# RÉNYI'S PARKING PROBLEM REVISITED

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ABSTRACT. Rényi's parking problem (or $1D$ sequential interval packing problem) dates back to 1958, when Rényi studied the following random process: Consider an interval $I$ of length $x$, and sequentially and randomly pack disjoint unit intervals in $I$ until the remaining space prevents placing any new segment. The expected value of the measure of the covered part of $I$ is $M(x)$, so that the ratio $M(x)/x$ is the expected filling density of the random process. Following recent work by Gargano *et al.* [GWML(2005)], we studied the discretized version of the above process by considering the packing of the $1D$ discrete lattice interval $\{1, 2, \ldots, n+2k-1\}$ with disjoint blocks of $(k+1)$ integers but, as opposed to the mentioned [GWML(2005)] result, our exclusion process is symmetric, hence more natural. Furthermore, we were able to obtain useful recursion formulas for the expected number of $r$-gaps $(0 \leq r \leq k)$ between neighboring blocks. We also provided very fast converging series and extensive computer simulations for these expected numbers, so that the limiting filling density of the long line segment (as $n \to \infty$) is Rényi's famous parking constant, $0.7475979203\ldots$.

## 1. INTRODUCTION

Rényi's Parking Problem (or $1D$ sequential interval packing problem) dates back to 1958 when Rényi [R(1958)] studied the probabilistic properties of the following random process: Consider an interval $I$ of length $x >> 1$ ($x$ will eventually tend to infinity), and sequentially and randomly pack disjoint unit intervals in $I$ as long as the remaining space permits placing any new unit segment in $I$. At each step of the packing process the position of the newly placed interval is chosen uniformly from the available space. Denote the expected value of the measure of the covered part by $M(x)$, so that the ratio $M(x)/x$ is the expected filling density of the "parking process". (The interval $I$ is the street curb, and the packed unit segments are the parked cars.) Rényi himself proves the following continuous recursion for $M(x)$

$$(1.1) \qquad M(x) = \begin{cases} 0 \text{ for } 0 \leq x < 1 \\ 1 + \frac{2}{x-1} \int_0^{x-1} M(y) dy \text{ for } x \geq 1, \end{cases}$$

and from this he deduces the asymptotic mean filling density

$$(1.2) \quad m = \lim_{x \to \infty} M(x)/x = \int_0^\infty \exp\left[-2\int_0^x \frac{1-e^{-y}}{y}dy\right]dx = 0.7475979203\ldots,$$

which number is now known as Rényi's Parking Constant. Rényi [R(1958)] further proves the asymptotic formula

$$(1.3) \qquad\qquad M(x) = mx + m - 1 + \mathcal{O}(x^{-n})$$

for every positive integer $n$, which was further improved by Dvoretzky and Robbins [DR(1964)] to

$$(1.4) \qquad\qquad M(x) = mx + m - 1 + \mathcal{O}\left[\left(\frac{2e}{x}\right)^{x-3/2}\right].$$

In that paper Dvoretzky and Robbins also prove that

$$\inf_{x \leq t \leq x+1} \frac{M(t)+1}{t+1} \leq m \leq \sup_{x \leq t \leq x+1} \frac{M(t)+1}{t+1}.$$

The first "discretized" version of the problem, namely the expected density derived from sequential packings of non-overlapping neighboring pairs of integer points, i.e., edges or bonds, selected at random on a long segment of a $1D$ lattice was first given by Page [P(1959)]. His results have been confirmed and extended in various ways by Downton [D(1961)], Mackenzie [M(1962)], Widom [W(1966)], and Solomon [S(1967)]. This random sequential addition model is pertinent when molecules are sequentially absorbed, and once absorbed, are fixed. The expected density derived by non-sequentially packing disjoint, indistinguishable, neighboring pairs of points on a linear lattice, each configuration being considered equally likely, was first given by Jackson and Montroll [JM(1958)], and extended to neighboring triplets by Fisher and Temperly [FT(1960)].

Following a more recent paper by Gargano *et al.* [GWML(2005)] we studied the discretized version of the above process by considering the sequential packing of the $1D$ discrete lattice interval $\{1, 2, \ldots, n + 2k - 1\}$ $(n \gg 1)$ with disjoint blocks of $k + 1$ consecutive integers but, as opposed to the approach in [GWML(2005)], our packing process is symmetric, hence more natural. Furthermore, we were able to obtain useful recursions for the expected number of $r$-gaps $(0 \leq r \leq k)$ between neighboring blocks (cars). The construction of such a recursion is one of the open problems listed at the end of [GWML(2005)].

