Stability for the inverse resonance problem for the CMV operator

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I am reporting on joint work with

- Roman Shterenberg (UAB) and
- Maxim Zinchenko (New Mexico).
Inverse resonance problems

- Brown and Eastham (1997 – 2000) found that the points of spectral concentration indicate the presence of resonances in the vicinity.
- Brown, Knowles, Naboko and myself have some contributions to uniqueness (Schrödinger problems with zero or algebro-geometric backgrounds, Jacobi problems with zero background).
- Iantchenko and Korotyaev (2011) have results for Jacobi problems with periodic background).
- Stability of the recovered potential for finite noisy resonance data for the Schrödinger equation (with Marletta, Naboko, Shterenberg; Bledsoe).
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Orthogonal polynomials on the real line

• Let $\mu$ be a probability measure on $\mathbb{R}$ all of whose moments are finite.

• By Gram-Schmidt this gives rise to a sequence of orthonormal polynomials $p_n$ (unique with positive leading coefficient).

• It is well known (and easy to see) that they satisfy a three-term recurrence:

$$a_{n-1}p_{n-1}(t) + b_n p_n(t) + a_n p_{n+1}(t) = tp_n(t)$$

where the $a_n \neq 0$.

• Thus, choosing the $p_n$ as a basis, multiplication by $t$ is represented by a three-diagonal semi-infinite matrix, i.e., a Jacobi matrix.

• Multiplication by $t$ is a self-adjoint operator.

• Spectral theory for Jacobi matrices allows to investigate the polynomials.
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- The representation of multiplication by the independent variable (using the orthogonal polynomials as a basis) leads only to a Hessenberg matrix.
Cantero, Moral, and Velázquez (CMV)

- In 2003 CMV suggested to use Laurent polynomials, i.e., polynomials in \( \mathbb{C}[z, 1/z] \).
- Apply Gram-Schmidt to \((1, 1/z, z, 1/z^2, z^2, \ldots)\) to produce orthonormal Laurent polynomials \( f_n \) (OLPs) instead of the standard OPs \( p_n \).
- In terms of this basis multiplication by \( z \) is represented by a five-diagonal matrix.
- Multiplication by \( z \) is now a unitary operator denoted by \( U \).
- The OLPs and the OPs are in a simple relationship:
  \[
  f_{2n}(z) = z^{-n}p_{2n}(z), \quad f_{2n+1}(z) = z^n p_{2n+1}(1/z).
  \]
- Starting from \((1, z, 1/z, z^2, 1/z^2, \ldots)\) the OLPs are
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- Starting from $(1, z, 1/z, z^2, 1/z^2, \ldots)$ the OLPs are

  $$g_n(z) = f_{n,*}(z) := \overline{f_n(1/z)}.$$
The 5-term recurrence

- Instead of a Jacobi matrix CMV obtained the matrix

\[
U = \begin{pmatrix}
-\alpha_1 & \rho_1 & 0 & & \\
-\rho_1 \alpha_2 & -\overline{\alpha_1} \alpha_2 & -\rho_2 \alpha_3 & \rho_2 \rho_3 & 0 \\
\rho_1 \rho_2 & \overline{\alpha_1} \rho_2 & -\overline{\alpha_2} \alpha_3 & \overline{\alpha_2} \rho_3 & 0 \\
0 & -\rho_3 \alpha_4 & -\overline{\alpha_3} \alpha_4 & -\rho_4 \alpha_5 & \rho_4 \rho_5 \\
\rho_3 \rho_4 & \overline{\alpha_3} \rho_4 & -\overline{\alpha_4} \alpha_5 & \overline{\alpha_4} \rho_5 & 0 \\
0 & & & & & \cdots \\
& & & & & \cdots \\
& & & & & \cdots
\end{pmatrix}
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where \( \alpha_n \in \mathbb{D}, \rho_n = \sqrt{1 - |\alpha_n|^2} \).
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\cdots & \cdots & \cdots & \cdots & \cdots
\end{pmatrix}
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• The \( \alpha_n \) are called Verblunsky coefficients.
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• We may think of \( U \) as a unitary operator in \( \ell^2(\mathbb{N}_0) \) or as mapping any sequence of complex numbers to another.
Factorization of $U$

