} c_i c_j\}$$

where γ is a normalization and (A_{ij}) is a real anti-symmetric matrix. Block-diagonalizing A we can re-express ρ is standard form

$$\rho = \frac{1}{2^m} \prod_{k=1}^{m} (I + i\lambda_k \tilde{c}_{2k-1} \tilde{c}_{2k}),$$

where $\tilde{c} = R^T c$ with R block-diagonalization of A. Here $\lambda \in [-1, 1]$. For Gaussian pure states $\lambda \in \{-1, 1\}$.

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<u>Check:</u> ρ is a pure state iff $\rho^2 = \rho$. So

$$\begin{aligned}
\rho^2 &= \frac{1}{2^{2m}} \prod_{k=1}^{m} (I + i\lambda_k \tilde{c}_{2k-1} \tilde{c}_{2k})^2 \\
&= \frac{1}{2^{2m}} \prod_{k=1}^{m} (I + \lambda_k^2 I + 2i\lambda_k \tilde{c}_{2k-1} \tilde{c}_{2k}) \\
&= \frac{1}{2^m} \prod_{k=1}^{m} \left(\frac{1 + \lambda^2}{2} I + i\lambda_k \tilde{c}_{2k-1} \tilde{c}_{2k} \right).
\end{aligned}$$

Therefore $\rho^2 = \rho$ iff $\lambda_k^2 = 1$.

Given a state $\rho \in \mathcal{C}_{2m}$, the correlation matrix M is

$$M_{ab} = \frac{i}{2}\mathrm{Tr}(\rho[c_a, c_b]), \text{ with } a, b = 1, .., 2m.$$

For $a = b$, we have $M_{aa} = 0$ and
for $a \neq b$, we have $M_{ab} = \frac{i}{2}\mathrm{Tr}(\rho c_a c_b - \rho c_b c_a) = i\mathrm{Tr}(\rho c_a c_b)$. Note that $M_{ab} = -M_{ba}$.

It can be block-diagonalized by $R \in SO(2m)$

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Fermionic linear optics (FLO) transformation maps Gaussian states into Gaussian states

$$U c_i U^* = \sum_j R_{ij} c_j$$

with $R \in SO(2m)$.
The total fermionic parity operator is invariant under FLO

$$U P_{all} = P_{all} U.$$

Lemma

[1] (de Melo, Ćwikliński, Terhal).
The correlation matrix M of any even density state $\rho \in \mathcal{C}_{2m}$ has eigenvalues $\pm i\lambda_k$, with $\lambda_k \in [-1, 1]$, $k = 1, ..., m$. Moreover $M^T M \leq I$ with equality iff ρ is a Gaussian pure state.

Dephasing procedure. Define the FLO transformation U_k, $k = 1, ..., m$ as

$$U_k c_{2k} U_k^* = -c_{2k}, \;\; U_k c_{2k-1} U^* = -c_{2k-1}, \;\; U_k c_i U_k^* = c_i \;\; \forall i \neq 2k-1, 2k.$$

It leaves the correlation matrix of ρ invariant. With $\rho_0 = \rho$, let

$$\rho_k = \frac{1}{2}(\rho_{k-1} + U_k \rho_{k-1} U_k^*).$$

After m steps, we get

$$\rho_m = \sum_k p_k \left|\psi_k\right\rangle \left\langle\psi_k\right|$$

where each $\left|\psi_k\right\rangle \left\langle\psi_k\right| = \frac{1}{2^m} \prod_{j=1}^m (I + i\beta_{kj} c_{2j-1} c_{2j})$ is a Gaussian state, i.e. $\beta_{kj} = \pm 1$. It is an eigenvector to all $ic_{2j-1}c_{2j}$, $j = 1, .., m$, since $ic_{2j-1}c_{2j} \left|\psi_k\right\rangle \left\langle\psi_k\right| = \beta_{kj} \left|\psi_k\right\rangle \left\langle\psi_k\right|$.
The correlation matrix of ρ_m is

$$M_{\rho_m} = R \bigoplus_{j=1}^m \begin{pmatrix} 0 & \sum_k p_k \beta_{jk} \\ -\sum_k p_k \beta_{jk} & 0 \end{pmatrix} R^T.$$

Therefore $M_{\rho_m}^T M_{\rho_m} = I$ iff ρ_m is pure Gaussian. Since $M_{\rho_m} = M_\rho$, ρ is pure Gaussian.

