Stability of local observables

in closed and open quantum systems

Angelo Lucia
anlucia@ucm.es

Universidad Complutense de Madrid

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Many-body evolutions

- Fix a finite metric graph Λ sufficiently regular (e.g., $\Lambda = \mathbb{Z}_N^D$, and N is called the *system size*).
- Algebra of observables: $\mathcal{A}_\Lambda = \bigotimes_{x \in \Lambda} B(\mathcal{H}_x)$, where $H_x \sim \mathbb{C}^d$.
- Local observables: $\mathcal{A}_X \subset \mathcal{A}_\Lambda$, for $X \subset \Lambda$ independent of N.

Definition (Many-body evolution)

A continuous (semi)group of completely positive, unital maps $\gamma_t : \mathcal{A}_\Lambda \to \mathcal{A}_\Lambda$

- Closed systems: γ_t is a unitary evolution and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(A) = i[H, \gamma_t(A)]$
- Open systems: $\|\gamma_t\| \leqslant 1$ and $\frac{\mathrm{d}}{\mathrm{d}t}\gamma_t(A) = \mathcal{L}(\gamma_t(A))$

In order for γ_t to be c.p. and unital, $\mathcal{L}$ has to satisfy the *Lindblad condition*.

Lindblad form

Definition (Lindblad form)

$$\mathcal{L}(A) = i[H, A] + \sum_j K_j^* A K_j - \frac{1}{2}\{K_j^* K_j, A\}$$

where H is Hermitian.

Remark

γ_t will not be an automorphism in general.

Local evolutions

Definition (Local evolution)

γ_t is said to be local if $\mathcal{L}$ can be written as a sum of local terms

$$\mathcal{L} = \sum_{x \in \Lambda} \mathcal{L}_x$$

where the support of $\mathcal{L}_x$ is finite: $\mathcal{L}_x = \mathcal{L}_x \otimes \mathbb{1}$, and $\mathcal{L}_x : \mathcal{A}_{B_x(r)} \to \mathcal{A}_{B_x(r)}$ and is Lindlad. r is called the range of the interaction.

Definition (Quasi-local evolution)

γ_t is said to be quasi-local if $\mathcal{L}$ can be written as a sum of local terms

$$\mathcal{L} = \sum_{x \in \Lambda} \sum_{r \geqslant 0} \mathcal{L}_x(r)$$

where the support of $\mathcal{L}_x(r)$ is $B_x(r)$ and $\|\mathcal{L}_x(r)\| \leqslant f(r)$. $f(r)$ is the decay rate and is assumed to be faster than any polynomial.

Expectation values of local observables

Expectation values

- In closed systems, we are usually interested in the ground states of H
- At times, we are also interested in other eigenstates of H
- They are all steady states of γ_t: $\omega \circ \gamma_t = \omega$
- In open systems, we are usually interested in the case where γ_t has a unique steady state ρ_∞ and no periodic points: $\rho_\infty \circ \gamma_t = \rho_\infty$ and moreover

$$\lim_{t \to \infty} \gamma_t(A) = \rho_\infty(A)\mathbb{1}$$

General problem

Given one such "interesting state" ρ, how sensitive is the expectation value of local observables $\rho(A)$ with respect to changes in $\mathcal{L}$?
Can it tolerate local perturbations, i.e. replacing $\mathcal{L}$ by $\mathcal{L}' = \mathcal{L} + E$, where

$$E = \sum_{x \in \Lambda} E_x$$

where $\|E_x\|$ is small (but $\|E\|$ is not)?

$$\left|\rho(A) - \rho'(A)\right| \leqslant k_Y \sup_x \|E_x\| \,, \quad A \in \mathcal{A}_Y$$

where k_Y is a constant independent of the system size.

Motivation

1. Quantum phase classification
   - The physicists definition of quantum phase is given in terms of smoothness of expectation values of observables
   - Local observables are what is reasonable to expect to be actually measurable in an experiment
   - The mathematicians definition requires the states to be groundstates of the extremes of a smooth path of gapped local Hamiltonians
2. Noise modelling
3. Using engineered dissipation to prepare interesting quantum states as fixed points of open system dynamics (entangled states, magic states, quantum codes, etc.)

