EIGENFUNCTIONS AT THE THRESHOLD ENERGIES
OF MAGNETIC DIRAC OPERATORS

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Discussed are ±m modes and ±m resonances of Dirac operators with vector potentials
\[ H_A = \alpha \cdot (D - A(x)) + m\beta, \] (1.1)
Here \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is the triple of 4 × 4 Dirac matrices
\[ \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \ (j = 1, 2, 3) \]
with the 2 × 2 zero matrix \( 0 \) and the triple of 2 × 2 Pauli matrices
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]

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1. Introduction
The introduction is devoted to exhibiting our results as well as to reviewing previous contributions in connection with the results in the present paper.

This paper is concerned with eigenfunctions and resonances at the threshold energies of Dirac operators with vector potentials

\[ H_A = \alpha \cdot (D - A(x)) + m\beta, \]
\[ D = \frac{1}{i} \nabla_x, \ x \in \mathbb{R}^3. \] (1.1)

Here \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) is the triple of 4 × 4 Dirac matrices
\[ \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \ (j = 1, 2, 3) \]
with the 2 × 2 zero matrix \( 0 \) and the triple of 2 × 2 Pauli matrices
\[ \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
The constant $m$ is assumed to be positive. Throughout the present paper, we assume that each component of the vector potential $A(x) = (A_1(x), A_2(x), A_3(x))$ is a real-valued measurable function. In addition to this, we shall later impose four different sets of assumptions on $A(x)$ under which the operator $-\alpha \cdot A(x)$ is relatively compact with respect to the free Dirac operator $H_0 = \alpha \cdot D + m\beta$. Therefore, under any set of assumptions to be made, the magnetic Dirac operator $H_A$ is a self-adjoint operator in the Hilbert space $L^2(\mathbb{R}^3)^4$, and the essential spectrum of $H_A$ is given by the union of the intervals $(-\infty, -m]$ and $[m, +\infty)$:

$$\sigma_{\text{ess}}(H_A) = (-\infty, -m] \cup [m, +\infty).$$

(1.2)

By the threshold energies of $H_A$, we mean the values $\pm m$, the edges of the essential spectrum $\sigma_{\text{ess}}(H_A)$. We shall see in Secs. 3–5 that the discrete spectrum of $H_A$ in the gap $(-m, m)$ is empty, although we should like to mention that this fact is well known by the result of Thaller [31, Theorem 7.1, p. 195] where smoothness of vector potentials is assumed though. In other words, there are no isolated eigenvalues with finite multiplicity in the spectral gap $(-m, m)$. In the present paper, this fact will be obtained as a by-product of Theorem 2.3 in Sec. 2, where we shall deal with an abstract Dirac operator, i.e. a supersymmetric Dirac operator. As a result, we shall have

$$\sigma(H_A) = \sigma_{\text{ess}}(H_A) = (-\infty, -m] \cup [m, +\infty)$$

under any set of the assumptions on $A(x)$ of the present paper.

In relation with the relative compactness of $-\alpha \cdot A(x)$ with respect to $H_0$, it is worthwhile to mention a work by Thaller [30], where he showed that (1.2) is true under the assumption that $|B(x)| \to 0$ as $|x| \to \infty$. Here $B(x)$ denotes the magnetic field: $B(x) = \nabla \times A(x)$. It is clear that the assumption that $|B(x)| \to 0$ does not necessarily imply the relative compactness of $-\alpha \cdot A(x)$ with respect to $H_0$. In Helffer et al. [16], they showed that (1.2) is true under much weaker assumptions on $B(x)$, which do not even need the requirement that $|B(x)| \to 0$ as $|x| \to \infty$; see also [31, §7.3.2].

It is generally expected that eigenfunctions corresponding to a discrete eigenvalue of $H_A$ decay exponentially at infinity (describing bound states), and that (generalized) eigenfunctions corresponding to an energy inside the continuous spectrum $(-\infty, -m] \cup [m, +\infty)$ behave like the sum of a plane and a spherical waves at infinity (describing scattering states). At the energies $\pm m$, on which we shall focus in the present paper, (generalized) eigenfunctions are expected to behave like $C_0 + C_1|x|^{-1} + C_2|x|^{-2}$ at infinity, where $C_j$, $j = 1, 2, 3$, are constant vectors in $\mathbb{C}^4$. If $C_0 = 0$ and $C_1 \neq 0$, then the (generalized) eigenfunctions become either of
$\pm m$ resonances, and if $C_0 = 0$ and $C_1 = 0$, then the (generalized) eigenfunctions become either of $\pm m$ modes. For the precise definitions of $\pm m$ resonances and $\pm m$ modes, see Definition 6.1 in Sec. 6, and Definition 3.1 in Sec. 3, respectively. As for the exponential decay of eigenfunctions, we refer the reader to works by Helffer and Parisse [17], Wang [32], and a recent work by Yafaev [34]. As for the generalized eigenfunctions corresponding to an energy in $(-\infty, -m) \cup (m, +\infty)$, we refer the reader to Yamada [36].

As mentioned above, our main concern is the threshold energies $\pm m$ of the magnetic Dirac operator $H_A$. These energies are of particular importance and of interest from the physics point of view. We should like to mention Pickl and D"urr [21], and Pickl [20], where they investigate generalized eigenfunctions not only at the energies $\pm m$ but also at the energies near $\pm m$, with the emphasis on the famous relativistic effect of the pair creation of an electron and a positron. It is worthwhile to note that $\pm m$ modes and $\pm m$ resonances play decisive roles in their results. In the same spirit as in [20, 21], Pickl and D"urr [22] mention the possibility of experimental verifications of the pair creation by combining lasers and heavy ions fields. Therefore, it is obvious that results on $\pm m$ modes and $\pm m$ resonances of magnetic Dirac operators are useful to understand the physics of the pair creations in such laser fields; see [22] for details.

The goal of the present paper is to derive a series of new results on $\pm m$ modes and $\pm m$ resonances of the magnetic Dirac operators $H_A$. Precisely speaking, we shall study asymptotic behaviors at infinity of the $\pm m$ modes, show sparseness of the set of vector potentials which give rise to the $\pm m$ modes, and establish non-existence of $\pm m$ resonances.

According to Pickl [20, Theorems 3.4 and 3.5], the behavior of the generalized eigenfunctions of Dirac operators near criticality largely depends on whether Dirac operators with critical potentials have $\pm m$ resonances or not. Since the modulus of their critical potentials are less than or equal to $C(1 + |x|)^{-2}$, we can actually conclude from Theorems 6.1 and 6.2 in Sec. 6 that the magnetic Dirac operators with the critical potentials have no $\pm m$ resonances. However, one has to pay attention to a slight difference between our definition of the threshold resonances (cf. Definition 6.1) and theirs (cf. [20, Definition 2.3 and the paragraph after it]).

Finally, we would like to mention that there is a striking difference between two- and three-dimensional Dirac operators with magnetic fields at the threshold energies. Compare the results in Sec. 6 of the present paper with those of Aharonov and Casher [1], where their arguments indicate that one can find magnetic Dirac operators in dimension two which possess threshold resonances. See also [33, Sec. 10].

The plan of the paper is as follows. In Sec. 2, we shall prepare a few results on a supersymmetric Dirac operator, which will be used in all the later sections. In Sec. 3, we shall investigate asymptotic behaviors at infinity of $\pm m$ modes of $H_A$, provided that it has $\pm m$ modes. Sparseness of the set of vector potentials $A(x)$ which yield $\pm m$ modes of $H_A$ will be discussed in Secs. 4 and 5 in different regimes. In Sec. 6, we shall prove that any $H_A$ does not have $\pm m$ resonances under a stronger
assumption than those made in the previous sections. Finally in Sec. 7, we shall give examples of vector potentials $A(x)$ which yield $\pm m$ modes of magnetic Dirac operators $H_A$, and shall show that these operators $H_A$ do not have $\pm m$ resonances. Then we shall propose an open question in relation with $\pm m$ resonances.

2. Supersymmetric Dirac Operators

This section is devoted to a discussion about spectral properties of a class of supersymmetric Dirac operators. We should like to remark that our approach appears to be in the reverse direction in the sense that we start with two Hilbert spaces, and introduce a supersymmetric Dirac operator on the direct sum of the two Hilbert spaces. We find this approach convenient for our purpose; see [31, Chap. 5] for the standard theory of the supersymmetric Dirac operator.