We also provided very fast (faster than any exponential) converging series for the expected number of $r$-gaps, and carried out extensive computer simulations for these expected numbers, indicating that the limiting filling density is indeed Rényi's famous parking constant $m = 0.7475979203\ldots$, also in the discrete parking problem.

It has to be noted, however, that our approach differs slightly, albeit just in minor technical terms, from the approach in [GWML(2005)]. Namely, instead of considering packings with disjoint $(k+1)$-blocks of consecutive integer lattice points, i.e., with $k$ consecutive edges or bonds between them, we consider the positions of the centers of these blocks. The available space for the centers is either the original integer lattice (when $k$ is even) or the original lattice shifted by $1/2$ units (when $k$ is odd), so that the distance between neighboring centers is always at least $k+1$, i.e., the gap between them contains at least $k$ points. This approach is clearly equivalent to that of [GWML(2005)].

Finally, in the paper [GWML(2005)] the authors consider the process in which not only the distances between neighboring centers of blocks are at least $k+1$ but, in addition, each center is distanced at least $k+2$ from at least one of its two neighbors (an asymmetric model). At the end of their paper among the open problems they list the need to study the symmetric model in which we drop the second lower bound requirement ($\geq k+2$). This is exactly the kind of model we are investigating in this paper.

## 2. The Model

For incoming cars (i.e. centers of $(k+1)$-blocks of consecutive integers as described in the introduction) there are $n+k-1$ parking slots, labelled as $1, 2, \ldots, n+k-1$ ($n, k \geq 1$), in a row that the cars, arriving one-by-one, want to occupy. The drivers have the desire that the distance between occupied parking slots is at least $k+1$, i.e., the gap between neighboring cars (and also the gap before the first car and after the last one) contains at least $k$ unoccupied slots. ($k$ being a fixed positive integer.) When a new car arrives, the driver considers all available slots and occupies one of them with equal probability. The process lasts as long as the cars can occupy parking slots.

At the end of the process there will be gaps of sizes $k, k+1, \ldots, 2k$. For any $r$, $k \leq r \leq 2k$, and for any positive integer $n$ let $a_n^{(r)}$ be the expected number of $r$-gaps produced by the above random process.

Since the events $A_i$ ($1 \leq i \leq n-k-1$) that the first arriving car occupies the slot $i+k$ are equally probable, pairwise exclusive, and their union is the sure event, one immediately gets the recursion formula

(2.1) $$a_n^{(r)} = \frac{2}{n-k-1} \sum_{i=1}^{n-k-1} a_i^{(r)}$$

for $n \geq k + 2$. Furthermore, the initial conditions

$$(2.2) \qquad a_n^{(r)} = \begin{cases} 1 \text{ if } n = r - k + 1 \\ 0 \text{ if } 1 \leq n \leq k + 1, \ n \neq r - k + 1 \end{cases}$$

hold true, $k \leq r \leq 2k$.

We take $s_n^{(r)} = \sum_{i=1}^{n} a_i^{(r)}$, $t_n^{(r)} = \dfrac{s_n^{(r)}}{n(n + 2k + 1)}$, so that

$$s_n^{(r)} = s_{n-1}^{(r)} + \frac{2}{n - k - 1} \cdot s_{n-k-1}^{(r)}$$

and

$$(2.3) \qquad n(n + 2k + 1)t_n^{(r)} = (n - 1)(n + 2k)t_{n-1}^{(r)} + 2(n + k)t_{n-k-1}^{(r)}$$

for $n \geq k + 2$, $k \leq r \leq 2k$.

For $n \geq 2$ define $u_n^{(r)} = t_n^{(r)} - t_{n-1}^{(r)}$. From (2.3) elementary calculation yields the $k$-step linear recursion

$$(2.4) \qquad u_n^{(r)} = \frac{-2(n + k)}{n(n + 2k + 1)} \cdot \sum_{i=1}^{k} u_{n-i}^{(r)}$$

for $n \geq k + 2$. The initial values $\left\{ u_n^{(r)} \mid 2 \leq n \leq k + 1 \right\}$ for the $\left( u_n^{(r)} \right)_{n=2}^{\infty}$ sequence are as follows:

$$(2.5) \qquad u_n^{(r)} = \begin{cases} 0 & \text{if } 2 \leq n \leq r - k \\ \frac{1}{(r-k+1)(r+k+2)} & \text{if } n = r - k + 1 \text{ and } r \geq k + 1 \\ \frac{1}{n(n+2k+1)} - \frac{1}{(n-1)(n+2k)} & \text{if } r - k + 2 \leq n \leq k + 1. \end{cases}$$

