A NOTE ON THE SIZE OF THE LARGEST BALL INSIDE A CONVEX POLYTOPE

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Abstract. Let $m > 1$ be an integer, B_m the set of all unit vectors of $\mathbb{R}^m$ pointing in the direction of a nonzero integer vector of the cube $[-1,\,1]^m$. Denote by s_m the radius of the largest ball contained in the convex hull of B_m. We determine the exact value of s_m and obtain the asymptotic equality $s_m \sim \frac{2}{\sqrt{\log m}}$.

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§1. INTRODUCTION

Let $m \geq 2$ be an integer, and consider the sets

$$A_m = \{-1,\,0,\,1\}^m \setminus \{\vec{0}\}, \text{ and } B_m = \left\{ \frac{v}{||v||} \,\middle|\, v \in A_m \right\}.$$

Let C_m be the convex hull of B_m, and s_m the radius of the largest ball contained in C_m. (Due to the apparent symmetries of C_m, such a largest ball is necessarily centered at the origin.) In the paper [B-M-S(2005)] (dealing with rotation numbers/vectors of billiards) we needed sharp lower and upper estimates for the extremal radius s_m. Here we determine the exact value of s_m which, of course, implies such estimates.

Theorem.

$$s_m = \left(\sum_{k=1}^{m} \frac{1}{\left(\sqrt{k} + \sqrt{k-1}\right)^2} \right)^{-1/2}$$

The Theorem implies that

$$\frac{1}{4} \log m < s_m^{-2} < \frac{1}{4} \log m + \frac{5}{4},$$

where log denotes the natural logarithm. As an immediate corollary, the quantity s_m is asymptotically equal to $\frac{2}{\sqrt{\log m}}$.

§2. Proof of the Theorem

The proof will be split into of a few lemmas. The first one of them is a trivial observation.

Lemma 1. The set of vertices B_m of the convex polytope C_m, and hence C_m itself, is invariant under the action of the full isometry group G of the cube $[-1, 1]^m$. (The group G is generated by all permutations of the coordinates in $\mathbb{R}^m$, and by all reflections across the coordinate hyperplanes.) $\square$

We will use the notation $v_k = \frac{1}{\sqrt{k}} \sum_{i=1}^{k} e_i$ $(k = 1, \ldots, m)$ for some specific vertices of C_m. (Here e_i stands for the i-th standard unit vector of $\mathbb{R}^m$.)

Lemma 2. The simplex S, spanned by the linearly independent vectors v_k $(k = 1, \ldots, m)$ as vertices, is a face of the polytope C_m whose outer normal vector is $u = (u_1, \ldots, u_m)$ with the coordinates $u_i = \sqrt{i} - \sqrt{i-1}$.

Proof. Consider the scalar product function $\langle v, u \rangle$ $(v \in B_m)$ restricted to the set B_m of vertices of the polytope C_m. Elementary inspection shows that this scalar product function can only attain its maximum value at the vertices v_k, and actually,

$$\langle v_k, u \rangle = 1 \tag{1}$$

for each $k = 1, \ldots, m$. This proves all claims of the lemma. $\square$

Lemma 3. For any face F of the polytope C_m there exists a congruence $g \in G$ such that $g(F) = S$.

Proof. Fix a non-zero vector $w = (w_1, \dots, w_m)$ whose ray $R(w) = \{\lambda w | \ \lambda \geq 0\}$ intersects the interior of the face F. By selecting w in a generic manner, we can assume that the absolute values $|w_i|$ of its coordinates are distinct and all different from zero. Therefore, by applying a suitable element $g \in G$, we can even assume that

$$w_1 > w_2 > \cdots > w_m > 0. \tag{2}$$

We claim that $g(F) = S$. Indeed, by (2) we have the linear expansion

$$w = \sum_{k=1}^{m} \sqrt{k}(w_k - w_{k+1})v_k.$$

of w in the basis $\{v_1, \dots, v_m\}$ with positive coefficients. (With the natural convention $w_{m+1} = 0$.) This proves that some positive multiple of w is a convex linear combination of the vertices of S with non-zero coefficients, so the face $g(F)$ shares an interior point with S. $\square$

It follows from the previous lemma that the radius s_m of the inscribed sphere is actually the distance between S and the origin. However, this distance is equal to $s_m = \langle u, e_1 \rangle / ||u|| = 1/||u||$ by (1). It is clear that

$$||u||^2 = \sum_{k=1}^{m} \frac{1}{\left(\sqrt{k} + \sqrt{k-1}\right)^2}.$$

finishing the proof of our theorem. $\square$

Define $R_m = \sum_{k=1}^{m} \frac{1}{k}$. For the asymptotic value of s_m we use the elementary fact that $\log m < R_m < \log m + 1$.

$$\frac{1}{4}\log m < \sum_{k=1}^{m} \frac{1}{4k} < \sum_{k=1}^{m} \frac{1}{\left(\sqrt{k} + \sqrt{k-1}\right)^2} = ||u||^2$$

$$< 1 + \sum_{k=2}^{m} \frac{1}{4(k-1)} < 1 + \frac{1}{4}(\log m + 1) = \frac{1}{4}\log m + \frac{5}{4}.$$

Remark 1. Let K be the convex cone generated by the vectors v_k, $k = 1, \dots, m$. The meaning of Lemma 3 is that the cones $g(K)$ $(g \in G)$ form a triangulation of the space $\mathbb{R}^m$. As a matter of fact, the intersections of the cones $g(K)$ with the standard $(m-1)$-simplex

$$S_{m-1} = \left\{ x \in \mathbb{R}^m | \ \sum_{i=1}^{m} x_i = 1, \ x_i \geq 0 \text{ for all } i \right\}$$

form the baricentric subdivision of S_{m-1}.

Remark 2. The following natural question has been considered in several papers, for instance in [B-F(1988)] and [B-W(2003)]. What is the maximal radius $r(m, N)$ of the inscribed ball of the convex hull of N points chosen from the unit ball of $\mathbb{R}^m$? In our case $N = 3^m - 1$ and one may wonder how close s_m and B_m are to the maximal radius and best arrangement. It turns out that they are very far: it follows from the results of [B-F(1988)], [R(1963)], and [B-W(2003)] that, in the given range $N = 3^m - 1$,

$$r(m, N) = \left(\frac{8}{9}\right)^{1/2} (1 + o(1))$$

as $m \to \infty$. So the optimal radius is much larger than s_m. This also shows that, as expected, B_m is far from being distributed uniformly on the unit sphere.

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