ON HILL'S EQUATION WITH A SINGULAR
COMPLEX-VALUED POTENTIAL

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[Received 22 October 1996]

1. Introduction

The differential equation \( y'' + qy = Ey \) is called Hill's equation if \( q \) is a real-valued, periodic, locally integrable function of a real variable. It is very well known that the spectrum of the operator associated with this equation consists of a countably infinite number of compact intervals of the real line. These intervals are called (spectral) bands and are generally separated by open intervals called gaps. In some instances, however, all but a finite number of gaps are empty resulting in a spectrum consisting of a finite number of compact bands and one closed interval extending to negative infinity. In this case one calls \( q \) a finite-band or finite-gap potential. The relationship of finite-band potentials with the hierarchy of Korteweg-de Vries equations has been the reason for a heightened interest in them during the past twenty years. For recent accounts see, for instance, the monographs by Dickey [8] and Belokolos et al. [3].

The first non-trivial example of a finite-band potential was given by Ince in 1940 [18]: consider the Lamé potentials

\[ q(x) = -g(g + 1) \rho(x) \omega(x, \omega') \]

where \( g \in \mathbb{N} \) and \( \rho(x, \omega, \omega') \) denotes Weierstrass's elliptic function with fundamental half-periods \( \omega \) and \( \omega' \). If \( \omega \) is real, \( \omega' \) is pure imaginary, and \( c = \omega' \), then \( q \) is real-valued, periodic and continuous. Ince's result, when translated to the present language, shows that \( q \) then is a finite-band potential with \( g \) non-empty gaps between the bands. This, however, raises immediately the question of what happens when \( c \neq \omega' \) or when the half-periods are more general. In these cases one encounters potentials which are complex-valued (but continuous) as well as potentials which have singularities of the form \( -g(g + 1)x^2 \) (and are real- or complex-valued).

Before giving an overview of the content of the paper I want to give a brief account of the history of the subject. Let \( q \) be a real, locally integrable, periodic function with period \( p \), the differential expression \( d^2dx^2 + q \), and \( H \) the self-adjoint operator in \( L^2(R) \) associated with \( L \). Then consider the following four statements.

(A) There are precisely \( 2g + 1 \) real numbers \( E_0, E_{-1}, \ldots, E_{-g} \) such that the spectrum of \( H \) is given by

\[ (-\infty, E_{-g}] \cup [E_{-g-1}, E_{-g-2}] \cup \cdots \cup [E_1, E_0]. \]

This paper is based upon work supported by the US National Science Foundation under Grant No. DMS-9401816.

1991 Mathematics Subject Classification: 34L40, 14H60.

(B) There are precisely $2g + 1$ real numbers $E_{g} < \ldots < E_{0}$ such that the differential equation $y'' + qy = Ey$ fails to have two linearly independent Floquet solutions if and only if $E \in \{E_{0}, \ldots , E_{2g}\}$.

(C) The Dirichlet boundary value problem on the interval $[x_{0}, x_{g} + p]$ has precisely $g$ eigenvalues which vary with $x_{0}$. These are located respectively in the closures of the $g$ non-empty gaps $(E_{2j}, E_{2j+1})$ for $j = 1, \ldots , g$.

(D) There exists a monic ordinary differential expression $P$ of order $2g + 1$ but none of smaller odd order such that $[P, L] = 0$, that is, $q$ is a solution of some equation in the stationary KdV hierarchy (which is also called the Novikov hierarchy). Moreover, $P$ and $L$ satisfy the algebraic relation $P^{g} = (L - E_{g}) \ldots (L - E_{0})$ where $E_{2g} < \ldots < E_{0}$.