The following formulas are immediate consequences of the definitions of the involved quantities.

$$(2.6) \qquad t_n^{(r)} = \frac{s_1^{(r)}}{2k + 2} + \sum_{i=2}^{n} u_i^{(r)}, \quad n \geq 1,$$

where

$$s_1^{(r)} = \begin{cases} 1 & \text{if } r = k \\ 0 & \text{if } k < r \leq 2k. \end{cases}$$

$$(2.7) \qquad s_n^{(r)} = n(n + 2k + 1)t_n^{(r)}, \quad n \geq 1,$$

$$(2.8) \qquad a_n^{(r)} = s_n^{(r)} - s_{n-1}^{(r)}, \quad n \geq 1,$$

where $s_0^{(r)} = 0$ by convention.

**Corollary 2.9.** *The limiting densities*

$$(2.10) \qquad D(k,r) = (r+1) \lim_{n \to \infty} \frac{a_n^{(r)}}{n} = 2(r+1)t_\infty^{(r)}$$

*exist for all $r$, $k \leq r \leq 2k$. Clearly*

$$(2.11) \qquad \sum_{r=k}^{2k} D(k,r) = 1.$$

*Proof.* According to the previous corollary $t_n^{(r)} = t_\infty^{(r)} + \mathcal{O}(a^n)$ with an arbitrarily small constant $a > 0$. Therefore

$$s_n^{(r)} = n(n+2k+1)[t_\infty^{(r)} + \mathcal{O}(a^n)] = n(n+2k+1)t_\infty^{(r)} + \mathcal{O}(a^n),$$

and

$$a_n^{(r)} = s_n^{(r)} - s_{n-1}^{(r)} = 2(n+k)t_\infty^{(r)} + \mathcal{O}(a^n).$$

$\square$

## 3. Fundamental calculations

From 2.4 for $n \geq k+2$ one gets

$$|u_n^{(r)}| \leq \frac{2(n+k)}{n(n+2k+1)} \cdot \sum_{i=1}^{k} |u_{n-i}^{(r)}| \leq \frac{2k}{n} \left( \frac{1}{k} \sum_{i=1}^{k} |u_{n-i}^{(r)}| \right),$$

so for the non-negative numbers $w_n^{(r)} = |u_n^{(r)}|$ $(n \geq 2)$ one obtains

$$(3.1) \qquad w_n^{(r)} \leq \frac{2k}{n} \left( \frac{1}{k} \sum_{i=1}^{k} w_{n-i}^{(r)} \right)$$

for $n \geq k+2$.

**Lemma 3.2.** *For $n \geq 2$ write $n = pk + s$ with $p \geq 0$ and $2 \leq s \leq k + 1$. We claim that the inequality*

$$(3.3) \qquad |u_n^{(r)}| = w_n^{(r)} \leq \frac{M \cdot 2^p}{p!}$$

*holds true with the constant*

$$M = M_{k,r} = \max \left\{ w_n^{(r)} \big| \, 2 \leq n \leq 2k \right\},$$

*depending only on $k$ and $r$.*

*Proof.* We define the auxiliary sequence $\left( w_n^{(r)} \right)' = w_n'$ $(n \geq 2)$ with the following recursion:

$$(3.4) \qquad w_n' = \begin{cases} M \text{ for } 2 \leq n \leq 2k \\ \frac{2}{n} \cdot \sum_{i=1}^k w_{n-i}' \text{ for } n \geq 2k + 1. \end{cases}$$