The supersymmetric Dirac operator $H$ which we shall consider in the present paper is defined as follows:

$$H := \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \quad \text{on } \mathcal{K} = \mathcal{H}_+ \oplus \mathcal{H}_-, \quad (2.1)$$

where $T$ is a densely defined operator from a Hilbert space $\mathcal{H}_+$ to another Hilbert space $\mathcal{H}_-$, $m$ is a positive constant, and the identity operators in $\mathcal{H}_+$ and $\mathcal{H}_-$ are both denoted by $I$ with an abuse of notation. We recall that the domain of $H$ is given by $\mathcal{D}(H) = \mathcal{D}(T) \oplus \mathcal{D}(T^*)$, and that the inner product of the Hilbert space $\mathcal{K}$ is defined by

$$(f,g)_\mathcal{K} := (\varphi^+,\psi^+)_{\mathcal{H}_+} + (\varphi^-,\psi^-)_{\mathcal{H}_-} \quad (2.2)$$

for

$$f = \begin{pmatrix} \varphi^+ \\ \varphi^- \end{pmatrix}, \quad g = \begin{pmatrix} \psi^+ \\ \psi^- \end{pmatrix} \in \mathcal{K}. \quad (2.3)$$

The first term on the right-hand side of (2.1) is called the supercharge and the second term the involution, and denoted by $Q$ and $\tau$ respectively:

$$Q = \begin{pmatrix} 0 & T^* \\ T & 0 \end{pmatrix}, \quad \tau = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}. \quad (2.4)$$

We now state the main results in this section, which are about the nature of eigenvectors of the supersymmetric Dirac operator (2.1) at the eigenvalues $\pm m$. We mention that $T$ does not need to be a closed operator, and that $T^*$ does not need to be densely defined, because we only focus on the eigenvalues $\pm m$ of $H$ and the corresponding eigenspaces. In the standard theory of the supersymmetric Dirac operator, $T$ is assumed to be a densely defined closed operator and $T^*$ needs to be densely defined; cf. [31, §5.2.2].

We should like to draw attention to the fact that Theorems 2.1 and 2.2 below are simply abstract restatements of Thaller [31, Theorem 7.1, p. 195], where he dealt with the magnetic Dirac operators under the assumption that $A_j \in C^\infty$. 
From the mathematically rigorous point of view, it is not appropriate to apply [31, Theorem 7.1] to the magnetic Dirac operators with non-smooth vector potentials. However, the vector potentials we shall treat in Secs. 3–5 are not smooth. In particular we shall deal, in Sec. 4, with vector potentials which can have local singularities. In this case, even self-adjointness of the magnetic Dirac operators is not trivial. Hence [31, Theorem 7.1] is not applicable to this case. These are the reasons why we need to generalize and restate [31, Theorem 7.1] in an abstract setting.

**Theorem 2.1.** Suppose that $T$ is a densely defined operator from $\mathcal{H}_+$ to $\mathcal{H}_-$. Let $H$ be a supersymmetric Dirac operator defined by (2.1).

(i) If $f = t(\varphi^+, \varphi^-) \in \text{Ker}(H - m)$, then $\varphi^+ \in \text{Ker}(T)$ and $\varphi^- = 0$.

(ii) Conversely, if $\varphi^+ \in \text{Ker}(T)$, then $f = t(\varphi^+, 0) \in \text{Ker}(H - m)$.

**Theorem 2.2.** Assume that $T$ and $H$ are the same as in Theorem 2.1.

(i) If $f = t(\varphi^+, \varphi^-) \in \text{Ker}(H + m)$, then $\varphi^+ = 0$ and $\varphi^- \in \text{Ker}(T^*)$.

(ii) Conversely, if $\varphi^- \in \text{Ker}(T^*)$, then $f = t(0, \varphi^-) \in \text{Ker}(H + m)$.

As immediate consequences, we have

**Corollary 2.1.** Assume that $T$ and $H$ are the same as in Theorem 2.1. Then

(i) $\text{Ker}(H - m) = \text{Ker}(T) \oplus \{0\}$, $\dim(\text{Ker}(H - m)) = \dim(\text{Ker}(T))$.

(ii) $\text{Ker}(H + m) = \{0\} \oplus \text{Ker}(T^*)$, $\dim(\text{Ker}(H + m)) = \dim(\text{Ker}(T^*))$.

The eigenspaces corresponding to the eigenvalues $\pm m$ of supersymmetric Dirac operators do not seem to have been explicitly formulated in the literature as in the form of Corollary 2.1. It is straightforward from this formulation that the eigenspaces of $H$ corresponding to the eigenvalue $\pm m$ are independent of $m$.

**Proof of Theorem 2.1.** We first prove Assertion (i). Let $f = t(\varphi^+, \varphi^-) \in \text{Ker}(H - m)$. We then have

$$
\begin{pmatrix}
0 & T^* \\
T & 0
\end{pmatrix}
\begin{pmatrix}
\varphi^+ \\
\varphi^-
\end{pmatrix} + m
\begin{pmatrix}
I & 0 \\
0 & -I
\end{pmatrix}
\begin{pmatrix}
\varphi^+ \\
\varphi^-
\end{pmatrix} = m
\begin{pmatrix}
\varphi^+ \\
\varphi^-
\end{pmatrix},
\tag{2.5}
$$

hence

$$
\begin{cases}
T^*\varphi^- + m\varphi^+ = m\varphi^+ \\
T\varphi^+ - m\varphi^- = m\varphi^-,
\end{cases}
\tag{2.6}
$$

which immediately implies that $T^*\varphi^- = 0$ and $T\varphi^+ = 2m\varphi^-$. It follows that

$$
\|T\varphi^+\|^2_{\mathcal{H}_-} = (T\varphi^+, T\varphi^+)_{\mathcal{H}_-} = (T\varphi^+, 2m\varphi^-)_{\mathcal{H}_-} = (\varphi^+, 2mT^*\varphi^-)_{\mathcal{H}_+} = 0.
\tag{2.7}
$$

Thus we see that $\varphi^+ \in \text{Ker}(T)$, and that $\varphi^- = (2m)^{-1}T\varphi^+ = 0$. 

We next prove Assertion (ii). Let $\varphi^+ \in \text{Ker}(T)$ and put $f := t(\varphi^+, 0)$. Then it follows that $Hf = t(m\varphi^+, T\varphi^+) = mf$.

We omit the proof of Theorem 2.2, which is quite similar to that of Theorem 2.1.

In connection with our applications to the magnetic Dirac operator $H_A$ in later sections, we should like to consider the case where the Hilbert space $H^+$ coincides with $H^−$ and $T$ is self-adjoint ($T^* = T$). In this case, the supersymmetric Dirac operator $H$ becomes of the form

$$H = \begin{pmatrix} 0 & T \\ T & 0 \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$$

(2.8)

in the Hilbert space $K = H \oplus H$, and it follows from Theorems 2.1 and 2.2 that the operator $H$ of the form (2.8) possesses of a simple but important equivalence:

$T$ has a zero mode $\Leftrightarrow$ $H$ has an $m$ mode

$\Leftrightarrow$ $H$ has a $-m$ mode

(2.9)

which is actually a well-known fact: see [31, Corollary 5.14, p. 155]. Here we say that $T$ has a zero mode if $0$ is an eigenvalue of $T$. In a similar manner, we say that $H$ has an $m$ mode (respectively, a $-m$ mode) if $m$ (respectively, $-m$) is an eigenvalue of $H$. Furthermore, Theorems 2.1 and 2.2 imply the following equivalence for a zero mode $\varphi$ of $T$:

$$T\varphi = 0 \Leftrightarrow H\begin{pmatrix} \varphi \\ 0 \end{pmatrix} = m\begin{pmatrix} \varphi \\ 0 \end{pmatrix} \Leftrightarrow H\begin{pmatrix} 0 \\ \varphi \end{pmatrix} = -m\begin{pmatrix} 0 \\ \varphi \end{pmatrix}.$$ 

(2.10)

We shall show in Theorem 2.3 below that a sufficient condition for the fact that $\sigma(H) = \sigma_{\text{ess}}(H) = (-\infty, -m] \cup [m, \infty)$ (2.11) is given by the inclusion $\sigma(T) \supset (0, \infty)$. Therefore $\pm m$ are always threshold energies of the supersymmetric Dirac operator $H$, provided that $\sigma(T) \supset (0, \infty)$.

Theorem 2.3. Let $T$ be a self-adjoint operator in the Hilbert space $H$. Suppose that $\sigma(T) \supset [0, +\infty)$. Then

$$\sigma(H) = (-\infty, -m] \cup [m, +\infty).$$

In particular, $\sigma_d(H) = \emptyset$, i.e. the set of discrete eigenvalues of $H$ with finite multiplicity is empty.