The equivalence of (A) and (B) follows from work of Hamel [16] (see § 14) in 1913. In 1909 Birkhoff [4] showed that Dirichlet eigenvalues are situated between adjacent periodic or semi-periodic eigenvalues which implies the equivalence of (A) and (C). The fact that two commuting ordinary differential expressions satisfy an algebraic relation as stated in (D) was first shown in 1923 by Burchnall and Chaundy [6] even if $q$ is complex-valued. Consequently, the polynomial $P^{g} = (L - E_{g}) \ldots (L - E_{0})$ is called a Burchnall–Chaundy polynomial. That (A) and (D) are equivalent is a discovery which was made only about twenty years ago by Novikov [25] and Dubrovin [9]. Around the same time finite-band potentials were also studied by its and Matveev [19], Lax [21], McKean and van Moerbeke [23] and others. Since the equations of the KdV hierarchy can be represented by Lax equations $q_{t} = [P, L]$, where $P$ is a suitable ordinary differential expression (see [20]), statement (D) gives the relation of finite-band potentials with the KdV hierarchy. If $q$ is a stationary solution of some equation in the KdV hierarchy, it is called an algebro-geometric potential (associated with the KdV hierarchy) in view of its relation to the appropriate Burchnall–Chaundy polynomial.

Now suppose that $q$ is complex-valued. The spectrum of $H$ is then not necessarily a subset of the real line any more but still a countable set of regular analytic arcs. Also neither the points $E$ where less than two linearly independent Floquet solutions of $y'' + qy = Ey$ exist nor the Dirichlet eigenvalues need to be real anymore. Finally, algebraic multiplicities of boundary value problems are no more a priori bounded. (An essential feature of real-valued locally integrable potentials is that geometric and algebraic multiplicities of eigenvalues of Dirichlet and (semi-)periodic boundary value problems coincide and hence that algebraic multiplicities of Dirichlet eigenvalues are 1 and that algebraic multiplicities of (semi-)periodic eigenvalues are at most 2.) Complex-valued potentials in the present context have been considered by Birnir [5] under the restriction that algebraic and geometric multiplicities still coincide. Gesztesy and I considered the general problem in [15] and proved that there are only finitely many spectral bands, that only finitely many Dirichlet eigenvalues vary with $x_{0}$, and that some monic differential expression of odd order commutes with $L$ provided that the case where $y'' + qy = Ey$ has less than two linearly independent Floquet solutions happens only at finitely many points $E$. This result and a theorem of Picard are the essential ingredients of the main theorem in [15]. If $q$ is an elliptic function, this

\footnote{However, see also papers by Liapounoff [22], who treated the case $y'' = \lambda p$ with a periodic function $p$ in 1899, and Haage [17], who corrects a mistake in Hamel’s paper.}
theorem states the equivalence of an appropriately generalized statement (D), that is, statement (D') below, with yet another seemingly unrelated property of \( q \), namely the following statement (E).

(E) For every \( E \in \mathbb{C} \) every solution of the differential equation \( y'' + qy = Ey \) is a meromorphic function of the independent variable \( x \).

A potential satisfying (E) is called a Picard potential.

While [15] shows that an appropriate generalization of statement (B) implies appropriate generalizations of statements (A), (C), and (D) when \( q \) is complex-valued, it is not true in general that the finite-band property implies any of the others. In fact, it is shown in § 8 that \( q = e^{2\pi i} \) has spectrum \((-\infty, 0)\) and no gaps even though \( q \) is not the stationary solution of any equation in the KdV hierarchy. In other words, the equivalence of the statements (A) and (D) is not a basic principle and breaks down when \( q \) becomes complex or singular; even so, its discovery was extraordinarily fruitful.

The goal of the present paper is now to explore the relations between generalizations of the statements (A)–(D) for the case of periodic, complex-valued potentials with inverse square singularities. Combining the most important results of the paper we arrive at the following theorem.

**Theorem.** Let \( \Sigma \) be an isolated periodic subset of \( \mathbb{R} \) and \( q \) a complex-valued function on \( \mathbb{R} - \Sigma \) with the following properties:

1. \( q \) is locally integrable in \( \mathbb{R} - \Sigma \);
2. for each \( \xi \in \Sigma \) there exist a positive integer \( g(\xi) \) and a positive real number \( b(\xi) \) such that \( q(\cdot) - g(\xi)(\cdot)g(\xi) + 1 \) is a bounded analytic continuation to the set \{ \( x \mid |x| < b(\xi), \Re(x) = \xi \Rightarrow \Im(x) > 0 \} \);
3. \( q \) is periodic with period \( p \).

Let \( L = \frac{d^2}{dx^2} + q \). Then the following statements are equivalent.