Since the expression on the right-hand-side of (3.1) is monotone increasing in its variables $w_{n-i}^{(r)}$ (the coefficients being positive), we immediately get the bounds

$$(3.5) \qquad w_n^{(r)} \leq w_n'$$

for $n \geq 2$. Also, it is clear from the recursion (3.4) that $M > w_{2k+2}' > w_{2k+3}' > w_{2k+4}' > \ldots$, so

$$(3.6) \qquad w_n' \leq \frac{2k}{n} \cdot w_{n-k}'$$

for $n \geq k + 2$. By an obvious induction, the inequalities (3.6) above yield

$$(3.7) \qquad w_n' \leq \frac{M(2k)^p}{n(n-k)(n-2k)\ldots(n-pk+k)} \leq \frac{M(2k)^p}{k^p p!} = \frac{M \cdot 2^p}{p!},$$

where $n = pk + s$ with $p \geq 0$ and $2 \leq s \leq k + 1$. $\qquad \square$

**Corollary 3.8.** *For the $n$-th error term*

$$R_n = \sum_{i \geq n} u_i^{(r)} = t_\infty^{(r)} - t_{n-1}^{(r)}$$

*of the absolutely convergent series*

$$(3.9) \qquad t_\infty^{(r)} = \lim_{i \to \infty} t_i^{(r)} = \frac{s_1^{(r)}}{2k+2} + \sum_{i=2}^\infty u_i^{(r)}$$

*we have the superexponential upper bound*

$$(3.10) \qquad |R_n| \leq \frac{Mke^2 \cdot 2^{p_n}}{p_n!},$$

*where $n = p_n k + s$ with $2 \leq s \leq k + 1$.*

*Proof.*

(3.11)
$$\left| \sum_{i \geq n} u_i^{(r)} \right| \leq \sum_{i \geq n} |u_i^{(r)}| \leq M \cdot \sum_{i \geq n} \frac{2^{p_i}}{p_i!}$$
$$\leq Mk \cdot \sum_{p \geq p_n} \frac{2^p}{p!} \leq \frac{Mke^2 \cdot 2^{p_n}}{p_n!},$$

according to the usual upper bound for the $p_n$-th error term of the Taylor expansion of the exponential function. □

## 4. Behavior of the Limiting Densities $D(k, r)$

We conjecture that, for a given $k$, $D(k, r)$ is decreasing in $r$, and $kD(k, 2k) > 0$ is separated from 0, uniformly in $k$. Please keep in mind 2.11, indicating that the proper normalization (to get non-zero limit) of the densities $D(k, r)$ is $kD(k, r)$. As follows, we present strong numerical evidence for this. Such numerical evidence is certainly feasible for, according to Corollary 3.8, the partial sums of the series (3.9) converge faster than any exponential function, therefore all the formulas (2.4), (2.6), (2.7), and (2.8) converge very fast with error terms that are easy to effectively estimate.

Of particular interest is the limiting cumulative distribution function

(4.1)
$$F(t) = \lim_{k \to \infty} \sum_{r=k}^{[(1+t)k]} D(k, r),$$

and the corresponding limiting density function

(4.2)
$$F'(t) = \lim_{k \to \infty} kD(k, [(1+t)k])$$

for $0 \leq t \leq 1$.