Closed systems

Local observables are stable if $H(s) = H + sE$ is uniformly gapped:

$$\inf_{s\in[0,1]} \operatorname{gap} H(s) \geqslant \gamma > 0$$

We reduce the problem to stability of the spectral gap (Frustration-free, pbc: [Michalakis, Pytel, 2011])

Open systems with unique fixed point

Local observables are stable if γ_t is fast mixing, i.e. if

$$t_\varepsilon = \inf \left\{ t > 0 \,|\, \sup_A \|\gamma_t(A) - \rho_\infty(A)\mathbb{1}\| \leqslant \varepsilon \|A\| \right\}$$

scales sub-linearly with system size [Cubitt, L., Michalakis, Perez-Garcia, 2013]

Quasi-adiabatic evolution [Hastings 2004]

Let $H(s)$ be a family of uniformly gapped local Hamiltonians. Let $P(s)$ the projector on the groundstate space of $H(s)$.

Quasi-adiabatic evolution

There exists a unitary $U(s)$ such that

$$P(s) = U(s)P(0)U(s)^*$$

and $U(s)$ has a quasi-local structure given by

$$\frac{\mathrm{d}}{\mathrm{d}s} U_s = iD(s)U(s), \quad U(0) = \mathbb{1}$$

with $D(s)$ being quasi-local Hermitian operator (i.e. a time-dependent Hamiltonian)

Also called spectral flow, since it is not specific of the groundstate.

L.R. bounds for quasi-adiabatic evolution

Let $\alpha_s(A) = U(s)^* A U(s)$ the dual of the spectral flow.

Lemma (Bachmann, Michalakis, Nachtergaele, Sims 2011)

For $A \in \mathcal{A}_X$, $B \in \mathcal{A}_Y$

$$\|[\alpha_s(A), B]\| \leqslant 2\,\|A\|\,\|B\|\, e^{vs - \mu\,\mathrm{dist}(X,Y)}$$

By denoting $1/\varepsilon$, where $\varepsilon = \sup_x \|E_x\|$, we have

$$H(s) = H + s\frac{E}{\varepsilon}, \quad s \in [0, \varepsilon]$$

in such a way that v, μ are independent of ε.

Local perturbations pertub locally

Remember that α_s was generated by $D(s) = \sum_x D_x(s)$
For $Y \subset \Lambda$, let α_s^Y be the evolution generated by $D^Y(s) = \sum_{x \in Y} D_x(s)$

Lemma

Let $A \in \mathcal{A}_X$. Let $X_r = \{x |\, \mathrm{dist}(x, X) \leqslant r\}$ Then

$$\left\| \alpha_s(A) - \alpha_s^{X_r}(A) \right\| \leqslant \|A\|\, e^{vs} g(r)$$

with $g(r)$ a fast decaying function.

Remark

$\alpha_s^{X_r}(A)$ is supported on X_r.

Stability

Theorem

For all $A \in \mathcal{A}_X$ *it holds that* $\left| \langle A \rangle_{P(0)} - \langle A \rangle_{P(s)} \right| \leqslant \delta(s)\, k_X \, \|A\|$ *where* $\delta(s)$ *grows polynomially in* s *and* $\langle A \rangle_{P(s)} = \operatorname{tr} AP(s)$

Proof.

By using the spectral flow we obtain

$$\begin{aligned} \langle A \rangle_{P(s)} &= \operatorname{tr} AP(s) = \operatorname{tr} AU(s)P(0)U(s)^* \\ &= \operatorname{tr} U(s)^* AU(s)P(0) = \operatorname{tr} \alpha_s(A)P(0) = \langle \alpha_s(A) \rangle_{P(0)} \end{aligned}$$

and therefore

$$\left| \langle A \rangle_{P(0)} - \langle A \rangle_{P(s)} \right| \leqslant \|\alpha_s(A) - A\| \leqslant \left\| \alpha_s(A) - \alpha_s^{X_r}(A) \right\| + \left\| \alpha_s^{X_r}(A) - A \right\|$$

Proof.

$$\left|\langle A\rangle_{P(0)} - \langle A\rangle_{P(s)}\right| \leqslant \left\|\alpha_s(A) - \alpha_s^{X_r}(A)\right\| + \left\|\alpha_s^{X_r}(A) - A\right\|$$