Proof. It follows from (2.8) that $\mathcal{D}(H^2) = \mathcal{D}(T^2) \oplus \mathcal{D}(T^2)$ and that

$$H^2 = \begin{pmatrix} T^2 + m^2I & 0 \\ 0 & T^2 + m^2I \end{pmatrix} \geq m^2 \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$ 

(2.12)

This inequality implies that $\sigma(H) \subset (-\infty, -m] \cup [m, +\infty)$.
To complete the proof, we shall prove the fact that \( \sigma(H) \supset (-\infty, -m] \cup [m, +\infty). \) To this end, suppose \( \lambda_0 \in (-\infty, -m] \cup [m, +\infty) \) be given. Since \( \sqrt{\lambda_0^2 - m^2} \geq 0, \) we see, by the assumption of the theorem, that \( \sqrt{\lambda_0^2 - m^2} \in \sigma(T). \) Therefore, we can find a sequence \( \{\psi_n\}_{n=1}^{\infty} \subset \mathcal{H} \) such that

\[
\|\psi_n\| = 1, \quad \psi_n \in \text{Ran} \left( \left( E_T \left( \nu_0 - \frac{1}{n}, \nu_0 + \frac{1}{n} \right) \right) \right), \quad \nu_0 := \sqrt{\lambda_0^2 - m^2}
\]  

for each \( n \), where \( E_T(\cdot) \) is the spectral measure associated with \( T \):

\[
T = \int_{-\infty}^{\infty} \lambda \, dE_T(\lambda).
\]

Here we have used a basic property of the spectral measure; see, for example [25, Proposition, p. 236]. It is straightforward to see that

\[
\|(T - \nu_0)\psi_n\| \to 0 \quad \text{as} \quad n \to \infty.
\]

We shall construct a sequence \( \{f_n\} \subset \mathcal{D}(H) = \mathcal{D}(T) \oplus \mathcal{D}(T) \) satisfying \( \|f_n\|_K = 1 \) and \( \|(H - \lambda_0)f_n\|_K \to 0 \) as \( n \to \infty. \) To this end, we choose a pair of real numbers \( a \) and \( b \) so that

\[
a^2 + b^2 = 1
\]

and that

\[
\begin{pmatrix} m & \nu_0 \\ \nu_0 & -m \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \lambda_0 \begin{pmatrix} a \\ b \end{pmatrix}.
\]

This is possible because the \( 2 \times 2 \) symmetric matrix in (2.17) has eigenvalues \( \pm \lambda_0. \) We now put

\[
f_n := \begin{pmatrix} a\psi_n \\ b\psi_n \end{pmatrix}.
\]

It is easy to see that \( \|f_n\|_K = 1. \) By using (2.16) and (2.17), we can show that

\[
\|(H - \lambda_0)f_n\|_K^2 = \|(m - \lambda_0)a\psi_n + bT\psi_n\|_\mathcal{H}^2
\]

\[
+ \|aT\psi_n - (m + \lambda_0)b\psi_n\|_\mathcal{H}^2
\]

\[
= \|b(-\nu_0 + T)\psi_n\|_\mathcal{H}^2 + \|a(T - \nu_0)\psi_n\|_\mathcal{H}^2
\]

\[
= \|(T - \nu_0)\psi_n\|_\mathcal{H}^2 \to 0 \quad \text{as} \quad n \to \infty.
\]

We thus have shown that \( \lambda_0 \in \sigma(H). \)

Here we briefly mention of the abstract Fouldy–Wouthuysen transformation \( U_{FW} \) in connection with Theorem 2.3. The transformation \( U_{FW} \) is a unitary operator in \( \mathcal{K}, \) and transforms the supersymmetric Dirac operator \( H \) of the form (2.8) into the diagonal form:

\[
U_{FW} H U_{FW}^* = \begin{pmatrix} \sqrt{T^2 + m^2} & 0 \\ 0 & -\sqrt{T^2 + m^2} \end{pmatrix}.
\]
Note that it is possible to prove (2.11) based on this unitary equivalence. For the abstract Fouldy–Wouthuysen transformation for the supersymmetric Dirac operator of the form (2.1), we refer the reader to Thaller [31, Chap. 5, §5.6].

In all the later sections, we shall apply the obtained results on the supersymmetric Dirac operator to the magnetic Dirac operator \( H_A \) of the form (1.1) in the Hilbert space \( \mathcal{K} = [L^2(\mathbb{R}^3)]^4 \), where we take \( T \) to be the Weyl–Dirac operator

\[
T_A = \sigma \cdot (D - A(x))
\]  

(2.19)

acting in the Hilbert space \( \mathcal{H} = [L^2(\mathbb{R}^3)]^2 \).

As was mentioned above (cf. (2.9) and (2.10)), the investigations of properties of \( \pm m \) modes of the magnetic Dirac operator \( H_A \) are reduced to the investigations of the corresponding properties of zero modes of the Weyl–Dirac operator \( T_A = \sigma \cdot (D - A(x)) \).

We have to emphasize the broad applicability of the supersymmetric Dirac operator in the context of the present paper. Namely, thanks to the generality of Theorems 2.1 and 2.2, we are able to utilize most of the existing works on the zero modes of the Weyl–Dirac operator \( T_A \) (cf. [2–8, 10–14, 19]) for the purpose of investigating \( \pm m \) modes of the magnetic Dirac operator \( H_A \).

3. Asymptotic Limits of \( \pm m \) Modes

In this section, we consider a class of magnetic Dirac operators \( H_A \) under Assumption(SU) below, and will focus on the asymptotic behaviors at infinity of \( \pm m \) modes of \( H_A \), assuming that \( \pm m \) are the eigenvalues of \( H_A \). In Sec. 7, we shall see that there exists infinitely many \( A \)’s such that the corresponding magnetic Dirac operators \( H_A \) have the threshold eigenvalues \( \pm m \).

We now introduce the terminology of \( \pm m \) modes for the magnetic Dirac operator \( H_A \).

**Definition 3.1 (Following [18]).** By an \( m \) mode (respectively, a \( -m \) mode), we mean an eigenfunction corresponding to the eigenvalue \( m \) (respectively, \( -m \)) of \( H_A \), provided that the threshold energy \( m \) (respectively \( -m \)) is an eigenvalue of \( H_A \).

**Assumption(SU).** Each element \( A_j(x) \) \((j = 1, 2, 3)\) of \( A(x) \) is a measurable function satisfying

\[
|A_j(x)| \leq C(x)^{-\rho} \quad (\rho > 1),
\]  

(3.1)

where \( C \) is a positive constant.

It is easy to see that under Assumption(SU) the Dirac operator \( H_A \) is a self-adjoint operator in the Hilbert space \( \mathcal{K} = [L^2(\mathbb{R}^3)]^4 \) with \( \text{Dom}(H_A) = [H^1(\mathbb{R}^3)]^4 \), where \( H^1(\mathbb{R}^3) \) denotes the Sobolev space of order 1. Also it is easy to see that under Assumption(SU) the Weyl–Dirac operator \( T_A \) is a self-adjoint operator in the Hilbert space \( \mathcal{H} = [L^2(\mathbb{R}^3)]^2 \) with \( \text{Dom}(T_A) = [H^1(\mathbb{R}^3)]^2 \). Since the operator
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\(-\sigma \cdot A(x)\) is relatively compact with respect to the operator \(T_0 := \sigma \cdot D\), and since \(\sigma(T_0) = \mathbb{R}\), it follows that \(\sigma(T_A) = \mathbb{R}\). Recalling that

\[
H_A = \begin{pmatrix} 0 & T_A \\ T_A & 0 \end{pmatrix} + m \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix},
\]

we can apply Theorem 2.3 to \(H_A\), and get

\(\sigma(H_A) = \sigma_{\text{ess}}(H_A) = (-\infty, -m] \cup [m, +\infty)\).

Hence \(\pm m\) are the threshold energies of the operator \(H_A\). Assuming that \(\pm m\) are the eigenvalues of \(H_A\), we find that the eigenspaces corresponding to the eigenvalues \(\pm m\) of \(H_A\) are given as the direct sum of Ker\((T_A)\) and the zero space \(\{0\}\) (cf. Corollary 2.1 in Sec. 2), and that these two eigenspaces themselves as well as their dimensions are independent of \(m\).

**Theorem 3.1.** Suppose that Assumption\((SU)\) is verified, and that \(m\) (respectively, \(-m\)) is an eigenvalue of \(H_A\). Let \(f\) be an \(m\) mode (respectively, a \(-m\) modes) of \(H_A\). Then there exists a zero mode \(\varphi^+\) (respectively, \(\varphi^-\)) of \(T_A\) such that for any \(\omega \in S^2\)

\[
\lim_{r \to \infty} r^2 f(r\omega) = \begin{pmatrix} u^+(\omega) \\ 0 \end{pmatrix} \quad \text{respectively,} \quad \begin{pmatrix} 0 \\ u^-(\omega) \end{pmatrix},
\]

where

\[
u^\pm(\omega) = \frac{i}{4\pi} \int_{\mathbb{R}^3} \{(\omega \cdot A(y))I_2 + i\sigma \cdot (\omega \times A(y))\} \varphi^\pm(y) dy,
\]

and the convergence is uniform with respect to \(\omega\).

Theorem 3.1 is a direct consequence of Corollary 2.1, together with [27, Theorem 1.2]. Note that under Assumption\((SU)\) every eigenfunction of \(H_A\) corresponding to either one of eigenvalues \(\pm m\) is a continuous function of \(x\) (cf. [28, Theorem 2.1]), therefore the expression \(f(r\omega)\) in (3.3) makes sense for each \(\omega\).

4. Sparseness of Vector Potentials Yielding \(\pm m\) Modes

In this section, we shall discuss the sparseness of the set of vector potentials \(A\) which give rise to \(\pm m\) modes of magnetic Dirac operators \(H_A\), in the spirit of Balinsky and Evans [5,6], where they investigated Pauli operators and Weyl–Dirac operators respectively.