Proposition

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Proof.

For $m = 1$, a state can be written as $\rho = |\psi\rangle\langle\psi| = \alpha I + \beta c_1 c_2$. Since $\mathrm{Tr}\rho = 1$, we get $\alpha = 1/2$. From $\rho^2 = \rho$ we get $(\alpha^2 - \beta^2) I + 2\alpha\beta c_1 c_2 = \alpha I + \beta c_1 c_2$. So $\beta = i/2$. So $|\psi\rangle\langle\psi|$ is Gaussian.

For $m = 2$, block-diagonalize the correlation matrix. Then any state can be written as $\rho = |\psi\rangle\langle\psi| = \alpha I + \sum_{k=1}^{2} \beta_k c_{2k-1} c_{2k} + P_{all}$. Apply dephasing procedure: $\rho_2 = \rho$. Since ρ_2 is a convex mixture of pure Gaussian states, $\rho = |\psi\rangle\langle\psi|$ is Gaussian.

For $m = 3$, after block-diagonalization, $|\psi\rangle\langle\psi| = \alpha I + \beta P_{all} + \sum_k \gamma_k c_{2k-1} c_{2k} + \sum_{i<j<k<l} \eta_{ijkl} c_i c_j c_k c_l$. Note that $P_{all} |\psi\rangle\langle\psi| = \pm |\psi\rangle\langle\psi|$ for even pure states. Apply dephasing procedure: $\rho_3 = \rho$ is a convex mixture of pure Gaussian states. So $|\psi\rangle\langle\psi|$ is Gaussian. $\square$

A **convex-Gaussian** state is

$$\rho = \sum_i p_i \sigma_i,$$

where σ_i are pure Gaussian states, $p_i \geq 0$ and $\sum_i p_i = 1$.

Proposition

[1] For any even state $\rho \in \mathcal{C}_{2m}$ there exists $\epsilon > 0$ such that $\rho_\epsilon = \epsilon\rho + (1-\epsilon)I/2^m$ is convex-Gaussian.

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Define
$$\Lambda = \sum_{i=1}^{2m} c_i \otimes c_i \in \mathcal{C}_{2m} \otimes \mathcal{C}_{2m}.$$

The operator Λ is invariant under $U \otimes U$ for any FLO transformation.

<u>Check</u>: for any FLO U with $R \in SO(2m)$ we have

$$\begin{aligned} U \otimes U\, \Lambda\, U^* \otimes U^* &= \sum_{i=1}^{2m} U c_i U^* \otimes U c_i U^* \\ &= \sum_{i,j,k} R_{ijc_j} \otimes R_{ik} c_k \\ &= \sum_j c_j \otimes c_j. \end{aligned}$$

Here we used that $\sum_i R_{ij} R_{ik} = \sum_i r^T_{ji} R_{ik} = (R^T R)_{jk} = \delta_{jk}$.

Gaussian-symmetric states

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\begin{aligned}
\|\Lambda(\rho \otimes \rho)\Lambda\|_1 &= \sum_{a,b=1}^{2m} \mathrm{Tr}[(c_a \otimes c_a)(\rho \otimes \rho)(c_b \otimes c_b)] \\
&= \sum_{a=1}^{2m} \mathrm{Tr}(c_a \rho c_a \otimes c_a \rho c_a) + \sum_{a \neq b} (\mathrm{Tr} c_a \rho c_b)^2 \\
&= 2m - \sum_{a \neq b} (\mathrm{Tr}\, i\, c_a c_b \rho)^2 \\
&= 2m - \sum_{a,b} (M_{ab})^2 = 2m - \sum_a \Big(- \sum_b M_{ab} M_{ba} \Big) \\
&= 2m - \sum_a \sum_b M^T_{ab} M_{ba} = 2m - \mathrm{Tr}\, M^T M.
\end{aligned}
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For a pure state we have $M^T M = I$, so $\Lambda(\rho \otimes \rho) = 0$. For a mixed Gaussian state or non-Gaussian state $M^T M < I$, so $\|\Lambda(\rho \otimes \rho)\Lambda\|_1 > 0$.