Every CMV matrix admits the following factorization

$$U = VW = \begin{pmatrix}
1 & -\alpha_2 & \rho_2 \\
-\alpha_2 & \rho_2 & \bar{\alpha}_2 \\
\rho_2 & \bar{\alpha}_2 & -\alpha_4 & \rho_4 \\
-\alpha_4 & \rho_4 & \bar{\alpha}_4 & \ddots \\
\rho_4 & \bar{\alpha}_4 & \ddots & \ddots
\end{pmatrix} \begin{pmatrix}
-\alpha_1 & \rho_1 \\
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\rho_3 & \bar{\alpha}_3 & \ddots
\end{pmatrix}.$$
CMV recursion

- Despite $U$ being 5-diagonal we have a second order problem.
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- Let $\beta_k = \alpha_k$ or $\beta_k = \bar{\alpha}_k$ and $\zeta_k = z$ or $\zeta_k = 1$ depending on whether $k$ is odd or even and
  
  $$T(z, k) = \frac{1}{\rho_k} \begin{pmatrix} \beta_k & \zeta_k \\ 1/\zeta_k & \beta_k \end{pmatrix}.$$
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  \[ T(z, k) = \frac{1}{\rho_k} \begin{pmatrix} \beta_k & \zeta_k \\ 1/\zeta_k & \beta_k \end{pmatrix}. \]
- Then [Gesztesy, Zinchenko 2006]
  \[ \begin{pmatrix} u \\ v \end{pmatrix}(k) = T(k) \begin{pmatrix} u \\ v \end{pmatrix}(k - 1), \quad k \in \mathbb{N} \]
  if and only if
  \[ Wu = zv \quad \text{and} \quad Vv = u + (v(z, 0) - u(z, 0))\delta_0. \]
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- Consequently $(U - z)u = z(v(z, 0) - u(z, 0))\delta_0$ (not necessarily in the operator sense).
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- Consequently $(U - z)u = z(v(z, 0) - u(z, 0))\delta_0$ (not necessarily in the operator sense).

- In particular $Uu = zu$ if $v(z, 0) = u(z, 0)$.
Initial value problems

- For $z \neq 0$ introduce the solutions $\vartheta(z, \cdot)$ and $\varphi(z, \cdot)$ of

$$
\begin{pmatrix}
u
\end{pmatrix}(k) = \begin{pmatrix}
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$$

with initial conditions $(-1, 1)^\top$ and $(1, 1)^\top$, respectively.
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- Note that

$$
\varphi(z, k) = \begin{pmatrix}0 & 1 \\ 1 & 0 \end{pmatrix} \varphi(1/\bar{z}, k) \quad \text{and} \quad \vartheta(z, k) = -\begin{pmatrix}0 & 1 \\ 1 & 0 \end{pmatrix} \vartheta(1/\bar{z}, k).
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• Also, if $\begin{pmatrix} u \\ v \end{pmatrix} = \varphi$, we have $v(z, 0) = u(z, 0)$ and hence $Uu = zu$. 
Weyl-Titchmarsh solutions

- Define, for $|z| \neq 1$,

$$u(z, \cdot) = 2z(U - z)^{-1}\delta_0 \in \ell^2(\mathbb{N}_0) \quad \text{and} \quad v(z, \cdot) = \frac{1}{z}\mathcal{W}u \in \ell^2(\mathbb{N}_0).$$
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- Then $(u, v)^\perp$ satisfies the CMV recursion and

\[
  \begin{pmatrix} u \\ v \end{pmatrix}(k) = \vartheta(k) + m(z) \varphi(k) =: \omega(k)
\]

when $m(z) = 1 + u(z, 0)$; this is the Weyl-Titchmarsh solution.
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• It follows that

$$m(z) = 1 + u(z, 0) = \langle \delta_0, (U + z)(U - z)^{-1}\delta_0 \rangle$$

is a Caratheodory function with representation

$$m(z) = \oint_{\partial \mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta).$$
Caratheodory and Schur functions

• An analytic function $f : \mathbb{D} \to \mathbb{C}$ is called a Caratheodory function, if $f(0) = 1$ and $\text{Re}(f) > 0$.

• An analytic function $g : \mathbb{D} \to \mathbb{C}$ is called a Schur function, if $|g| < 1$.

• If $f$ is Caratheodory and $g$ is Schur, then $(f - 1)/(f + 1)$ is Schur and $(1 + g)/(1 - g)$ is Caratheodory.

• Note also that, by Schwarz’s lemma, $z \mapsto g(z)/z$ is a Schur function, if $g$ is a Schur function and $g(0) = 0$.

• The Möbius transform $z \mapsto S(w, z) = (z + w)/(1 + wz)$ maps $\mathbb{D}$ to itself, if $|w| < 1$.

