Stability for the inverse resonance problem for the CMV operator

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I am reporting on joint work with

- Roman Shterenberg (UAB) and
- Maxim Zinchenko (New Mexico).

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- Iantchenko and Korotyaev (2011) have results for Jacobi problems with periodic background).
- Stability of the recovered potential for finite noisy resonance data for the Schrödinger equation (with Marletta, Naboko, Shterenberg; Bledsoe).

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- Thus, choosing the p_n as a basis, multiplication by t is represented by a three-diagonal semi-infinite matrix, i.e., a Jacobi matrix.
- Multiplication by t is a self-adjoint operator.
- Spectral theory for Jacobi matrices allows to investigate the polynomials.

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- The representation of multiplication by the independent variable (using the orthogonal polynomials as a basis) leads only to a Hessenberg matrix.

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- The OLPs and the OPs are in a simple relationship:

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- Starting from $(1, z, 1/z, z^2, 1/z^2, \ldots)$ the OLPs are

$$g_n(z) = f_{n,*}(z) := \overline{f_n(1/\overline{z})}.$$

The 5-term recurrence

- Instead of a Jacobi matrix CMV obtained the matrix

$$U = \begin{pmatrix} -\alpha_1 & \rho_1 & 0 & & & & \\ -\rho_1\alpha_2 & -\overline{\alpha}_1\alpha_2 & -\rho_2\alpha_3 & \rho_2\rho_3 & & \mathbf{0} & \\ \rho_1\rho_2 & \overline{\alpha}_1\rho_2 & -\overline{\alpha}_2\alpha_3 & \overline{\alpha}_2\rho_3 & 0 & & \\ & 0 & -\rho_3\alpha_4 & -\overline{\alpha}_3\alpha_4 & -\rho_4\alpha_5 & \rho_4\rho_5 & \\ & & \rho_3\rho_4 & \overline{\alpha}_3\rho_4 & -\overline{\alpha}_4\alpha_5 & \overline{\alpha}_4\rho_5 & 0 \\ \mathbf{0} & & \ddots & \ddots & \ddots & \ddots & \ddots \end{pmatrix}$$

where $\alpha_n \in \mathbb{D}$, $\rho_n = \sqrt{1 - |\alpha_n|^2}$.

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- We may think of U as a unitary operator in $\ell^2(\mathbb{N}_0)$ or as mapping any sequence of complex numbers to another.

Factorization of U

Every CMV matrix admits the following factorization

$$U = VW = \begin{pmatrix} 1 & & & & & \\ & -\alpha_2 & \rho_2 & & & \\ & \rho_2 & \overline{\alpha}_2 & & & \\ & & & -\alpha_4 & \rho_4 & \\ & & & \rho_4 & \overline{\alpha}_4 & \\ & & & & & \ddots \end{pmatrix} \begin{pmatrix} -\alpha_1 & \rho_1 & & & \\ \rho_1 & \overline{\alpha}_1 & & & \\ & & -\alpha_3 & \rho_3 & \\ & & \rho_3 & \overline{\alpha}_3 & \\ & & & & \ddots \end{pmatrix}.$$

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- Let $\beta_k = \alpha_k$ or $\beta_k = \overline{\alpha}_k$ and $\zeta_k = z$ or $\zeta_k = 1$ depending on whether k is odd or even and
$$T(z,k) = \frac{1}{\rho_k} \begin{pmatrix} \beta_k & \zeta_k \\ 1/\zeta_k & \overline{\beta}_k \end{pmatrix}.$$

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- Then [Gesztesy, Zinchenko 2006]

$$\begin{pmatrix} u \\ v \end{pmatrix}(k) = T(k) \begin{pmatrix} u \\ v \end{pmatrix}(k-1), \quad k \in \mathbb{N}$$

if and only if

$$Wu = zv \quad \text{and} \quad Vv = u + (v(z,0) - u(z,0))\delta_0.$$

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- Consequently $(U - z)u = z(v(z,0) - u(z,0))\delta_0$ (not necessarily in the operator sense).
- In particular $Uu = zu$ if $v(z,0) = u(z,0)$.

Initial value problems

- For $z \neq 0$ introduce the solutions $\vartheta(z, \cdot)$ and $\varphi(z, \cdot)$ of

$$\begin{pmatrix} u \\ v \end{pmatrix}(k) = T(k) \begin{pmatrix} u \\ v \end{pmatrix}(k-1)$$

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- Note that

$$\varphi(z, k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\varphi(1/\overline{z}, k)} \text{ and } \vartheta(z, k) = - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\vartheta(1/\overline{z}, k)}.$$

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- Also, if $\begin{pmatrix} u \\ v \end{pmatrix} = \varphi$, we have $v(z, 0) = u(z, 0)$ and hence $Uu = zu$.

Weyl-Titchmarsh solutions

- Define, for $|z| \neq 1$,

$$u(z,\cdot) = 2z(U-z)^{-1}\delta_0 \in \ell^2(\mathbb{N}_0) \quad \text{and} \quad v(z,\cdot) = \frac{1}{z} W u \in \ell^2(\mathbb{N}_0).$$

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- Then $(u,v)^\perp$ satisfies the CMV recursion and

$$\begin{pmatrix} u \\ v \end{pmatrix}(k) = \vartheta(k) + m(z)\varphi(k) =: \omega(k)$$

when $m(z) = 1 + u(z,0)$; this is the Weyl-Titchmarsh solution.