We shall make the following assumption:

**Assumption\((BE)\).** \(A_j \in L^3(\mathbb{R}^3)\) for \(j = 1, 2, 3\).

Under Assumption\((BE)\) Balinsky and Evans [6, Lemma 2] showed that \(-\sigma \cdot A\) is infinitesimally small with respect to \(T_0 = \sigma \cdot D\) with Dom\((T_0) = [H^1(\mathbb{R}^3)]^2\) (see (4.5) below). This fact enables us to define the self-adjoint realization \(T_A\) in the Hilbert space \(\mathcal{H} = [L^2(\mathbb{R}^3)]^2\) as the operator sum of \(T_0\) and \(-\sigma \cdot A\), thus Dom\((T_A) = [H^1(\mathbb{R}^3)]^2\). It turns out that under Assumption\((BE)\), \(-\alpha \cdot A\) is infinitesimally small
Lemma 4.2. Let Assumption (BE) be satisfied. Then $\sigma(T_A) = \mathbb{R}$.

We shall prepare a few lemmas for the proof of Proposition 4.1.

Lemma 4.1. Let $z \in \mathbb{C} \setminus \mathbb{R}$. Then $\langle D \rangle^{1/2}(T_0 - z)^{-1}$ is a bounded operator in $\mathcal{H}$. Moreover we have

$$\text{Ran}(\langle D \rangle^{1/2}(T_0 - z)^{-1}) \subset [H^{1/2}(\mathbb{R}^3)]^2. \quad (4.1)$$

Proof. It is sufficient to show the conclusions of the lemma for $z = -i$. Let $\varphi \in \text{Dom}(T_0)$. Then we have

$$\| (T_0 + i)\varphi \|_{\mathcal{H}}^2 = \int_{\mathbb{R}^3} |((\sigma \cdot \xi) + iI_2)\tilde{\varphi}(\xi)|^2 d\xi = \int_{\mathbb{R}^3} |(\xi|^2 + 1)|\tilde{\varphi}(\xi)|^2 d\xi = \|\langle D \rangle \varphi\|_{\mathcal{H}}^2, \quad (4.2)$$

where we have used the anti-commutation relation $\sigma_j \sigma_k + \sigma_k \sigma_j = 2\delta_{jk}I_2$ in the second equality. It follows from (4.2) that

$$\|\varphi\|_{\mathcal{H}} = \|\langle D \rangle(T_0 + i)^{-1}\varphi\|_{\mathcal{H}} \quad (4.3)$$

for all $\varphi \in \mathcal{H}$. Furthermore, we see that

$$\|\langle D \rangle^{1/2}(T_0 + i)^{-1}\varphi\|_{\mathcal{H}} \leq \|\langle D \rangle^{1/2}(T_0 + i)^{-1}\varphi\|_{H^{1/2}(\mathbb{R}^3)^2} = \|\langle D \rangle(T_0 + i)^{-1}\varphi\|_{\mathcal{H}} = \|\varphi\|_{\mathcal{H}}, \quad (4.4)$$

It is evident that (4.4) proves the conclusions of the lemma for $z = -i$. \qed

Lemma 4.2. If $\varphi \in [H^{1/2}(\mathbb{R}^3)]^2$, then $(\sigma \cdot A)\langle D \rangle^{-1/2}\varphi \in \mathcal{H}$.

Proof. By [6, Lemma 2], we see that for any $\epsilon > 0$, there exists a constant $k_\epsilon > 0$ such that for all $\varphi \in \text{Dom}(T_0)$

$$\| (\sigma \cdot A)\varphi \|_{\mathcal{H}} \leq \epsilon \|T_0\varphi\|_{\mathcal{H}} + k_\epsilon \|\varphi\|_{\mathcal{H}}. \quad (4.5)$$

By virtue of the fact that $\langle D \rangle^{-1/2}\varphi \in \text{Dom}(T_0)$ for $\varphi \in [H^{1/2}(\mathbb{R}^3)]^2$, it follows from (4.5) that

$$\| (\sigma \cdot A)\langle D \rangle^{-1/2}\varphi \|_{\mathcal{H}} \leq \epsilon \|T_0 + i\langle D \rangle^{-1/2}\varphi\|_{\mathcal{H}} + k_\epsilon \|\langle D \rangle^{-1/2}\varphi\|_{\mathcal{H}} \leq \epsilon \|\langle D \rangle^{1/2}\varphi\|_{\mathcal{H}} + k_\epsilon \|\varphi\|_{\mathcal{H}} < +\infty,$$

where we have used (4.2) and the fact that $\|\langle D \rangle^{1/2}\varphi\|_{\mathcal{H}} = \|\varphi\|_{[H^{1/2}(\mathbb{R}^3)]^2}$. \qed
Lemma 4.3. \( \langle D \rangle^{-1}(\sigma \cdot A)(D)^{-1/2} \) is a compact operator in \( \mathcal{H} \).

Proof. One can make a factorization
\[
\langle D \rangle^{-1}(\sigma \cdot A)(D)^{-1/2} = \left( \frac{|D|^{1/2}}{\langle D \rangle} \right) \left( \frac{1}{|D|^{1/2}}(\sigma \cdot A) \frac{1}{|D|^{1/2}} \right) \left( \frac{|D|^{1/2}}{\langle D \rangle} \right).
\]
(4.6)

It is obvious that the first term and the last term on the right-hand side of (4.6) are bounded operators in \( \mathcal{H} \). Then it follows from (4.6) and [6, Lemma 1] that the conclusion of the lemma holds true.

Lemma 4.4. Let \( z \in \mathbb{C} \setminus \mathbb{R} \). Then \( (T_A - z)^{-1}(D)|_{\mathcal{H}^1(\mathbb{R}^3)^2} \) can be extended to a bounded operator \( \tilde{R}_A(z) \) in \( \mathcal{H} \). Moreover
\[
(T_A - z)^{-1}(D) = \tilde{R}_A(z)(D)^{-1} \varphi \quad \text{for } \forall \varphi \in \mathcal{H}.
\]
(4.7)

Proof. We first show that \( \langle D \rangle (T - z)^{-1} \) is a closed operator in \( \mathcal{H} \). To this end, suppose that \( \{ \varphi_j \} \) is a sequence in \( \mathcal{H} \) such that \( \varphi_j \to 0 \) in \( \mathcal{H} \) and \( \langle D \rangle (T - z)^{-1} \varphi_j \to \psi \) in \( \mathcal{H} \) as \( j \to \infty \). Then \( \{ (T - z)^{-1} \varphi_j \} \) is a Cauchy sequence in \( [H^1(\mathbb{R}^3)]^2 \), hence there exists a \( \tilde{\psi} \in [H^1(\mathbb{R}^3)]^2 \) such that
\[
(T - z)^{-1} \varphi_j \to \tilde{\psi} \quad \text{in } [H^1(\mathbb{R}^3)]^2 \quad \text{as } j \to \infty.
\]
(4.8)

Since the topology of \( [H^1(\mathbb{R}^3)]^2 \) is stronger than that of \( \mathcal{H} \), (4.8) implies that
\[
(T - z)^{-1} \varphi_j \to \tilde{\psi} \quad \text{in } \mathcal{H} \quad \text{as } j \to \infty.
\]
(4.9)

On the other hand, since \( \varphi_j \to 0 \) in \( \mathcal{H} \), and since \( (T - z)^{-1} \) is a bounded operator in \( \mathcal{H} \), we see that
\[
(T - z)^{-1} \varphi_j \to 0 \quad \text{in } \mathcal{H}
\]
(4.10)
as \( j \to \infty \). Combining (4.9) and (4.10), we see that \( \tilde{\psi} = 0 \). This fact, together with (4.8), \( \langle D \rangle (T - z)^{-1} \varphi_j \to 0 \) in \( \mathcal{H} \) as \( j \to \infty \). Hence \( \psi = 0 \). We have thus shown that \( \langle D \rangle (T - z)^{-1} \) is a closed operator. Noting that \( \text{Dom}(\langle D \rangle (T - z)^{-1}) = \mathcal{H} \), we can conclude from the Banach closed graph theorem that \( \langle D \rangle (T - z)^{-1} \) is a bounded operator in \( \mathcal{H} \), which will be denoted by \( Q_A(z) \).