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Corollary

[1] For an even state $\rho \in \mathcal{C}_{2m}$, $\Lambda(\rho \otimes \rho) = 0$ iff ρ is a pure Gaussian state.

For every Gaussian state ψ, the state $|\psi,\psi\rangle$ is contained in the null space of Λ. Define a 'FLO twirl'

$$\mathcal{S}(\rho) = \int_{FLO} dU\ U \otimes U \rho U^* \otimes U^*.$$

Lemma

[1], [4] (Terhal, V.)
The projector onto the null-space of Λ is $\Pi_{\Lambda=0} = \binom{2m}{m}\mathcal{S}(|0,0\rangle\langle 0,0|)$. Thus the states $|\psi,\psi\rangle$ where ψ is Gaussian span the null space of Λ.

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Proof.
To show that $\Pi = \binom{2m}{m}\mathcal{S}(|0,0\rangle\langle 0,0|)$, we need to show that for any X,

$$\mathrm{Tr}(X\,\Pi) = \binom{2m}{m}\mathrm{Tr}\Big(X\mathcal{S}(|0,0\rangle\langle 0,0|)\Big).$$

Need only to consider $\mathcal{S}(X)$ instead of X.
The invariant subspace of $\mathcal{S}$ is spanned by $I \otimes I, \Lambda, ..., \Lambda^{2m}$ operators. [4]
For any $i \neq 0$ we have

$$\mathrm{Tr}\,\Lambda^i\Pi_{\Lambda=0} = 0 \quad \text{and} \quad \mathrm{Tr}\,\Lambda^i\mathcal{S}(|0,0\rangle\langle 0,0|) = \mathrm{Tr}\,\mathcal{S}(\Lambda^i)\,|0,0\rangle\langle 0,0| = 0.$$

Overall prefactor:

$\Lambda = \sum_{i=1}^{2m} c_i \otimes c_i$, each term $c_i \otimes c_i$ has eigenvalue $\mu_i = \pm 1$.
Then eigenvalues of Λ are $\sum_{i=1}^{2m} \mu_i$ with the projector onto the eigenstates

$$P_{\vec{\mu}} = \frac{1}{2^{2m}} \prod_{i=1}^{2m} (I + \mu_i c_i \otimes c_i).$$

The null space is spanned by $\binom{2m}{m}$ eigenvectors $P_{\vec{\mu}}$ such that $\sum_{i=1}^{2m} \mu_i = 0$. Therefore $\mathrm{Tr}\,\Pi = \binom{2m}{m}$.
Thus states $|\psi, \psi\rangle$ with ψ Gaussian span the null space of Λ.
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Lemma

[4] If a state τ is such that

$$\|\Lambda(\tau \otimes \tau)\Lambda\|_1 \leq \epsilon,$$

then there exists a Gaussian state $|\psi\rangle$ close to τ, i.e. such that

$$\|\tau - |\psi\rangle \langle\psi|\,\|_1 \leq \epsilon' = \sqrt{2m(1 - \sqrt{1-\epsilon})}.$$

Proof

Let M_τ be a correlation matrix of state τ.

$$M_\tau = R \bigoplus_{j=1}^{m} \begin{pmatrix} 0 & \lambda_j \\ -\lambda_j & 0 \end{pmatrix} R^T.$$

Then

$$\|\Lambda\tau \otimes \tau\Lambda\|_1 = 2m - \operatorname{Tr} M_\tau^T M_\tau = 2m - \sum_j \lambda_j^2 \leq \epsilon.$$

So $\sum_j \lambda_j^2 \geq 2m - \epsilon$. Since every $\lambda_j^2 \leq 1$, for every j, $\lambda_j^2 \geq 1 - \epsilon$.