• If $g$ is a Schur function with $g(0) = -w$ and $|w| < 1$ then $z \mapsto 1/z S(w, g(z))$ is a Schur function.
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- If \( g \) is a Schur function with \( g(0) = -\overline{w} \) and \( |w| < 1 \) then
  \[
  z \mapsto \frac{1}{z} S(w, g(z))
  \]
  is a Schur function.
The Schur algorithm I

- Define

\[ \Phi_{2k}(z) = \frac{1}{z} \frac{\omega_1(z, 2k)}{\omega_2(z, 2k)} \quad \text{and} \quad \Phi_{2k+1}(z) = \frac{\omega_2(z, 2k + 1)}{\omega_1(z, 2k + 1)} \]
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- Initial conditions: \( \Phi_0(z) = \frac{1}{z} \frac{m(z)-1}{m(z)+1} \).
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- Hence \( \Phi_1 \) is a Schur function and knowledge about \( m''(0) \) gives \( \Phi_1(0) = -\overline{\alpha_2} \in \mathbb{D} \).
The Schur algorithm II

- Deleting 2 rows and 2 columns from the matrix $U$ gives a similar problem whose Weyl-Titchmarsh function is a multiple of the original one truncated by the first two elements.

$\Phi_2(0) = -\alpha_3$ and $\Phi_3(0) = -\alpha_4$

Conclusion: The $\Phi_k$ are Schur functions with $\Phi_k(0) = -\alpha_{k+1}$ for $k \in \mathbb{D}$.
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• Assume

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for some $\eta > 0$ and $\gamma > 1$. 
Jost solutions I

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- The Volterra equation [W., Zinchenko 2010]

\[
F(z, k) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \sum_{n=k+1}^{\infty} \begin{pmatrix} 0 & \alpha_n \zeta_n \\ -\alpha_n z^{n-k-1} \zeta_{k+1} & 0 \end{pmatrix} F(z, n), \quad k \in \mathbb{N}_0
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  \[
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  satisfies the CMV recursion.
Jost solutions II

• $|z| < 1$. 

• $\nu(z, \cdot)$ is in $\ell^2(N)$; it is then called the Jost solution for CMV.

• $\nu$ must be a multiple of the Weyl-Titchmarsh solution $\vartheta + m \varphi$.

• Define the Jost function $\psi_0$ by $\nu = \psi_0(\vartheta + m \varphi)$.

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• $m$ has a meromorphic extension to all of $\mathbb{C}$ (denoted by $M$).

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• From asymptotics

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\det(\nu(z, 2k), \tilde{\nu}(z, 2k)) \approx \det \begin{pmatrix} 0 & 2z^{-k} \\ 2z^k & 0 \end{pmatrix} = -4.
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- But

$$\det(\nu(z, 0), \tilde{\nu}(z, 0)) = \psi_0(z)\overline{\psi_0(1/\bar{z})} \det \begin{pmatrix} M(z) - 1 & \overline{M(1/\bar{z})} - 1 \\ M(z) + 1 & \overline{M(1/\bar{z})} + 1 \end{pmatrix}$$

$$= 2\psi_0(z)\overline{\psi_0(1/\bar{z})}(M(z) - \overline{M(1/\bar{z})}).$$
Recall 4 = 2\psi_0(z)\bar{\psi}_0(1/z)(M(z) - \overline{M(1/z)}).
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• An analytic function in the unit disk is (up to an additive constant) determined by its real part on the unit circle.

$$m(z) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} \text{Re}(M(e^{it})) dt.$$
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The inverse resonance problem

Theorem

The location of the resonances (accounting for multiplicities) determine the Verblunsky coefficients uniquely.
Sketch of proof

- The Verblunsky coefficients are given by the Schur functions as
  \[ \alpha_{k+1} = -\Phi_k(0). \]
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- \( \psi_0(z) = \psi_0(0) \prod_{k=1}^{\infty} (1 - z/z_k) \) where the \( z_k \) are the resonances.
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- \( \psi_0(z) = \psi_0(0) \prod_{k=1}^{\infty} (1 - z/z_k) \) where the \( z_k \) are the resonances.
- \( |\psi_0(0)| \) is determined since

  \[ 1 = m(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|\psi_0(e^{it})|^2} \, dt. \]
Zinchenko’s improvements

- Growth order: \( \rho(\psi) = \inf\{\tau > 0 : |\psi(z)| \leq e^{\tau|z|} \text{ eventually} \} \)

- Decay rate: \( \delta(\alpha) = \sup\{\tau : |\alpha_k| \leq e^{-\tau k \log(k)} \text{ eventually} \} \)

- Our condition on \( \alpha \) gives \( \delta(\alpha) = \infty \)

- Theorem [Zinchenko 2014]: \( \delta(\alpha) > 0 \iff \psi_0 \) has entire extension and \( \rho(\psi_0) = \frac{1}{\delta(\alpha)} \)

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Distribution of resonances

- Assume $|\alpha_k| \leq \eta e^{-k\gamma}$ for some $\eta > 0$ and $\gamma > 1$ and
  $\prod_{j=1}^{\infty}(1 - |\alpha_j|) \geq 1/Q$ for some $Q > 1$. 