We now put \( \tilde{R}_A(z) := Q_A(\overline{z})^* \), where \( Q_A(\overline{z})^* \) denotes the adjoint operator of \( Q_A(\overline{z}) \). Then for any \( \varphi \in \mathcal{H} \) and any \( \psi \in [H^1(\mathbb{R}^3)]^2 \), we have
\[
(\varphi, \tilde{R}_A(z)\psi)_\mathcal{H} = (Q_A(\overline{z})\varphi, \psi)_\mathcal{H}
= ((D)(T - \overline{z})^{-1} \varphi, \psi)_\mathcal{H}
= (\varphi, (T - z)^{-1}(D)\psi)_\mathcal{H}.
\]
(4.11)

It follows from (4.11) that
\[
\tilde{R}_A(z)\psi = (T - z)^{-1}(D)\psi
\]
(4.12)
for all \( \psi \in [H^1(\mathbb{R}^3)]^2 \). Replacing \( \psi \) in (4.12) with \( \langle D \rangle^{-1} \varphi, \varphi \in \mathcal{H} \), we get (4.7). \( \square \)
Proof of Proposition 4.1. Since $\sigma(T_0) = \sigma_{\text{ess}}(T_0) = \mathbb{R}$, it is sufficient to show that

$$\sigma_{\text{ess}}(T_A) = \sigma_{\text{ess}}(T_0).$$  \hfill (4.13)

To this end, we shall prove that the difference $(T_A + i)^{-1} - (T_0 + i)^{-1}$ is a compact operator in $\mathcal{H}$. Then, this fact implies (4.13); see [26, Corollary 1, p. 113].

We see that

$$(T_A + i)^{-1} - (T_0 + i)^{-1} = (T_A + i)^{-1}(\sigma \cdot A)(T_0 + i)^{-1} = \tilde{R}_A(-i)\{\langle D \rangle^{-1/2}(\sigma \cdot A)(D)^{-1/2}\}\{(D)^{1/2}(T_0 + i)^{-1}\},$$ \hfill (4.14)

where we have used Lemma 4.4 in (4.14). It follows from Lemmas 4.1-4.4 that (4.14) makes sense as a product of three bounded operators in $\mathcal{H}$ and that the product is a compact operator in $\mathcal{H}$. 

Proposition 4.1, together with Theorem 2.3, gives the following result on the spectrum of the magnetic Dirac operator $H_A$.

**Theorem 4.1.** Let Assumption(BE) be satisfied. Then

$$\sigma(H_A) = \sigma_{\text{ess}}(H_A) = (-\infty, -m] \cup [m, \infty).$$

We now state the main results in this section, which are concerned with the eigenspaces corresponding to the threshold eigenvalues of the magnetic Dirac operator $H_A$.

**Theorem 4.2.** Let Assumption(BE) be satisfied. Then

(i) $\text{Ker}(H_A - m)$ is non-trivial if and only if $\text{Ker}(H_A + m)$ is non-trivial; in other words,

$$\{ A \in [L^3(\mathbb{R}^3)]^3 | \text{Ker}(H_A - m) \neq \{0\} \} = \{ A \in [L^3(\mathbb{R}^3)]^3 | \text{Ker}(H_A + m) \neq \{0\} \}.$$

(ii) There exists a constant $c$ such that

$$\dim(\text{Ker}(H_A - m)) = \dim(\text{Ker}(H_A + m)) \leq c \int_{\mathbb{R}^3} |A(x)|^3 dx.$$ \hfill (4.15)

Moreover, the dimension of $\text{Ker}(H_A \mp m)$ is independent of $m$.

(iii) The set $\{ A \in [L^3(\mathbb{R}^3)]^3 | \text{Ker}(H_A \mp m) = \{0\} \}$ contains an open dense subset of $[L^3(\mathbb{R}^3)]^3$.

**Proof.** By Corollary 2.1, we see that

$$\text{Ker}(T_A) \text{ is trivial} \iff \text{Ker}(H_A - m) \text{ is trivial} \iff \text{Ker}(H_A + m) \text{ is trivial.}$$ \hfill (4.16)

Assertion (i) is equivalent to (4.16). Assertion (ii) follows from Corollary 2.1 and [6, Theorem 3]. Assertion (iii) follows from Corollary 2.1 and [6, Theorem 2].
Remark 4.1. Assertions (i) and (ii) of Theorem 4.2 mean the following facts: The threshold energy $m$ is an eigenvalue of $H_A$ if and only if the threshold energy $-m$ is an eigenvalue of $H_A$. If this is the case, their multiplicity are the same.

Remark 4.2. As for the best constant in the inequality (4.15), see [6, Theorem 3].

5. The Structure of the Set of Vector Potentials Yielding $\pm m$ Modes

In this section, we shall discuss a property of non-locality of magnetic vector potentials as well as the sparseness of the set of vector potentials $A$ which give rise to $\pm m$ modes of $H_A$ in the spirit of Elton [11], where he investigated Weyl–Dirac operators.

We make the following assumption:

Assumption(E). Each $A_j$ ($j = 1, 2, 3$) is a real-valued continuous function such that $A_j(x) = o(|x|^{-1})$ as $|x| \to \infty$.

It is straightforward to see that under Assumption(E), $-\sigma \cdot A$ is a bounded self-adjoint operator in the Hilbert space $H = [L^2(\mathbb{R}^3)]^2$. Hence we can define the self-adjoint realization $T_A$ with $\text{Dom}(T_A) = [H^1(\mathbb{R}^3)]^2$ as the operator sum of $T_0$ and $-\sigma \cdot A$.

Also, it is straightforward to see that $-\alpha \cdot A$ is a bounded self-adjoint operator in the Hilbert space $K = [L^2(\mathbb{R}^3)]^4$, hence we can define the self-adjoint realization $H_A$ with $\text{Dom}(H_A) = [H^1(\mathbb{R}^3)]^4$ in $K$ as the operator sum of $H_0$ and $-\alpha \cdot A$. Therefore, in the same way as in Sec. 5, we can regard $H_A$ as a supersymmetric Dirac operator, and apply the results in Sec. 2 to $H_A$.

We note that under Assumption(E), $(-\sigma \cdot A)(T_0 + i)^{-1}$ is a compact operator in $H$. Hence, in the same way as in the proof of Proposition 4.1, we can show that $\sigma(T_A) = \mathbb{R}$. This fact, together with Theorem 2.3, implies the following result.

Theorem 5.1. Let Assumption(E) be satisfied. Then

$$\sigma(H_A) = \sigma_{\text{ess}}(H_A) = (-\infty, -m] \cup [m, \infty).$$

To state the main results in this section, we need to introduce the following notation:

$$\mathcal{A} := \{ A | A \text{ satisfies Assumption(E)} \}.$$  \hspace{1cm} (5.1)

We regard $\mathcal{A}$ as a Banach space with the norm

$$||A||_{\mathcal{A}} = \sup_x \langle x \rangle |A(x)|.$$

Theorem 5.2. Let Assumption(E) be satisfied. Define

$$Z^\pm_k = \{ A \in \mathcal{A} | \dim(\text{Ker}(H \mp m)) = k \}$$

for $k = 0, 1, 2, \ldots$. Then

(i) $Z^+_k = Z^-_k$ for all $k$. 

(ii) $\mathcal{Z}_0^\pm$ is an open and dense subset of $\mathcal{A}$.
(iii) For any $k$ and any open subset $\Omega(\neq \emptyset)$ of $\mathbb{R}^3$ there exists an $A \in \mathcal{Z}_k^\pm$ such that $A \in [C_0^\infty(\Omega)]^3$.

**Proof.** Assertion (i) is a direct consequence of Corollary 2.1. Assertions (ii) and (iii) follow from Corollary 2.1 and [11, Theorem 1].

It is of some interest to point out a conclusion following from Theorem 3.1 and Assertion (iii) of Theorem 5.2. Namely, there are (at least) countably infinite number of vector potentials $A$ with compact support such that the corresponding Dirac operators $H_A$ have $\pm m$ modes $f^\pm$ with the property (3.4). The $\pm m$ modes $f^\pm$ behave like $|f^\pm(x)| \asymp |x|^{-2}$ for $|x| \to \infty$, in spite of the fact that the vector potentials and the corresponding magnetic fields vanish outside bounded regions. It is obvious that this phenomenon describes a certain kind of non-locality.

Also, it is of some interest to mention that $H_A$ does not have $\pm m$ resonances if the support of vector potential $A$ is compact. This is an immediate consequence of Theorem 6.1 in the next section.

6. Non-Existence of $\pm m$ Resonances

In this section, we will work in bigger Hilbert spaces than $\mathcal{H} = [L^2(\mathbb{R}^3)]^2$ and $\mathcal{K} = [L^2(\mathbb{R}^3)]^4$. Therefore, the results on the supersymmetric Dirac operators in Sec. 2 are not applicable in this section.

In this section, we shall occasionally write the inner product of $\mathcal{H}$ as

$$(\varphi, \psi)_{\mathcal{H}} = \int_{\mathbb{R}^3} (\varphi(x), \psi(x))_{\mathbb{C}^2} dx$$

for $\varphi, \psi \in \mathcal{H}$, where $(\cdot, \cdot)_{\mathbb{C}^2}$ denotes the inner product of $\mathbb{C}^2$.

We need to introduce weighted $L^2$ spaces in order to deal with $\pm m$ resonances, which do not belong to the Hilbert space $\mathcal{K}$. By $L^{2,s}(\mathbb{R}^3)$, we mean the weighted $L^2$ space defined by

$$L^{2,s}(\mathbb{R}^3) := \{u | \langle x \rangle^s u \in L^2(\mathbb{R}^3)\} \quad (s \in \mathbb{R})$$

where $\langle x \rangle = \sqrt{1 + |x|^2}$, and we set

$$\mathcal{L}^{2,s} = [L^{2,s}(\mathbb{R}^3)]^4.$$

**Definition 6.1.** By an $m$ resonance (respectively, a $-m$ resonance), we mean a function $f \in \mathcal{L}^{2,-s}\backslash \mathcal{K}$, $0 < s \leq 3/2$, such that $H_A f = mf$ (respectively, $H_A f = -mf$) in the distributional sense.