To support these conjectures, calculations were conducted for a wide range of $k$ values, and the results for $k = 2^{20}$ are presented in Figures 1, 2, and 3.
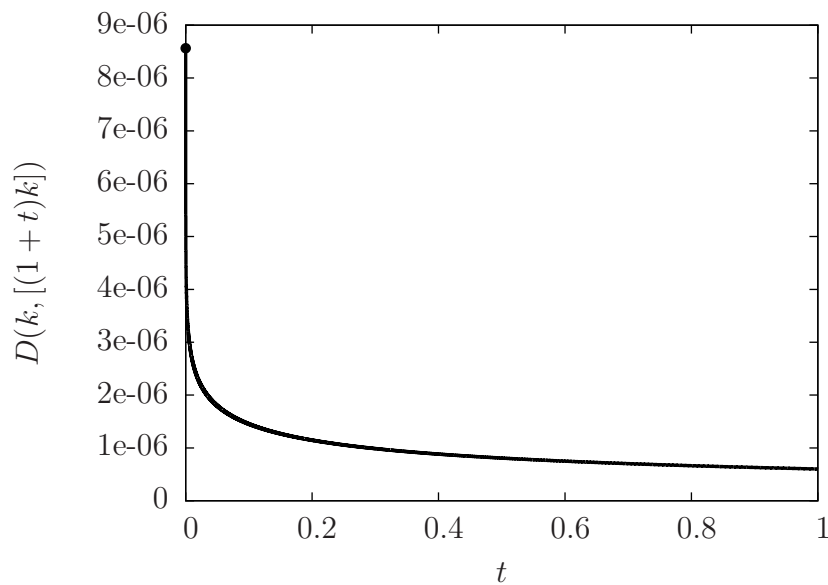
FIGURE 1. Plot of the values of $D(k, r)$ for $k = 2^{20}$ versus a normalized axis $t = (r - k)/k$. The maximum value obtained is denoted by the symbol at $t = 0$.
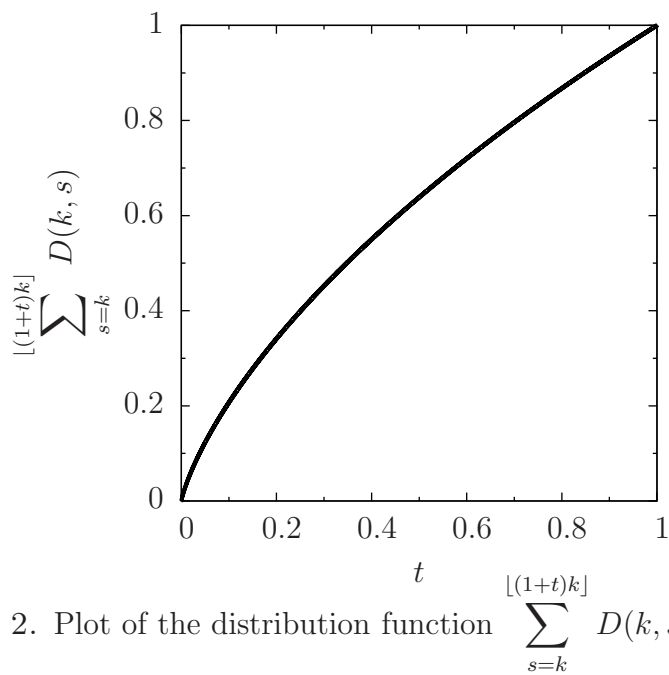


FIGURE 2. Plot of the distribution function $\sum\limits_{s=k}^{\lfloor (1+t)k \rfloor} D(k, s)$ for $k = 2^{20}$.
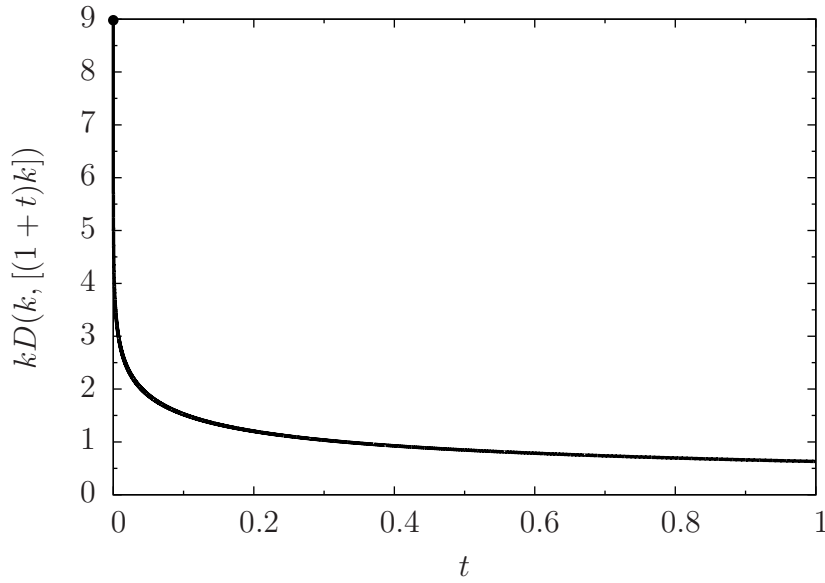
FIGURE 3. Plot of the density function $kD(k, [(1 + t)k])$ for $k = 2^{20}$. The maximum value is at $t = 0$ and is marked with the symbol.