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- From Jensen’s formula

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N(r) \leq A_1 + \frac{(4 \log r)^p}{2}
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where $p = \gamma/(\gamma - 1)$ and $A_1$ depends only on $Q$, $\eta$, and $\gamma$. 


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- This implies

$$\sum_{|z_n| > R} \frac{1}{|z_n|} = \int_R^\infty \frac{dN(t)}{t} \leq \frac{A_1}{R} + \frac{4^p}{2} \Gamma(p + 1, \log R).$$
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  \]

• Asymptotics of $\Gamma(p + 1, \cdot)$ give
  \[
  \sum_{|z_n| > R} \frac{1}{|z_n|} \leq A_2 \frac{\left(\log R\right)^p}{R}
  \]
  where $A_2$ depends only on $Q$, $\eta$, and $\gamma$. 
Suppose $\alpha$ and $\tilde{\alpha}$ are two sequences of Verblunsky coefficients with super-exponential decay as before. Assume that the resonances in some ball of radius $R$, if there are any, are respectively $\varepsilon$-close. Then there is a constant $A_0$, depending only on $\gamma$, $\eta$, and $Q$, such that

$$|\alpha_n - \tilde{\alpha}_n| \leq A_0 \left( \varepsilon + \frac{\log R}{R} \right)^{1/\log(6eQ^2)}$$

for all $n \in \mathbb{N}$. 
Sketch of proof

• \( |\alpha_k - \tilde{\alpha}_k| \leq |\Phi_{k-1}(0) - \tilde{\Phi}_{k-1}(0)| \leq \|\Phi_{k-1} - \tilde{\Phi}_{k-1}\|_1 \) by the mean value theorem \((\|f\|_p = \int_{-\pi}^{\pi} |f|^p dt/(2\pi))\).
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- $|1 + M(z)| \geq \text{Re}(1 + M(z)) \geq 1$.

- If $|\text{Re } f(0)| = |\text{Im } f(0)|$ then $\text{Re } f$ and $\text{Im } f$ have the same 2-norm.

- We need to estimate $\|\text{Re } M - \text{Re } \tilde{M}\|_2 = \|\psi_0|^{-2} - |	ilde{\psi}_0|^{-2}\|_2$. 
Sketch of proof

• \(|\alpha_k - \tilde{\alpha}_k| \leq |\Phi_{k-1}(0) - \tilde{\Phi}_{k-1}(0)| \leq \|\Phi_{k-1} - \tilde{\Phi}_{k-1}\|_1\) by the mean value theorem (\(\|f\|_p^p = \int_{-\pi}^{\pi} |f|^p dt / (2\pi)\)).

• \(\|\Phi_{k-1} - \tilde{\Phi}_{k-1}\|_1 \leq 6Q^2\|\Phi_{k-2} - \tilde{\Phi}_{k-2}\|_1\) by the Schur algorithm.

• \(\Phi_0(z) - \tilde{\Phi}_0(z) = \frac{2}{z} \frac{M(z) - \tilde{M}(z)}{(1 + M(z))(1 + \tilde{M}(z))}\).

• \(|1 + M(z)| \geq \text{Re}(1 + M(z)) \geq 1\).

• If \(|\text{Re} f(0)| = |\text{Im} f(0)|\) then \(\text{Re} f\) and \(\text{Im} f\) have the same 2-norm.

• We need to estimate \(\|\text{Re} M - \text{Re} \tilde{M}\|_2 = \||\psi_0|^{-2} - |\tilde{\psi}_0|^{-2}\|_2\).

• Hence we need to compare

\[
\psi_0(z) = \psi_0(0) \prod_{n=1}^{\infty} (1 - z/z_n) \quad \text{and} \quad \tilde{\psi}_0(z) = \tilde{\psi}_0(0) \prod_{n=1}^{\infty} (1 - z/\tilde{z}_n).
\]