A pure Gaussian state $|\psi\rangle$ is

$$M_\psi = R \bigoplus_{j=1}^{m} \begin{pmatrix} 0 & \beta_j \\ -\beta_j & 0 \end{pmatrix} R^T,$$

such that $\beta_j = \operatorname{sign} \lambda_j$.

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such that $\beta_j = \mathrm{sign}\, \lambda_j$.
Consider the fidelity

$$F(\tau, |\psi\rangle) = \langle\psi|\, \tau \,|\psi\rangle\,.$$

Apply a dephasing procedure to τ. Since the correlation matrix M_τ stays invariant, the fidelity $F(\tau, |\psi\rangle)$ stays the same.
After dephasing the state has the form

$$\tau_m =: \sum_k p_k \,|\phi_k\rangle\,\langle\phi_k|\,,$$

where $|\phi_k\rangle\,\langle\phi_k| = \frac{1}{2^m}\prod_{j=1}^m (I + i\beta_j^k c_{2j-1}c_{2j})$ and $|\phi_0\rangle\,\langle\phi_0| := |\psi\rangle\,\langle\psi|$.

Then $F(\tau, |\psi\rangle) = p_0$ and so

$$\|\tau - |\psi\rangle\langle\psi|\,\|_1 \leq 2\sqrt{1 - p_0}.$$

For every $j = 1, ..., m$ we have

$$\lambda_j = M_\tau(2j-1, 2j) = \mathrm{Tr}(ic_{2j-1}c_{2j}\tau_m) = \sum_k \beta_j^k p_k.$$

Note that

$$\lambda_j = 1 - 2\sum_{k:\beta_j^k = -1} p_k = 2\sum_{k:\beta_j^k = 1} p_k - 1.$$

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For every $j = 1, ..., m$ we have $1 - \epsilon \leq \lambda_j^2 \leq 1$. Consider two possible cases:

1. If $\sqrt{1-\epsilon} \leq \lambda_j \leq 1$, we have $\sum_{\beta_j^k=-1} p_k \leq \frac{1}{2}(1 - \sqrt{1-\epsilon})$. Note that p_0 is not in the sum, since $\beta_j^0 = 1$.
2. If $-\sqrt{1-\epsilon} \geq \lambda_j \geq -1$, we have $\sum_{\beta_j^k=1} p_k \leq \frac{1}{2}(1 - \sqrt{1-\epsilon})$. Note that p_0 is not in the sum, since $\beta_j^0 = -1$.

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Summing all inequalities (for every j) we obtain $\sum_{j=1}^m \sum_{k:\beta_j^k=-\,\mathrm{sign}\,\lambda_j} p_k \leq \frac{m}{2}(1-\sqrt{1-\epsilon})$. There is no p_0 in the sum and the only p_k absent in the sum is p_0, so

$$1 - p_0 = \sum_{k\neq 0} p_k \leq \frac{m}{2}(1-\sqrt{1-\epsilon}),$$

therefore

$$\|\tau - |\psi\rangle\langle\psi|\,\|_1 \leq 2\sqrt{1-p_0} \leq \sqrt{2m(1-\sqrt{1-\epsilon})}.$$

Q.E.D.

Finding convex-Gaussian states

Let $\rho = \sum_i p_i \sigma_i$ with σ_i pure Gaussian. Then there exists a symmetric extension $\rho_{ext} = \sum_i p_i \sigma_i^{\otimes n} \in \mathcal{C}_{2m}^{\otimes n}$, which is annihilated by $\Lambda^{k,l}$ and $\mathrm{Tr}_{2,..,n} \rho_{ext} = \rho$.

Program. *Input:* $\rho \in \mathcal{C}_{2m}$ and an integer $n \geq 2$.

Question: Is there a $\rho_{ext} \in \mathcal{C}_{2m}^{\otimes n}$ s.t.