(B') The differential equation \( Ly = Ey \) fails to have two linearly independent regular Floquet solutions (cf. Definition 6.1) if and only if \( E \) is any of the (not necessarily distinct) points \( E_0, \ldots, E_g \). The set \{ \( E_0, \ldots, E_g \) \} contains all those points \( E \) where \( Ly = Ey \) has less than two linearly independent Floquet solutions (and possibly others).

(C') The Dirichlet boundary value problem on the interval \([x_0, x_0 + p]\) has precisely \( g \) eigenvalues (counting multiplicities) which vary with \( x_0 \).

(D') There exists a monic ordinary differential expression \( P \) of order \( 2g + 1 \), but none of smaller odd order, such that \( [P, L] = 0 \), that is, \( q \) is a stationary solution of some equation in the KdV hierarchy. The expressions \( P \) and \( L \) satisfy the algebraic relation \( P^2 = (L - E_0) \cdots (L - E_g) \).

Also, if the hypotheses on \( q \) and any (and hence all) of the statements (B')–(D') are satisfied then

1. \( q \in C^\infty(\mathbb{R} - \Sigma) \);
2. the conditional stability set of \( L \) (cf. § 4) consists of finitely many regular analytic arcs;
3. \( q \) is a constant or the Neumann boundary value problem on the interval \([x_0, x_0 + p]\) has precisely \( g + 1 \) eigenvalues (counting multiplicities) which vary with \( x_0 \).

To emphasize the difference between the statements (B) and (B') observe firstly that points \( E \) at which less than two linearly independent Floquet solutions of \( y'' + q y = Ey \) exist have to be included in \( \{ E_0, \ldots, E_2 \} \), perhaps with higher multiplicities. It might happen, however, that the equation \( y'' + q y = Ey \) has two linearly independent Floquet solutions but that nevertheless the point \( E \) has to be included in \( \{ E_0, \ldots, E_2 \} \). This is the case when two suitably normalized linearly independent Floquet solutions of \( y'' + q y = \lambda y \) converge to linearly dependent Floquet solutions of \( y'' + q y = Ey \) when \( \lambda \) approaches \( E \) even though two independent Floquet solutions exist for \( E \) (which never happens when \( q \) is real and locally integrable). The points where less than two regular Floquet solutions exist can be found as the zeros of the Wronskian of the suitably normalized Floquet solutions which is a function of the spectral parameter. This approach was used in [12, 13, 14] to determine which elliptic functions satisfy stationary equations of the KdV hierarchy and to find the associated algebraic curve.

It is an essential feature of linear differential equations with locally integrable coefficients that initial value problems have a unique solution on the entire real line. This property is indispensable for the analysis we will perform. When \( q \) may be analytically continued into a cut vicinity of a singular point, that is, a point where \( q \) has an inverse square singularity, solutions of the differential equation may be continued as well thus allowing for unique solutions of initial value problems even in the presence of such singularities. Sections 2 and 3 show how an almost standard approach via integral equations (see, for example, Yosida [28]) can be used to obtain various properties of solutions of initial value problems in the cases of one and many singular points, respectively. We will be interested particularly in the behaviour of the solutions as the initial point and the spectral parameter vary.

The theory of periodic differential equations is called Floquet theory. A standard reference is Eastham [10]. In § 4 we will extend Floquet theory to Hill’s equation with potentials of inverse square singularities, define a generalized conditional stability set and show that it has band structure, that is, that it consists of (possibly countably many) regular analytic arcs. (In the absence of any singularities, the conditional stability set coincides with the spectrum of the maximal operator associated with \( L \) in \( L^2(\mathbb{R}) \).) We will characterize the endpoints of the bands as those eigenvalues of the periodic or semi-periodic boundary value problem which have an odd algebraic multiplicity. We also obtain the asymptotic behaviour of Floquet eigenvalues as \( E \) tends to infinity.

Sturm–Liouville eigenvalue problems on \( [x_0, x_0 + p] \) are studied in § 5. The algebraic multiplicities of their eigenvalues will be decomposed into movable (that is, depending on \( x_0 \) and immovable (that is, independent of \( x_0 \)) parts. We then study the relationship between multiplicities of eigenvalues of different Sturm–Liouville problems.

In § 6 we prove that the set of all regular Floquet solutions forms a line bundle on some open possibly singular Riemann surface.

