Solving the Camassa-Holm Equation by Inverse Scattering

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I am reporting on joint work with:

- Christer Bennewitz (Lund)
- Malcolm Brown (Cardiff)

The Camassa-Holm equation

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- where ψ is deviation from the free surface
- and κ is a dispersion coefficient (may be scaled to 0 or 1).
- Introducing $w = \psi_{xx} - \psi + \kappa$ we may write more concisely

$$w_t + 2\psi_x w + \psi w_x = 0.$$

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- CH is the compatibility condition for the linear equations

$$-u_{xx} + \frac{1}{4}u = \lambda w u$$

and

$$u_t - \left(\frac{1}{2\lambda} - \psi\right) u_x - \frac{1}{2}\psi_x u = 0$$

(recall $w = \psi_{xx} - \psi + \kappa$).

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w(\cdot, 0) & \xrightarrow{\;\textit{scattering}\;} & \text{scattering data} \\
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- The reflection coefficient and norming constants evolve by multiplying with the exponentials $\exp(\pm ikt/\lambda)$.

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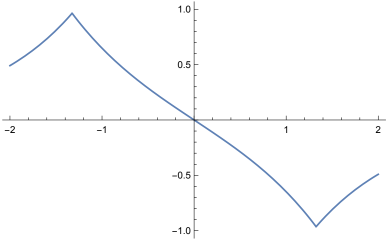
- The condition $w > 0$ removes the interesting cases (wave breaking, wave collisions, peakons).



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- Issue 3: Perturbations get multiplied by λ.

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- $\overline{E} = E^{**}$ and $E^* = \overline{E}^*$.
- E is called symmetric if $E \subset E^*$ and self-adjoint if $E = E^*$.

Application to Sturm-Liouville

- $T_1 = \{(u, f) \in \mathcal{H}_1 \oplus \mathcal{H}_1 : -u'' + qu = wf \text{ a.e.}\}$ is a closed linear self-adjoint relation.

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- If $w = 0$, then $\mathcal{H}_\infty = \mathcal{H}_1$, $\mathcal{H} = \{0\}$.
- $T = T_1 \cap \mathcal{H} \oplus \mathcal{H}$ is a densely defined operator on $\mathcal{H}$.

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- Transmission and reflection coefficients:

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$$\|\hat{u}\|^2_{\mathcal{J}} = \int_{q_0}^{\infty} (|\hat{u}_+(s)|^2 + |\hat{u}_-(s)|^2) \frac{|\mathfrak{T}(s)|}{4\pi st} dt + \sum_n \frac{|\hat{u}_+(k_n)|^2}{\|f_+(\cdot, k_n)\|^2}.$$

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- $u \in D_T$ implies $(\mathcal{J}(Tu))(\lambda) = \lambda(\mathcal{J} u)(\lambda)$, i.e., the Jost transform diagonalizes T.

Main Theorem

Theorem (BBW (JDE 2012))

Suppose two operators T and $\check{T}$ are given and that their scattering matrices (and hence eigenvalues) and norming constants are identical.

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Conversely, if the coefficients of T and $\check{T}$ are related in this way then scattering data of T and $\check{T}$ coincide.

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  - r' is locally absolutely continuous;
  - $s(x) - x$ and $r(x) - 1$ tend to 0 as x tends to $\pm\infty$.

Idea of proof — Outline I

- q_0, $|\mathfrak{T}|$, eigenvalues and norming constants determine $L^2_{\mathcal{J}} = L^2_{\breve{\mathcal{J}}}$.

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- We need to show that $\mathcal{U}$ defined here is a Liouville transform.

$$
\begin{array}{ccc}
\mathcal{H} & \xrightarrow{\quad \mathcal{U} \quad} & \breve{\mathcal{H}} \\
 & \searrow^{\mathcal{J}} \qquad {}^{\breve{\mathcal{J}}^*}\nearrow & \\
 & L^2_{\mathcal{J}} &
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- Choose $s(a) = \breve{a}$ (and $s(b) = \breve{b}$) (high-energy asymptotics).
- $r(x) = u(x)/\breve{u}(s(x))$
- Show that $\mathcal{L}_{r,s} \in \mathcal{S}$ and that $\mathcal{U}^{-1} = \mathcal{L}_{r,s}$.

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- w **is uniquely determined from the scattering data.**

Thank you for your attention!