- $\mathrm{Tr} \rho_{ext} = 1$
- $\rho_{ext} \geq 0$
- $\mathrm{Tr}_{2,\ldots,n} \rho_{ext} = \rho$
- $\Lambda^{k,l} \rho_{ext} = 0, \forall k \neq l$

Output: Yes, then provide ρ_{ext}, or No.

Since the null space of $\Lambda^{k,l}$ is spanned by $|\psi, \psi\rangle_{k,l}$ where $|\psi\rangle$ is Gaussian, the intersection of all null-spaces of $\Lambda^{k,l}$ is spanned by vectors $|\psi\rangle^{\otimes n}$.

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The program can be done in the standard form of semi-definite program:

$$\begin{aligned} \text{minimize} \quad & c^T x \\ \text{subject to} \quad & F_0 + \sum_i x_i F_i \geq 0 \\ & Ax = b, \end{aligned}$$

here $x \in \mathbb{R}^d$, $c \in \mathbb{R}^d$ is a given vector, $\{F_i\}_{i=0,\ldots,d}$ are given symmetric matrices and $A \in \mathbb{R}^{p\times d}$ with $rank(A) = p$ and $b \in \mathbb{R}^p$ are given.

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Let $\rho = \sum_i p_i \sigma_i$ with σ_i pure Gaussian. Then there exists a symmetric extension $\rho_{ext} = \sum_i p_i \sigma_i^{\otimes n} \in \mathcal{C}_{2m}^{\otimes n}$, which is annihilated by $\Lambda^{k,l}$ and $\mathrm{Tr}_{2,..,n}\rho_{ext} = \rho$.

Program. *Input:* $\rho \in \mathcal{C}_{2m}$ and an integer $n \geq 2$.

Question: Is there a $\rho_{ext} \in \mathcal{C}_{2m}^{\otimes n}$ s.t.

- $\mathrm{Tr}\rho_{ext} = 1$
- $\rho_{ext} \geq 0$
- $\mathrm{Tr}_{2,\ldots,n}\rho_{ext} = \rho$
- $\Lambda^{k,l}\rho_{ext} = 0, \forall k \neq l$

Output: Yes, then provide ρ_{ext}, or No.

Since the null space of $\Lambda^{k,l}$ is spanned by $|\psi,\psi\rangle_{k,l}$ where $|\psi\rangle$ is Gaussian, the intersection of all null-spaces of $\Lambda^{k,l}$ is spanned by vectors $|\psi\rangle^{\otimes n}$.

Theorem

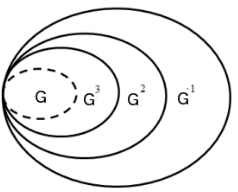
[1] If an even state $\rho \in \mathcal{C}_{2m}$ has an n-Gaussian-symmetric extension <u>for all</u> n, then ρ is convex-Gaussian, in other words, there exists a sequence of convex-Gaussian states $\omega_1, \omega_2, \ldots \in \mathcal{C}_{2m}$ s.t.

$$\lim_{n\to\infty} \|\rho - \omega_n\|_1 = 0.$$

Theorem

[4] Suppose that an even state $\rho \in \mathcal{C}_{2m}$ has an extension to n number of parties, where n is sufficiently large. Then there exists a convex-Gaussian state $\sigma(n)$ such that

$$\|\rho - \sigma(n)\|_1 \leq \epsilon(n) := 4\frac{2^m}{n} + \sqrt{2m\Big(1 - \sqrt{1 - 4\|\Lambda\|_1^2 \frac{2^m}{n}}\Big)}.$$



Denote G the set of convex-Gaussian states and $G^n \ni \rho$ iff $\exists \rho^n_{ext} \in \mathcal{C}_{2m}^{\otimes n}$. Then

$$G^1 \supset G^2 \supset ... \supset G \quad \text{and} \quad \lim_{n\to\infty} G^n = G.$$

Proof
Denote $\rho^n := \rho_{ext}$ to n parties. The state ρ^n is symmetric, therefore use quantum de Finetti theorem:

Theorem

[3] (Christandl, König, Mitchison, Renner).
Let $|\Psi\rangle$ be a symmetric state on n parties and let $\zeta^k = Tr_{k+1,\dots,n} |\Psi\rangle\langle\Psi|$. Then there exists a probability distribution $\{\alpha_j\}$ and states $\{\tau_j\}$ such that

$$\|\zeta^k - \sum_j \alpha_j \tau_j \otimes \tau_j\|_1 \leq \epsilon$$

Therefore there exists $\alpha_j \geq 0$, $\sum_j \alpha_j = 1$, and states $\tau_j \in \mathcal{C}_{2m}$, s.t.

$$\|\rho^n_{1,2} - \sum_j \alpha_j \, \tau_j \otimes \tau_j\|_1 \leq \tilde{\epsilon} = 4\frac{2^m}{n}.$$

Applying $\Lambda^{1,2}$ we obtain

$$\|\Lambda\|_1^2 \tilde{\epsilon} =: \hat{\epsilon} \geq \|\Lambda \sum_j \alpha_j \, \tau_j(n) \otimes \tau_j(n) \Lambda\|_1 = \sum_j \alpha_j \|\Lambda \, \tau_j(n) \otimes \tau_j(n) \, \Lambda\|_1.$$

Therefore for every $j = 1, ...m$ we obtain

$$\|\Lambda \tau_j(n) \otimes \tau_j(n) \Lambda\|_1 \leq \epsilon.$$

For every j the state $\tau_j(n)$ is ϵ'-close to a pure Gaussian state $\left|\psi_j(n)\right\rangle$. Therefore the state ρ is close to the convex-Gaussian state:

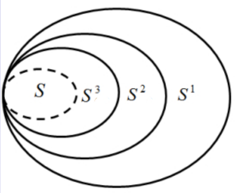
$$\begin{aligned} \|\rho - \sum_j \alpha_j \left|\psi_j(n)\right\rangle \left\langle\psi_j(n)\right| \|_1 &\leq \|\rho - \sum_j \alpha_j \tau_j(n)\|_1 + \sum_j \alpha_j \|\tau_j(n) - \left|\psi_j(n)\right\rangle \left\langle\psi_j(n)\right| \|_1 \\ &\leq \tilde{\epsilon} + \epsilon'. \end{aligned}$$

Q.E.D.

Complementary criterion

Separability criteria

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be a finite dimensional Hilbert space. Denote S the set of separable operators, i.e. the conical combination of all pure product states $\{|\psi_A\rangle\langle\psi_A| \otimes |\psi_B\rangle\langle\psi_B|\}$.



$\rho_{AB} \in S^N$ iff $\exists \rho_{AB^N} \in \mathcal{B}(\mathcal{H}_{AB^N})$ s.t.

1. $\rho_{AB^N} \geq 0$
2. $\mathrm{Tr}_{B^{N-1}}(\rho_{AB^N}) = \rho_{AB}$
3. ρ_{AB^N} is Bose symmetric in $\mathcal{H}_B^N$, i.e. $\rho_{AB^N}(\mathbb{1} \otimes P_{sym}^N) = \rho_{AB^N}$.

The sequence $\{S^N\}_{N=1}^{\infty}$ converges to S from outside [5] (Doherty et al.):

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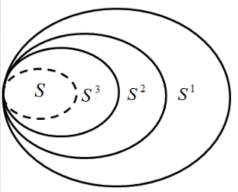

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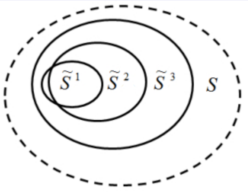

Define sets $\tilde{S}^N$ as:

$$\tilde{S}^N = \{\frac{N}{N+d}\rho_{AB} + \frac{1}{N+d}\rho_A \otimes I_B : \rho_{AB} \in S^N\}.$$

The sequence $\{\tilde{S}^N\}_{N=1}^{\infty}$ converges to S from inside [6] (Navascues et al.)

$$\tilde{S}^1 \subset \tilde{S}^2 \subset ... \subset S, \text{ with } \overline{\lim_{N\to\infty} S^N} = S.$$

Program. Given a state $\rho_{AB} \in \mathcal{B}(\mathcal{H})$ determine whether it is separable.