None of the material in §§ 3–6 is restricted to finite-band potentials or stationary solutions of equations in the KdV hierarchy. These potentials are the subject of §§ 7 and 8. In the former, we repeat the well-known treatment of the
KdV hierarchy and commuting differential expressions in order to facilitate easy reference. In the latter, we prove the above theorem and discuss the example \( q = e^{2i\epsilon} \) which shows that finite-band potentials are not necessarily algebro-geometric.

The role played by \( \cos(k(x-x_0)) \) and \( \sin(k(x-x_0))/k \) in the case of locally integrable potentials is played by certain combinations of Bessel functions when inverse square singularities are present. Some important properties of Bessel functions are listed in the appendix, again for the convenience of the reader.

2. Treatment of a singular point

The substitutions \( z = kx, \ y(x) = x^{1/2}w(z) \), and \( 4b^2 = 1 - 4\hat{a} \) transform the differential equation

\[
y'' + \frac{\hat{a}}{x} y = -k^2 y
\]

into Bessel’s equation

\[
z^2 w'' + zw' + (z^2 - \rho^2)w = 0.
\]

Hence a fundamental system of solutions of \( y'' + \hat{a}y/x^2 = -k^2 y \) is given by Bessel functions \( J_r \) and \( Y_r \) by

\[
\hat{c}_0(-k^2, x_0, x) = \frac{1}{2\pi} \left( \frac{x}{x_0} \right)^{1/2} (J_r(kx)Y_r(kx_0) - J_r(kx_0)Y_r(kx))
\]

\[
+ \frac{\hat{a}}{2} k(x_0)^{1/2} (J_r(kx)Y_r'(kx_0) - J_r'(kx_0)Y_r(kx))
\]

and

\[
\hat{\delta}_0(-k^2, x_0, x) = \frac{1}{2\pi} (x_0)^{1/2} (J_r(kx)Y_r'(kx_0) - J_r'(kx_0)Y_r(kx)),
\]

where \( x_0 \) is any non-zero complex number. Note that \( \hat{c}_0(E, x_0, \cdot) \) and \( \hat{\delta}_0(E, x_0, \cdot) \) are those solutions of \( y'' + \hat{a}y/x^2 = E \) which satisfy initial conditions \( y(x_0) = 1, \ y'(x_0) = 0 \) and \( y'(x_0) = 1, \ y(x_0) = 0 \), respectively. In particular, they coincide with \( \cos(k(x-x_0)) \) and \( \sin(k(x-x_0))/k \) when \( \hat{a} = 0 \). In \( \hat{\Omega}_0 = \{ x \in \mathbb{C} \colon \text{Re}(x) > 0 \implies \text{Im}(x) > 0 \} \), the functions \( \hat{c}_0(E, x_0, \cdot) \) and \( \hat{\delta}_0(E, x_0, \cdot) \) are single-valued and analytic. The function

\[
\hat{\Phi}_0(E, y_0, y_0', x_0, \cdot) = y_0 \hat{c}_0(E, x_0, \cdot) + y_0' \hat{\delta}_0(E, x_0, \cdot)
\]

is therefore the unique solution of the initial value problem \( y'' + \hat{a}y/x^2 = E y \), \( y(x_0) = y_0, \ y'(x_0) = y_0' \) in \( \hat{\Omega}_0 \).

For any choice of \( x \) and \( x_0 \) in \( \hat{\Omega}_0 \) the functions \( \hat{c}_0(\cdot, x_0, x) \) and \( \hat{\delta}_0(\cdot, x_0, x) \), as well as their derivatives with respect to \( x \), that is, \( \hat{c}_0'(\cdot, x_0, x) \) and \( \hat{\delta}_0'(\cdot, x_0, x) \), are entire functions of order \( \frac{1}{2} \).

Upon interchanging the roles played by \( x_0 \) and \( x \) one obtains the following relationships:

\[
\hat{c}_0(E, x_0, x) = \hat{\delta}_0(E, x, x_0), \quad \hat{\delta}_0(E, x_0, x) = -\hat{c}_0(E, x, x_0), \quad \hat{c}_0'(E, x_0, x) = -\hat{\delta}_0'(E, x, x_0).
\]
Next we consider perturbations of the potential $\alpha x^2$ which are locally integrable and analytic near zero. More precisely, for some real number $b > 0$ let $V = \{ x \in \mathbb{C} : |x| < b, \text{Re}(x) = 0 \Rightarrow \text{Im}(x) > 0 \}$ and $\bar{\Omega} = V \cup (\mathbb{R} - \{0\})$. Then suppose that $\hat{q}_1 : \bar{\Omega} \to \mathbb{C}$ satisfies the following three conditions:

1. $\hat{q}_1 \in L^2_{in}(\mathbb{R} - \{0\})$.
2. $\hat{q}_1$ is analytic in $V$.
3. $|\hat{q}_1|$ is bounded in $V$.