We would like to caution that in Definition 6.1 one has to take the meaning of $H_A f = \pm mf$ in the distributional sense, because of the reason that $\pm m$ resonances do not belong to the Hilbert space $\mathcal{K}$, hence do not belong to the domain of the self-adjoint realization of $H_A$. For this reason, we let $H_A$ stand for the formal differential
operator throughout this section, in spite of the fact that $H_A$ has the unique self-adjoint realization in $\mathcal{K}$ under the assumption of Theorem 6.1 below. We hope this will not cause any confusion.

**Theorem 6.1.** Assume that each element $A_j(x)$ $(j = 1, 2, 3)$ of $A(x)$ is a measurable function satisfying
\[ |A_j(x)| \leq C|x|^{-\rho} \quad (\rho > 3/2), \tag{6.1} \]
where $C$ is a positive constant. Suppose that $f = \frac{t}{2}(\varphi^+, \varphi^-)$ belongs to $L^2_{\text{loc}}(\mathbb{R}^3)$ for some $s$ with $0 < s < \min(1, \rho - 1)$ and satisfies $H_A f = m f$ (respectively, $H_A f = -m f$) in the distributional sense. Then $f \in [H^1(\mathbb{R}^3)]^4$ and $\varphi^- = 0$ (respectively, $\varphi^+ = 0$).

Theorem 6.1 implies the non-existence of $\pm m$ resonances in the sense of Definition 6.1, as well as in the sense described in the following theorem.

**Theorem 6.2.** Let $A(x)$ satisfy the same assumption as in Theorem 6.1. Suppose that $f$ belongs to $[L^2_{\text{loc}}(\mathbb{R}^3)]^4$ and satisfies either equation of $H_A f = \pm m f$ in the distributional sense. In addition, suppose that $f$ has the asymptotic expansion
\[ f(x) = C_1|x|^{-1} + C_2|x|^{-2} + o(|x|^{-2}) \tag{6.2} \]
as $|x| \to \infty$, where $C_1$ and $C_2$ are constant vectors in $\mathbb{C}^4$. Then $C_1 = 0$.

**Proof.** It follows from (6.2) that $f \in L^2_{\text{loc}}$ for any $s$ with $1/2 < s < 1$. This fact, together with the assumptions of the theorem, enables us to apply Theorem 6.1 and to conclude that $f \in [H^1(\mathbb{R}^3)]^4$. In particular, $f \in \mathcal{K}$, which leads to the fact that $C_1 = 0$. \hfill \Box

We shall give a proof of Theorem 6.1 only for $m$ resonances, since the proof for $-m$ resonances is similar. Roughly speaking, we will mimic the idea of the proof of Assertion (i) of Theorem 2.1. Therefore we need the Weyl–Dirac operator $T_A = \sigma \cdot (D - A(x))$ again. However, we are not allowed to use the Weyl–Dirac operator as a self-adjoint operator in the Hilbert space $\mathcal{H}$, but only allowed to use it as a formal differential operator instead. This is because $\pm m$ resonances do not belong to $\mathcal{K}$. This fact causes complication, in a certain extent, in the proof of Theorem 6.1.

We begin the proof of Theorem 6.1 with a lemma whose proof will be given after the proof of the theorem. The proof of the lemma is lengthy.

**Lemma 6.1.** Under the hypotheses of Theorem 6.1, $\varphi^\pm$ have the following properties:

(i) $(\sigma \cdot D)\varphi^+ \in \mathcal{H}$, $(\sigma \cdot A)\varphi^+ \in \mathcal{H}$ and $\varphi^- \in [H^1(\mathbb{R}^3)]^2$.

(ii) $(\sigma \cdot D)\varphi^+, \varphi^-)_{\mathcal{H}} = ((\sigma \cdot A)\varphi^+, \varphi^-)_{\mathcal{H}}$. 

Proof of Theorem 6.1. Let $f$ satisfy $H_A f = mf$ in the distributional sense. Then we have

$$
\begin{cases}
    m\varphi^+ + \sigma \cdot (D - A(x))\varphi^- = m\varphi^+ \\
    \sigma \cdot (D - A(x))\varphi^+ - m\varphi^- = m\varphi^-
\end{cases}
$$

(6.3)

in the distributional sense, which immediately implies

$$
\sigma \cdot (D - A(x))\varphi^- = 0
$$

(6.4)

and

$$
\sigma \cdot (D - A(x))\varphi^+ = 2m\varphi^-.
$$

(6.5)

In view of Lemma 6.1, it follows from (6.5) that

$$
4m^2\|\varphi^-\|_H^2 = 2m(2m\varphi^-,\varphi^-)_H = 2m(\sigma \cdot (D - A)\varphi^+,\varphi^-)_H = 2m\{(\sigma \cdot D)\varphi^+,\varphi^-)_H - ((\sigma \cdot A)\varphi^+,\varphi^-)_H\} = 0.
$$

(6.6)

Hence $\varphi^- = 0$. This fact, together with (6.5), means that

$$
\sigma \cdot (D - A(x))\varphi^+ = 0
$$

(6.7)

in the distributional sense. It follows from [28, Theorem 2.2] that $\varphi^+ \in [H^1(\mathbb{R}^3)]^2$. (Note that the hypothesis $0 < s < \min(1, \rho - 1)$ is stronger than the one imposed in [28, Theorem 2.2].) This implies that $f \in [H^1(\mathbb{R}^3)]^4$, because $\varphi^- = 0$ as was shown above.

Before proving Lemma 6.1, we should like to remark that (6.4) and (6.5) follow directly from the hypothesis that $H_A f = mf$ in the distributional sense. Therefore we are allowed to use (6.4) and (6.5) in the proof of Lemma 6.1 below.

Proof of Lemma 6.1. Since $\rho - s > 1$ by assumption, we see that

$$
(\sigma \cdot A)\varphi^+ \in [L^{2,\rho-s}(\mathbb{R}^3)]^2 \subset \mathcal{H}.
$$

(6.8)

It follows from (6.4) and [28, Theorem 2.2] that $\varphi^- \in [H^1(\mathbb{R}^3)]^2$. This fact, together with (6.5) and (6.8), implies that $(\sigma \cdot D)\varphi^+ = 2m\varphi^- + (\sigma \cdot A)\varphi^+ \in \mathcal{H}$. Thus Assertion (i) is proved.

In order to prove Assertion (ii), we need to introduce a cutoff function. Let $\chi$ be a function in $C^\infty(\mathbb{R})$ such that $0 \leq \chi \leq 1$, $\chi(r) = 1$ ($r \leq 1$), and $\chi(r) = 0$ ($r \geq 2$). Set

$$
\chi_n(x) = \chi(|x|/n) \quad (n = 1, 2, 3, \ldots).
$$

(6.9)
It is evident that
\[
((\sigma \cdot D)\varphi^+, \varphi^-)_H = \lim_{n \to \infty} ((\sigma \cdot D)\varphi^+, \chi_n\varphi^-)_H. \tag{6.10}
\]

Let \( \{\varepsilon_j\}_{0 < \varepsilon < 1} \) be Friedrichs’ mollifier, i.e. \( \varepsilon_j(x) := \varepsilon^{-3}j(x/\varepsilon) \), where \( j \in C^\infty_0(\mathbb{R}^3) \) and \( \|j\|_{L^1} = 1 \). Since \( \chi_n\varphi^- \in H \), we see that \( j_\varepsilon * (\chi_n\varphi^-) \) converges to \( \chi_n\varphi^- \) in \( H \) as \( \varepsilon \downarrow 0 \). Hence, for each \( n \), we have
\[
((\sigma \cdot D)\varphi^+, \chi_n\varphi^-)_H = \lim_{\varepsilon \downarrow 0}((\sigma \cdot D)\varphi^+, j_\varepsilon * (\chi_n\varphi^-))_H. \tag{6.11}
\]

It is straightforward that \( j_\varepsilon * (\chi_n\varphi^-) \in [C^\infty_0(\mathbb{R}^3)]^2 \) and that
\[
\supp(j_\varepsilon * (\chi_n\varphi^-)) \subset \{x \mid |x| \leq 2n + 1\}. \tag{6.12}
\]

Appealing to the definition of the distributional derivatives, we get
\[
((\sigma \cdot D)\varphi^+, j_\varepsilon * (\chi_n\varphi^-))_H = \int_{\mathbb{R}^3} (\varphi^+(x), (\sigma \cdot D)(j_\varepsilon * (\chi_n\varphi^-))(x))_C^2 \, dx. \tag{6.13}
\]