The density function $kD(k, [(1 + t)k])$ exhibits interesting behavior at $t = 0$. To demonstrate this, Figure 4 presents data for $kD(k, k)$ spanning many decades of $k$.
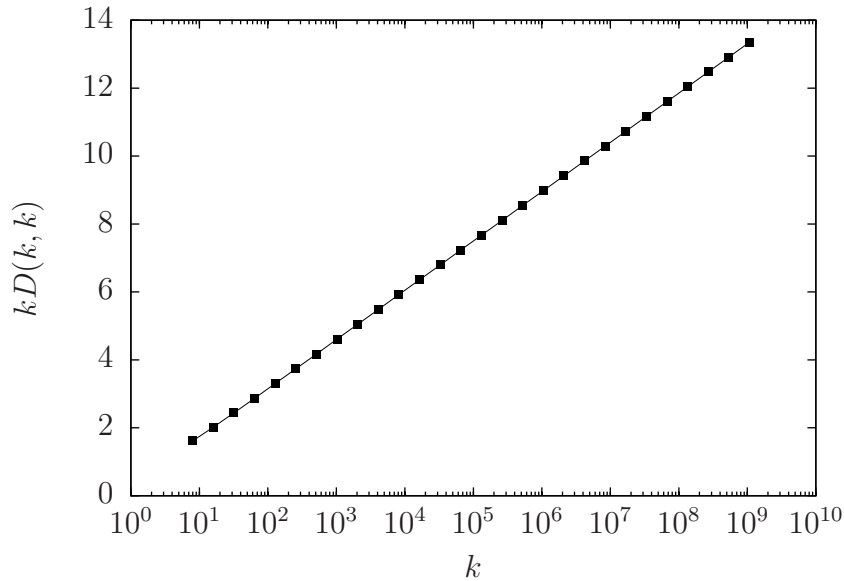


FIGURE 4. Plot of the growth of $kD(k, k)$ as $k$ is increased. The values of $k$ used were $2^n$, where $3 \leq n \leq 30$.

The results given in Figure 4 show that $kD(k,k)$ grows at a logarithmic rate with $k$. Similarly, we investigate the behavior of $kD(k,2k)$ in Figure 5.
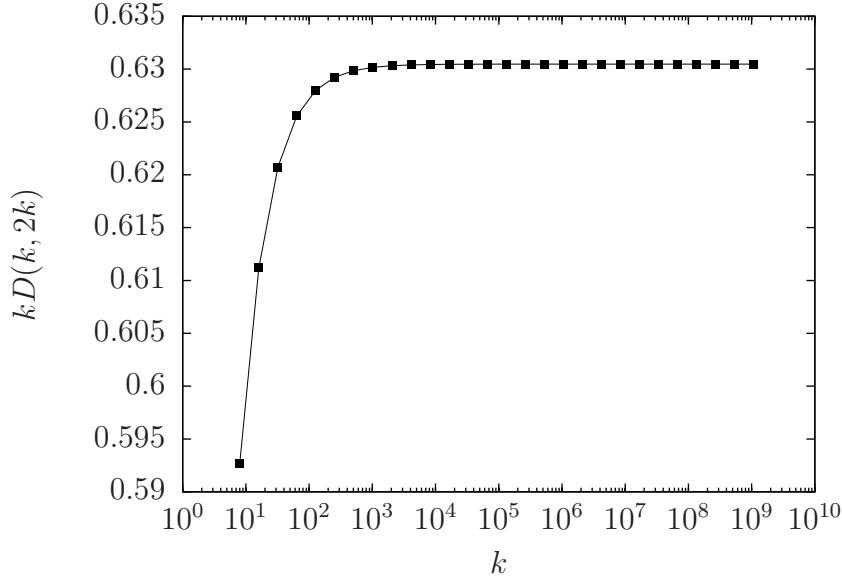


FIGURE 5. Plot of $kD(k,2k)$ as $k$ is increased. The values of $k$ used were $2^n$, where $3 \le n \le 30$.

The results in Figure 5 support the conjecture that the sequence $\{kD(k,2k)\}$ converges to a number $0.6304735\ldots$ in a monotone increasing fashion as $k \to \infty$. Figures 3, 4, and 5 serve as strong numerical evidence for the claim that the limiting density function $F'(t)$ of (4.2) continuously decreases from infinity at $t = 0$ to a positive constant $0.6304735\ldots$. Furthermore, it is worth noting that these pictures are in pretty good harmony with the results of §4 of [Man(1976)] on the distribution of the gap lengths.

The number $D(k) = \sum_{r=k}^{2k} \dfrac{k+1}{r+1} D(k,r)$ has a special meaning: It is the limiting filling density of cars (i. e. $(k+1)$-blocks) getting a parking slot, as $n \to \infty$. Clearly $\dfrac{k+1}{2k+1} \le D(k) \le 1$.