1. Check whether $\rho_{AB} \in S^k$. If ”NO”, then ρ is entangled. Done.
   If ”YES”, then go to 2.
2. Check whether $\rho_{AB} \in \tilde{S}^k$. If ”YES”, then it is separable. Done.
   If ”NO”, repeat steps for $k = k + 1$.

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We have that

$$G^1 \supset G^2 \supset \ldots \supset G.$$

Define $\tilde{G}^n = \{\epsilon_n \rho + (1 - \epsilon_n) I/2^m : \rho \in G^n\} \subset G$.
Problem. Show that

$$\overline{\lim_{n\to\infty} \tilde{G}^n} = G.$$

Proposition

[1] For any even state $\rho \in \mathcal{C}_{2m}$ there exists $\epsilon > 0$ such that $\rho_\epsilon = \epsilon\rho + (1 - \epsilon)I/2^m$ is convex-Gaussian.

I.e. need to show that

$$\epsilon_n \to 1 \text{ as } n \to \infty, \text{ for } \rho_n \in G^n.$$

Extension to $\mathcal{C}_{2mn}$

The isomorphism J between $\mathcal{C}_{2mn}$ and $\mathcal{C}_{2m}^{\otimes n}$ is the following:

$$J(c_j) = I \otimes ... \otimes c_j \otimes P_{all} \otimes ... \otimes P_{all}, \text{ for } 2m(k-1)+1 \leq j \leq 2mk,$$

where c_j stands on the k-th component and P_{all} acts on each $\mathcal{C}_{2m}$.
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For $n = 2$, for any state

$$\mu = \sum_k \alpha_k Z_k^1 Z_k^2 \ \in \mathcal{C}_{4m},$$

where Z^1 acts on $c_1, \ldots, c_{2m}$ and Z^2 acts on $c_{2m+1}, \ldots, c_{4m}$. Applying J this state gets mapped to:

$$J(\mu) = \sum_k \alpha_k Z_k^1 \otimes P_{all}^{\epsilon_k} Z_k^2 \ \in \mathcal{C}_{2m} \otimes \mathcal{C}_{2m},$$

where $\epsilon_k = 0$, if Z_k^1 is even and $\epsilon_k = 1$, if Z_k^1 is odd.

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Any state

$$\sigma = \sum_k \beta_k Z_k^1 \otimes Z_k^2 \ \in \mathcal{C}_{2m} \otimes \mathcal{C}_{2m}$$

gets mapped onto a state

$$J^{-1}(\sigma) = \sum_k \beta_K Z_k^1 P_{all}^{\epsilon_k} Z_k^2 \ \in \mathcal{C}_{4m},$$

where $P_{all}^{\epsilon_k}$ acts on $c_{2m+1} ... c_{4m}$.

The operator $\Lambda^{k,l}$ is equivalent to Γ acting on $\mathcal{C}_{2mn}$: for any $1 \leq k \neq l \leq n$,

$$\Gamma^{k,l} = \sum_{j=1}^{2m} (-1)^{m-j} \; c_{2m(k-1)+j} \;\; c_{2m(l-1)+1} \cdots \hat{c}_{2m(l-1)+j} \cdots c_{2ml}.$$

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For an even state $\rho = \sum_j \alpha_j Z_j^1 \;\in \mathcal{C}_{2m}$ the extension $\rho_{ext} \;\in \mathcal{C}_{2mn}$ has the following form

$$\rho_{ext} = 2^{-m(n-1)} \rho + Corr.$$

Every term in the correlations $Corr = \sum_j \beta_j Z_j^1 ... Z_j^n$ acts nontrivially on the last $(n-1)$ spaces spanned by c_{2m+1} ... c_{2mn}. Here Z^k acts on the space spanned by $c_{2m(k-1)+1} \cdots c_{2mk}$.

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