For each \( n \) and \( \varepsilon \), we have
\[
(\sigma \cdot D)(j_\varepsilon * (\chi_n\varphi^-))(x) = \int_{\mathbb{R}^3} (\sigma \cdot D)(j_\varepsilon(x - y))\chi_n(y)\varphi^-(y) \, dy
\]
\[
= \int_{\mathbb{R}^3} -(\sigma \cdot D_y)(j_\varepsilon(x - y))\chi_n(y)\varphi^-(y) \, dy
\]
\[
= \int_{\mathbb{R}^3} j_\varepsilon(x - y)(\sigma \cdot D_y)(\chi_n(y)\varphi^-(y)) \, dy
\]
\[
= j_\varepsilon * \{(\sigma \cdot D)(\chi_n\varphi^-)\}(x). \tag{6.14}
\]

In the third equality of (6.14), we have regarded \( j_\varepsilon(x - \cdot) \) as a function in \( C^\infty_0(\mathbb{R}^3) \) and have appealed to the definition of the distributional derivatives with respect to \( y \) variable. Note that
\[
(\sigma \cdot D)(\chi_n\varphi^-) = \{(\sigma \cdot D)\chi_n\}\varphi^- + \chi_n(\sigma \cdot D)\varphi^- \tag{6.15}
\]

Combining (6.13)–(6.15), we obtain
\[
((\sigma \cdot D)\varphi^+, j_\varepsilon * (\chi_n\varphi^-))_H = \int_{\mathbb{R}^3} (\varphi^+(x), j_\varepsilon * \{(\sigma \cdot D)\chi_n\}\varphi^-)(x) \, C^2 \, dx
\]
\[
+ \int_{\mathbb{R}^3} (\varphi^+(x), j_\varepsilon * [\chi_n(\sigma \cdot D)\varphi^-])(x) \, C^2 \, dx. \tag{6.16}
\]
We examine the limit of each integral on the right-hand side of (6.16) as \( \varepsilon \downarrow 0 \). As for the first integral, we have

\[
\left| \int_{\mathbb{R}^3} (\varphi^+(x), j_x \ast [\{(\sigma \cdot D)\chi_n\} \varphi^-](x)) dx \right|
\]

\[
- \int_{\mathbb{R}^3} (\varphi^+(x), \{(\sigma \cdot D)\chi_n\}(x) \varphi^- (x)) dx
\]

\[
\leq \int_{|x| \leq 2n + 1} |\varphi^+(x)| dx
\]

\[
\times |j_x \ast [\{(\sigma \cdot D)\chi_n\} \varphi^-](x) - \{(\sigma \cdot D)\chi_n\}(x) \varphi^- (x)| dx
\]

\[
\leq |||\varphi^+||_{C^2} ||j_x||_{L^2(|x| \leq 2n + 1)} ||\varphi^-||_{C^2} \rightarrow 0 \quad (\varepsilon \downarrow 0),
\]

(6.17)

since \( \{(\sigma \cdot D)\chi_n\} \varphi^- \in \mathcal{H} \). In the first inequality (6.17) we have used the Schwarz inequality in \( C^2 \), and in the second inequality the Schwarz inequality in \( L^2 \). Therefore

\[
\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} (\varphi^+(x), j_x \ast [\{(\sigma \cdot D)\chi_n\} \varphi^-](x)) dx
\]

\[
= \int_{\mathbb{R}^3} (\varphi^+(x), \{(\sigma \cdot D)\chi_n\}(x) \varphi^- (x)) dx.
\]

(6.18)

In a similar manner, we see that

\[
\lim_{\varepsilon \downarrow 0} \int_{\mathbb{R}^3} (\varphi^+(x), j_x \ast [\chi_n(\sigma \cdot D)\varphi^-](x)) dx
\]

\[
= \int_{\mathbb{R}^3} (\varphi^+(x), \chi_n(\sigma \cdot D)\varphi^-) dx,
\]

(6.19)

where we have used the fact that \( \varphi^- \in [H^1(\mathbb{R}^3)]^2 \). (Recall that this fact was shown in Assertion (i) of the lemma.) It follows from (6.11), (6.16), (6.18) and (6.19) that

\[
((\sigma \cdot D)\varphi^+, \chi_n\varphi^-)_{\mathcal{H}} = \int_{\mathbb{R}^3} (\varphi^+(x), \{(\sigma \cdot D)\chi_n\}(x) \varphi^- (x)) dx
\]

\[
+ \int_{\mathbb{R}^3} (\varphi^+(x), \chi_n(\sigma \cdot D)\varphi^-) dx.
\]

(6.20)

To estimate the first integral on the right-hand side of (6.20), we need the fact that

\[
\{(\sigma \cdot D)\chi_n\}(x) = \frac{1}{n} \chi'(\frac{|x|}{n}) \frac{1}{i} (\sigma \cdot \omega) \quad (\omega = x/|x|),
\]

(6.21)
Eigenfunctions at the Threshold Energies of Magnetic Dirac Operators

Note that \( \text{supp}(\sigma \cdot D)\chi_n \subset \{ x \mid n \leq |x| \leq 2n \} \) and that \( \sigma \cdot \omega \) is a unitary matrix. Hence we have

\[
\left| \int_{\mathbb{R}^3} (\phi^+ (x), \{ (\sigma \cdot D)\chi_n \}(x)\phi^- (x))c^2 dx \right|
\]

\[
\leq \frac{1}{n} \left( \sup_{r > 0} |\chi' (r)| \right) \int_{n \leq |x| \leq 2n} |\phi^+ (x)|c^2 |\phi^- (x)|c^2 dx
\]

\[
\leq \frac{1}{n} \left( \sup_{r > 0} |\chi' (r)| \right) \left\{ \int_{n \leq |x| \leq 2n} \langle x \rangle^{-s} |\phi^+ (x)|^2 c^2 dx \right\}^{1/2}
\]

\[
\times \left\{ \int_{n \leq |x| \leq 2n} \langle x \rangle^{2s} |\phi^- (x)|^2 c^2 dx \right\}^{1/2}
\]

\[
\leq \frac{1}{n} \left( \sup_{r > 0} |\chi' (r)| \right) \| |\phi^+ |c^2 \|_{L^2,-s} \times (1 + 4n^{-2})^{s/2} \| \phi^- \|_{\mathcal{H}}
\]

\[
\rightarrow 0 \quad (n \rightarrow \infty), \quad (6.22)
\]

since \( s < 1 \) by assumption of Theorem 6.1. Thus the first integral on the right-hand side of (6.20) tends to 0 as \( n \rightarrow \infty \).

We now investigate the limit of the second integral on the right-hand side of (6.20) as \( n \rightarrow \infty \). It follows from (6.4) that

\[
\int_{\mathbb{R}^3} (\phi^+ (x), \chi_n (x)(\sigma \cdot D)\phi^- (x))c^2 dx - \int_{\mathbb{R}^3} ((\sigma \cdot A)(x)\phi^+ (x), \phi^- (x))c^2 dx
\]

\[
= \int_{\mathbb{R}^3} ((\chi_n (x) - 1)(\sigma \cdot A)(x)\phi^+ (x), \phi^- (x))c^2 dx, \quad (6.23)
\]

where we have used the fact that \( (\sigma \cdot A)(x) \) is a Hermitian matrix for each \( x \). Noting (6.8), we find that the absolute value of the right-hand side of (6.23) is less than or equal to

\[
\| (\chi_n - 1)(\sigma \cdot A)\phi^+ \|_{\mathcal{H}} \| \phi^- \|_{\mathcal{H}}, \quad (6.24)
\]

which obviously tends to 0 as \( n \rightarrow \infty \). Combining this fact with (6.20), (6.22) and (6.23), we obtain

\[
\lim_{n \rightarrow \infty} ((\sigma \cdot D)\phi^+, \chi_n \phi^-)_{\mathcal{H}} = ((\sigma \cdot A)\phi^+, \phi^-)_{\mathcal{H}}. \quad (6.25)
\]

Assertion (ii) of the lemma is a direct consequence of (6.10) and (6.25).

7. Examples, Concluding Remarks and an Open Question

We shall give examples of vector potentials \( A(x) \) which yield \( \pm m \) modes but do not give rise to \( \pm m \) resonances. The basic idea in this section is to exploit the equivalences (2.9), (2.10), and to apply Theorem 6.1. It turns out that beautiful spectral
properties are common to all the examples of the magnetic Dirac operators in this section. See Properties (i)--(iv) of Examples 7.1 and 7.2.