Particularly interesting is the limiting packing density

$$D = \lim_{k \to \infty} D(k).$$

Clearly $1/2 \le D \le 1$. The behavior of $D$ was investigated numerically and the results are presented in Figure 6. The obtained numerical evidence supports the claim that $D = m = 0.7475979203\ldots$ is Rényi's famous parking constant.
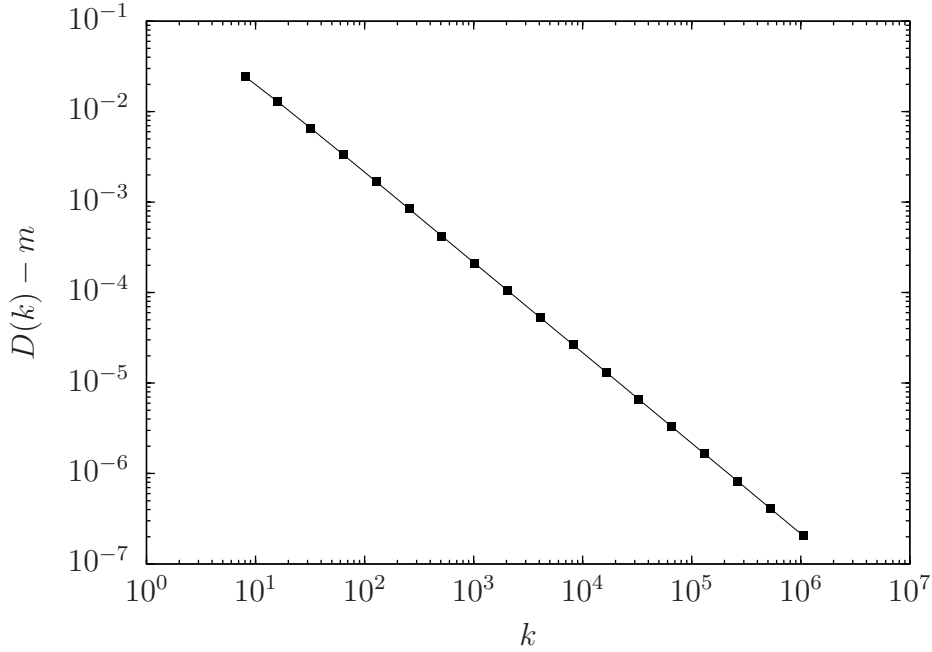
FIGURE 6. Plot depicting the difference between the calculated values
of $D(k)$ and Rényi's constant $m$ (to machine precision) versus $k$. The
values of $k$ used were $2^n$, where $3 \leq n \leq 20$.

4.1. **Example. Detailed Calculations for $k = 1$.** If we take the case $k = r = 1$,
(2.4) and (2.5) yield

$$u_2^{(1)} = -\frac{3}{20}, \quad u_n^{(1)} = \frac{-2(n+1)}{n(n+3)} u_{n-1}^{(1)}$$

for $n \geq 3$, thus $u_n^{(1)} = \frac{3(n+1)(-2)^{n-1}}{(n+3)!}$ for $n \geq 2$, so

$$D(1,1) = 4t_\infty^{(1)} = 1 + 12 \cdot \sum_{n=2}^{\infty} \frac{(n+3-2)(-2)^{n-1}}{(n+3)!}$$

$$(4.3) \qquad = 1 + 12 \cdot \sum_{n=2}^{\infty} \frac{(-2)^{n-1}}{(n+2)!} + 12 \cdot \sum_{n=2}^{\infty} \frac{(-2)^n}{(n+3)!}$$

$$= 1 - \frac{3}{2} \left( \sum_{n=2}^{\infty} \frac{(-2)^{n+2}}{(n+2)!} + \sum_{n=2}^{\infty} \frac{(-2)^{n+3}}{(n+3)!} \right) = 1 - 3e^{-2}$$

by obvious analysis technique.

Consequently, $D(1,2) = 1 - D(1,1) = 3e^{-2}$, according to (2.11). Finally, the exact value of the filling density $D(1)$ is $D(1) = D(1,1) + \frac{2}{3}D(1,2) = 1 - e^{-2}$. The filling density $D(1) = 1 - e^{-2}$ is in agreement with the filling density $(1 - e^{-2})/2$ obtained by Page [P(1959)]. The factor 2 discrepancy is due to the fact that we count each car with weight $k + 1 = 2$ (the space each car occupies), whereas Page uses weight 1.

*Remark.* Whoever is interested in repeating the computer calculations, re-generating the numerical plots, or checking the details in the source code of our programs, can directly send us an e-mail message. We will be more than happy to provide the code.

## Acknowledgements

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