only one full measure connected ergodic component for the flow and the theorem is proved.

In the talk a brief historical introduction on the conjecture and the main ideas of this proof and of the proof of the Main Lemma were given.

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Introduction to Hyperbolic Billiards

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Semi-dispersing billiards are defined as follows: We take a C^2-smooth, connected, Riemannian manifold M without boundary and with a positive injectivity radius ρ, and we remove from M finitely many compact, geodesically convex subsets $B_i \subset M$ (the so-called scatterers, $i = 1, \cdots, n$) that have C^1-smooth boundaries ∂B_i. The billiard table (confiuration space) is the set

$$B = M \setminus \bigcup_{i=1}^{n} \mathrm{Int(B_i)}.$$

We assume that B is compact. The billiard flow $(M, \{\Phi^t\}\, \mu)$ describes the uniform motion (with unit speed) of a point particle in B along geodesic lines, enduring elastic reflections when hitting a boundary component ∂B_i of B. If a trajectory ever hits an intersection $B_i \cap B_j$ $(i \neq j)$, then such a trajectory is simply undefined. We always assume that all sectional curvatures κ of M are bounded above by a real number K.

It has been very well known since the early studies of mathematical billiards by Ya. G. Sinai in the 1960s [S63], [S70], that obtaining upper bounds (in particular, finiteness) for the number of collisions in terms of the length of trajectory segments plays a pivotal role in studying the fine ergodic and statistical properties of such systems. Such bounds are especially useful in effectively estimating the topological entropy of hard ball systems, as Burago, Ferleger and Kononenko showed in 1998, [BFK98].

Our goal is to review the main results in this area of research by also giving the audience a glimps into the intricate geometric tools developed to tackle such problems. We will be discussing the geometric aspects (of the proofs) of the results below.

One of the early results is due to L. N. Vaserstein [V79] and G. Galperin [G81].

Theorem. If a natural *non-degeneracy condition* (see below) holds true for the semi-dispersing billiard flow $(M, \{\Phi^t\}, \mu)$, then in any trajectory the number of collisions during any finite time interval is finite.

Definition. The billiard table B (of a semi-dispersing billiard) is *non-degenerate* in an open subset U of M with the constant $C > 0$ if for every non-empty subset $I \subset \{1, 2, \cdots, n\}$ and for every $y \in (U \cap B) \setminus \bigcap_{j \in I} B_j$

$$\max \left\{ \frac{\operatorname{dist}(y, B_k)}{\operatorname{dist}(y, \cap_{j \in I} B_j)} \,\middle|\, k \in I \right\} \geq C, \tag{1}$$

whenever $\bigcap_{j \in I} B_j \neq \emptyset$. (We note that this is a local geometric property.)

Definition. B is *non-degenerate* in an open domain U of M if there exist constants $\delta > 0$, $C > 0$ such that B is non-degenerate with the constant C in any δ-ball of U.

In 1998 Burago, Ferleger, and Kononenko [BFK98] proved the following crucial result.

Theorem. Assume B is non-degenerate in an open neighborhood $U \subset M$ of a point $x \in \partial B$. Then there exists a neighborhood U_x of x (in M) and a number $P_x > 0$ such that every billiard trajectory entering U_x leaves it after making at most P_x collisions.

As an immediate corollary, we get that
Corollary. For every nondegenerate semi-dispersing billiard there exists a constant $P > 0$ such that every trajectory of the billiard flow makes no more than $P \cdot (t + 1)$ collisions during any time interval of length t.

For open ball systems in the euclidean space $\mathbb{R}^k$, the same authors also proved in [BFK98] the following theorem.
Theorem. The number of collisions of N elastic balls in $\mathbb{R}^k$ is not larger than

$$\left(32 \cdot \sqrt{\frac{m_{\max}}{m_{\min}}} \cdot \frac{r_{\max}}{r_{\min}} \cdot N^{3/2} \right)^{N^2}.$$

Here $m_{\max}$ ($m_{\min}$) denotes the maximum (minimum) mass of the particles, whereas $r_{\max}$ ($r_{\min}$) is the maximum (minimum) value of the radii.

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