**Example 7.1** ([19]). Let

\[ A_{LY}(x) = 3\langle x \rangle^{-4}((1 - |x|^2)w_0 + 2(w_0 \cdot x)x + 2w_0 \times x) \quad (7.1) \]

where \( \phi_0 = \mathbf{i}(1, 0) \) (\( \phi_0 \) can be any unit vector in \( \mathbb{C}^2 \)) and

\[ w_0 = \phi_0 \cdot (\sigma \phi_0) := ((\phi_0, \sigma_1 \phi_0)_{\mathbb{C}^2}, (\phi_0, \sigma_2 \phi_0)_{\mathbb{C}^2}, (\phi_0, \sigma_3 \phi_0)_{\mathbb{C}^2}) \quad (7.2) \]

Here \( w_0 \cdot x \) and \( w_0 \times x \) denotes the inner product and the exterior product of \( \mathbb{R}^3 \) respectively. Then the magnetic Dirac operator

\[ H_{LY} := H_{ALY} = \alpha \cdot (D - A_{LY}(x)) + m\beta \]

has the following properties:

(i) \( \sigma(H_{LY}) = \sigma_{ess}(H_{LY}) = (-\infty, -m] \cup [m, \infty) \); 
(ii) \( H_{LY} \) has \( \pm m \) modes. Moreover, the point spectrum of \( H_{LY} \) consists only of \( \pm m \), i.e. \( \sigma_p(H_{LY}) = \{-m, m\} \); 
(iii) \( H_{LY} \) does not have \( \pm m \) resonances; 
(iv) \( H_{LY} \) is absolutely continuous on \( (-\infty, -m) \cup (m, \infty) \).

We shall show these properties one-by-one. It is easy to see that \(-\sigma \cdot A_{LY}(x)\) is relatively compact perturbation of \( T_0 = \sigma \cdot D \), hence the Weyl–Dirac operator

\[ T_{LY} := T_{ALY} = \sigma \cdot (D - A_{LY}(x)) \]

is a self-adjoint operator in the Hilbert space \( \mathcal{H} = [L^2(\mathbb{R}^3)]^2 \) with the domain \( [H^1(\mathbb{R}^3)]^2 \). Since the spectrum of the operator \( T_0 \) equals the whole real line, we see that \( \sigma(T_{LY}) = \mathbb{R} \). Property (i) immediately follows from Theorem 2.3.

We shall show Property (ii). According to Loss and Yau [19, Sec. II], the Weyl–Dirac operator \( T_{LY} \) has a zero mode \( \varphi_{LY} \) defined by

\[ \varphi_{LY}(x) = \langle x \rangle^{-3}(I_2 + i\sigma \cdot x)\phi_0. \quad (7.3) \]

It follows from (2.9) and (2.10) that \( \langle \varphi_{LY}, 0 \rangle \) (respectively, \( \langle 0, \varphi_{LY} \rangle \)) is an \( m \) mode (respectively, \( -m \) mode) of \( H_{LY} \). Hence \( \sigma_p(H_{LY}) \supset \{-m, m\} \). On the other hand, it follows from Yamada [35] that \( H_{LY} \) has no eigenvalue in \((\infty, \infty) \cup (m, \infty) \). (Note that the vector potential \( A_{LY} \) satisfies the assumption of [35, Proposition 2.5].) This fact, together with Property (i), implies that \( \sigma_p(H_{LY}) \subset \{-m, m\} \). Summing up, we get Property (ii). Since \( |A_{LY}(x)| \leq C|x|^{-2} \), Property (iii) follows from Theorem 6.1. Property (iv) is a direct consequence of [35, Corollary 4.2]. As for absolutely continuity and limiting absorption principle for Dirac operators, see also [36, 9, 24].

**Remark 7.1.** Since \( A_{LY} \) is \( C^\infty \), one can apply Thaller [31, Theorem 7.1, p. 195] to conclude that \( \langle \varphi_{LY}, 0 \rangle \) (respectively, \( \langle 0, \varphi_{LY} \rangle \)) is an \( m \) mode (respectively a \( -m \) mode) of \( H_{LY} \). This fact is also mentioned in [30].
Remark 7.2. As was pointed out in [19, Sec. II], one sees that \( \text{div} A_{LY} \neq 0 \), and one can find, by a gauge transformation, a vector potential \( \tilde{A}_{LY} \) which satisfies \( \text{div} \tilde{A}_{LY} = 0 \) and \( \text{rot} \tilde{A}_{LY} = \text{rot} A_{LY} \) and yields a zero mode \( \tilde{\varphi}_{LY} \) of \( \sigma \cdot (D - \tilde{A}_{LY}(x)) \).

In fact, defining

\[
\tilde{A}_{LY} := A_{LY} + \nabla \chi_{LY}, \quad \tilde{\varphi}_{LY} := e^{i\chi_{LY}} \varphi_{LY}
\]

with

\[
\chi_{LY}(x) := \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{1}{|x-y|} \left( \text{div} A_{LY} \right)(y) dy,
\]

we observe that \( \tilde{A}_{LY} \) and \( \tilde{\varphi}_{LY} \) have the desired properties mentioned above. Moreover, we can show that \( |\tilde{A}_{LY}(x)| \leq C(x)^{-\frac{2}{3}}(\in L^6(\mathbb{R}^3)) \). Hence, the magnetic Dirac operator \( H_{\tilde{A}_{LY}} = \alpha \cdot (D - \tilde{A}_{LY}(x)) + m\beta \) shares the Properties (i)–(iv) of Example 7.1 with \( H_{A_{LY}} = \alpha \cdot (D - A_{LY}(x)) + m\beta \). This same idea is applicable to the vector potentials \( A^{(\ell)} \) in Example 7.2 below.

Example 7.2 ([2]). In the same spirit as in Example 7.1, we can show the existence of countably infinite number of vector potentials with which the magnetic Dirac operators have the Properties (i)–(iv) in Example 7.1.

In fact, we shall exploit a result on the Weyl–Dirac operator by Adam, Muratori and Nash [2], where they construct a series of vector potentials \( A^{(\ell)} (\ell = 0, 1, 2, \ldots) \), each of which gives rise a zero mode \( \psi^{(\ell)}(x) \) of the Weyl–Dirac operator \( T^{(\ell)} := \sigma \cdot (D - A^{(\ell)}(x)) \). The idea of [2] is an extension of that of [19, Sec. II]. Indeed \( A^{(0)} \) and \( \psi^{(0)} \) give the same vector potential and zero mode as in (7.1) and (7.3). For \( \ell \geq 1 \), the construction of the zero mode \( \psi^{(\ell)}(x) \) is based on an anzatz (see [2, Sec. II], (7)) and the definition of \( A^{(\ell)} \) is given by

\[
A^{(\ell)}(x) = \frac{h^{(\ell)}(x)}{\left| \psi^{(0)}(x) \right|^2} \{ \psi^{(\ell)}(x) \cdot (\sigma \psi^{(\ell)}(x)) \}, \tag{7.4}
\]

where \( h^{(\ell)}(x) \) is a real valued function defined as

\[
h^{(\ell)}(x) = \frac{c_\ell}{\langle x \rangle^2} \tag{7.5}
\]

and \( \psi^{(\ell)}(x) \cdot (\sigma \psi^{(\ell)}(x)) \) is defined in the same way as in (7.2). (For the definition of \( h^{(\ell)}(x) \), see [29].) By the same arguments as in Example 7.1, we can deduce that the magnetic Dirac operator \( H^{(\ell)} := \alpha \cdot (D - A^{(\ell)}(x)) + m\beta \), \( \ell = 0, 1, 2, \ldots \), has the Properties (i)–(iv) of Example 7.1.

Section 3 was based upon our results on supersymmetric Dirac operators in Sec. 2 of the present paper and those of [27]. It turned out that all \( \pm m \) mode have the same asymptotic limit at infinity, i.e. \( \sim |x|^{-2} \) as \( |x| \to \infty \). This means that the asymptotic limits of \( \pm m \) modes of the magnetic Dirac operator are the same as those of zero modes of the Weyl–Dirac operator. Section 4 was based upon our results on supersymmetric Dirac operators in Sec. 2 and those of Balinsky and Evans [6] on the Weyl–Dirac operator. Section 5 was based upon our results on
supersymmetric Dirac operators in Sec. 2 and those of Elton [11] on the Weyl–Dirac operator. In each section from Secs. 3–5, we made a different assumption on the vector potentials. It is meaningful to compare these assumptions with each other. To this end, imitating (5.1), we introduce the following notation

\[ A_{SU} := \{ A \mid A \text{ satisfies Assumption(SU)} \}, \]

\[ A_{BE} := \{ A \mid A \text{ satisfies Assumption(BE)} \}. \]

We then have

\[ A_{SU} \subseteq A_{BE}, \]

\[ A \setminus A_{SU} \neq \emptyset, \quad A_{SU} \setminus A \neq \emptyset, \]

\[ A \setminus A_{BE} \neq \emptyset, \quad A_{BE} \setminus A \neq \emptyset. \]

In Secs. 4 and 5, it was shown that the set of vector potentials which give rise to ±m modes is scarce in each regime. The non-existence of ±m resonances was proved in Sec. 6 under the assumption that \( |A_j(x)| \leq C(x)^{-\rho}, \rho > 3/2. \) Based on the results in Sec. 6, it follows that all the examples of vector potentials in this section do not have ±m resonances. A natural question arises:

Is there a vector potential \( A \) which satisfies \( |A_j(x)| \leq C(x)^{-\rho}, \rho > 0, \) and yields ±m resonances of the magnetic Dirac operator \( HA? \)

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**References**