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- It follows that

$$m(z) = 1 + u(z,0) = \langle \delta_0, (U+z)(U-z)^{-1}\delta_0 \rangle$$

is a Caratheodory function with representation

$$m(z) = \oint_{\partial\mathbb{D}} \frac{\zeta + z}{\zeta - z} d\mu(\zeta).$$

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- If g is a Schur function with $g(0) = -\overline{w}$ and $|w| < 1$ then

$$z \mapsto \frac{1}{z} S(w, g(z))$$

is a Schur function.

The Schur algorithm I

- Define

$$\Phi_{2k}(z) = \frac{1}{z}\frac{\omega_1(z,2k)}{\omega_2(z,2k)} \quad \text{and} \quad \Phi_{2k+1}(z) = \frac{\omega_2(z,2k+1)}{\omega_1(z,2k+1)}$$

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- Using the Neumann series

$$m(z) = \langle \delta_0, (U+z)(U-z)^{-1}\delta_0\rangle = 1+2\sum_{n=1}^{\infty}\langle \delta_0, U^{-n}\delta_0\rangle = 1-2\overline{\alpha_1}z+\ldots$$

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- $m(0)=1$ implies Φ_0 is a Schur function, $m'(0) = -2\overline{\alpha_1}$ implies $\Phi_0(0) = -\overline{\alpha_1} \in \mathbb{D}$.

- Hence Φ_1 is a Schur function and knowledge about $m''(0)$ gives $\Phi_1(0) = -\overline{\alpha_2} \in \mathbb{D}$.

The Schur algorithm II

- Deleting 2 rows and 2 columns from the matrix U gives a similar problem whose Weyl-Titchmarsh function is a multiple of the original one truncated by the first two elements.

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- Conclusion: The Φ_k are Schur functions with $\Phi_k(0) = -\overline{\alpha_{k+1}} \in \mathbb{D}$

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- Then
$$\nu(z,k) = 2z^{\lceil k/2 \rceil} \Big(\prod_{j=k+1}^{\infty} \rho_j^{-1} \Big) \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{k+1} F(z,k)$$
satisfies the CMV recursion.

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- ψ_0 cannot have zeros in $\mathbb{D}$. Those outside are called resonances.

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- For the sequence $\tilde\nu(z,k) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overline{\nu(1/\overline{z},k)}$ is also a solution of the CMV recursion (and square integrable for $|z|>1$).

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- But

$$\begin{aligned}\det(\nu(z,0), \tilde{\nu}(z,0)) &= \psi_0(z)\overline{\psi_0(1/\overline{z})} \det \begin{pmatrix} M(z)-1 & \overline{M(1/\overline{z})}-1 \\ M(z)+1 & \overline{M(1/\overline{z})}+1 \end{pmatrix} \\ &= 2\psi_0(z)\overline{\psi_0(1/\overline{z})}(M(z) - \overline{M(1/\overline{z})}).\end{aligned}$$

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The inverse resonance problem

Theorem

The location of the resonances (accounting for multiplicities) determine the Verblunsky coefficients uniquely.

Sketch of proof

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- $\psi_0(z) = \psi_0(0) \prod_{k=1}^{\infty} (1 - z/z_k)$ where the z_k are the resonances.
- $|\psi_0(0)|$ is determined since

$$1 = m(0) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{1}{|\psi_0(\mathrm{e}^{it})|^2} dt.$$

$\square$

Zinchenko's improvements

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- Theorem [Zinchenko 2014]:
  If ψ_0 is entire, of finite growth order and without zeros in $\overline{\mathbb{D}}$, then it is the Jost function of a unique CMV operator.

Distribution of resonances

- Assume $|\alpha_k| \leq \eta \mathrm{e}^{-k^\gamma}$ for some $\eta > 0$ and $\gamma > 1$ and $\prod_{j=1}^{\infty}(1 - |\alpha_j|) \geq 1/Q$ for some $Q > 1$.

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- Asymptotics of $\Gamma(p+1, \cdot)$ give

$$\sum_{|z_n|>R} \frac{1}{|z_n|} \leq A_2 \frac{(\log R)^p}{R}$$

where A_2 depends only on Q, η, and γ.

Stability

Suppose α and $\breve{\alpha}$ are two sequences of Verblunsky coefficients with super-exponential decay as before. Assume that the resonances in some ball of radius R, if there are any, are respectively ε-close. Then there is a constant A_0, depending only on γ, η, and Q, such that

$$|\alpha_n - \breve{\alpha}_n| \leq A_0 \left(\varepsilon + \frac{(\log R)^p}{R} \right)^{1/\log(6\mathrm{e}Q^2)}$$

for all $n \in \mathbb{N}$.

Sketch of proof

- $|\alpha_k - \breve{\alpha}_k| \leq |\Phi_{k-1}(0) - \breve{\Phi}_{k-1}(0)| \leq \|\Phi_{k-1} - \breve{\Phi}_{k-1}\|_1$ by the mean value theorem ($\|f\|_p^p = \int_{-\pi}^{\pi} |f|^p dt/(2\pi)$).

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- We need to estimate $\|\operatorname{Re} M - \operatorname{Re} \breve{M}\|_2 = \||\psi_0|^{-2} - |\breve{\psi}_0|^{-2}\|_2$.

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- Hence we need to compare

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