We can make the first term arbitrarily small by Lieb-Robison bounds, choosing $r \sim O(s)$

Then

$$\begin{aligned}\left\|\alpha_s^{X_r}(A) - A\right\| &\leqslant \int_0^s \left\|[D^{X_r}(\tau), A]\right\| \mathrm{d}\tau \\ &\leqslant \sum_{x\in X_r} \int_0^s \|[D_x(\tau), A]\| \, \mathrm{d}\tau \leqslant \delta(s)\, k_X \, \|A\|\end{aligned}$$

$\square$

Open system case

In the closed system case, we used two important facts related to the spectral flow:

1. α_s connects $P(0)$ to $P(s)$ in finite time
2. α_s can be localized by L.R. : finite times $\to$ finite volumes

For open systems, spectral flow is not available. (Unless somebody has an idea to prove me wrong...)

But if we have a unique fixed point, then

$$\eta_t(A) = \frac{\|\gamma_t(A) - \rho_\infty(A)\mathbb{1}\|}{\|A\|} \to 0$$

Let γ_t and γ'_t be the original and the perturbed evolutions, respectively. We can apply Duhamel formula:

$$\left\|\gamma_t(A) - \gamma'_t(A)\right\| \leqslant \int_0^t \left\|\gamma'_{t-s}(A) E \gamma_s(A)\, \mathrm{d}s\right\| \leqslant \sum_x \int_0^t \|E_x \gamma_s(A)\|\, \mathrm{d}s$$

where we have assumed $\|\gamma'_t\| \leqslant 1$.
Let us focus on one fixed $x \in \Lambda$.

We can split the integral at a time $t_0 = t_0(x)$ to be determined later:

1. short times: dissipative version of Lieb-Robinson bounds

$$\int_0^{t_0} \|E_x \gamma_s(A)\| \, \mathrm{d}s \leqslant \|E_x\| \, \|A\| \, \frac{1}{\mu v} e^{\mu(d - v t_0)}$$

   with $d = \operatorname{dist}(x, \operatorname{supp} A)$

2. long times: We use the fact $E_x(\mathbb{1}) = \mathcal{L}_x(\mathbb{1}) - \mathcal{L}'_x(\mathbb{1}) = 0$

$$\int_{t_0}^{t} \|E_x \gamma_s(A)\| \, \mathrm{d}s = \int_{t_0}^{t} \|E_x[\gamma_s(A) - \rho_\infty(A)\mathbb{1}]\| \, \mathrm{d}s$$

$$\leqslant \|E_x\| \int_{t_0}^{t} \|\gamma_s(A) - \rho_\infty(A)\mathbb{1}\| \, \mathrm{d}s \leqslant \|E_x\| \int_{t_0}^{\infty} \eta_s(A) \, \mathrm{d}s$$

By taking $t_0(x) = 2d(x)/v$ where $d(x) = \mathrm{dist}(x, \operatorname{supp} A)$ we have that

$$\left\|\gamma_t(A) - \gamma'_t(A)\right\| \leqslant c\,\|A\|\,(\sup_x \|E_x\|) \sum_{x\in\Lambda} e^{-\mu d(x)} + \int_{2d(x)/v}^{\infty} \eta_s(A)\,\mathrm{d}s$$

Theorem

Let $A \in \mathcal{A}_Y$. If $\int_x^\infty \eta_s(A)\,\mathrm{d}s$ is summable over Λ and is independent of system size, then

$$\left\|\gamma_t(A) - \gamma'_t(A)\right\| \leqslant k_Y\,\|A\| \sup_x \|E_x\|$$

Rapid mixing

Remember that

$$t_\varepsilon = \inf\{t > 0 \mid \sup_A \eta_t(A) \leqslant \varepsilon\}$$

Theorem

If $t_\varepsilon \leqslant \mathcal{O}(\log^m |\Lambda| + \log \frac{1}{\varepsilon})$ for some $m > 0$, then for each $A \in \mathcal{A}_Y$ it holds that

$$\eta_t(A) \leqslant c_A e^{-\gamma t}$$