**Theorem 2.1.** If $\hat{q}_1$ satisfies the above hypotheses then there exists a function $\hat{\phi} : \mathbb{C} \times \bar{\Omega} \to \mathbb{C}$ with the following properties.

(a) The function $\hat{\phi}(E, y_0, y_0', x_0, \cdot)$ is the unique solution of the integral equation

$$y(x) = \hat{\phi}_0(E, y_0, y_0', x_0, x) - \int_{x_0}^{x} s_0(E, x', x) \hat{q}_1(x') y(x') \, dx',$$

where $\gamma_s$ is a piecewise continuously differentiable, simple path in $\bar{\Omega}$ connecting $x_0$ and $x$. In particular, $\hat{\phi}(E, y_0, y_0', x_0, x) = y_0$.

(b) The function $\hat{\phi}(E, y_0, y_0', x_0, \cdot)$ is analytic in $V$ and continuously differentiable in $\mathbb{R} - \{0\}$. In fact, $\hat{\phi}(E, y_0, y_0', x_0, x)$ (where the prime denotes differentiation with respect to the last argument) is given by

$$\hat{\phi}'(E, y_0, y_0', x_0, x) = \hat{\phi}_0(E, y_0, y_0', x_0, x) - \int_{x_0}^{x} s_0(E, x', x) \hat{q}_1(x') \hat{\phi}(E, y_0, y_0', x_0, x') \, dx'$$

and therefore $\hat{\phi}'(E, y_0, y_0', x_0, x_0) = y_0$. The function $\hat{\phi}''(E, y_0, y_0', x_0, \cdot)$ is locally absolutely continuous in $\mathbb{R} - \{0\}$.

(c) For almost all $x$ in $\mathbb{R} - \{0\}$, the function $\hat{\phi}''(E, y_0, y_0', x_0, x)$ is given by

$$\hat{\phi}''(E, y_0, y_0', x_0, x) = (E - \alpha x^2 - \hat{q}_1(x)) \hat{\phi}(E, y_0, y_0', x_0, x),$$

that is, $\hat{\phi}(E, y_0, y_0', x_0, \cdot)$ is the unique solution of the initial value problem

$$y'' + (\alpha x^2 + \hat{q}_1(x)) y = E y, \quad y(x_0) = y_0, \quad y'(x_0) = y_0'.$$

(d) The functions $\hat{\phi}(E, y_0, y_0', \cdot, x)$ and $\hat{\phi}'(E, y_0, y_0', \cdot, \cdot)$ are locally absolutely continuous in $\mathbb{R} - \{0\}$. Moreover,

$$\frac{\partial \hat{\phi}(E, y_0, y_0', x_0, x)}{\partial x_0} = \hat{\phi}(E, y_0, y_0', x_0, x) = \hat{\phi}(E, -y_0', \alpha x_0^2 + \hat{q}_1(x_0) - E y_0, x_0)$$

and

$$\frac{\partial \hat{\phi}'}{\partial x_0}(E, y_0, y_0', x_0, x_0) = \hat{\phi}'(E, y_0, y_0', x_0, x_0) = \hat{\phi}'(E, -y_0', \alpha x_0^2 + \hat{q}_1(x_0) - E y_0, x_0, x_0).$$

(e) There exist positive constants $C$ and $\Lambda$ depending on $x_0$ and $x_0$, but not on $E$, $y_0$, and $y_0'$, such that

$$|\hat{\phi}(E, y_0, y_0', x_0, x)| \leq C e^{Re(\sqrt{E})|x-x_0|} |y_0| |\sqrt{E}| + |y_0'|$$

and

$$|\hat{\phi}'(E, y_0, y_0', x_0, x)| \leq C e^{Re(\sqrt{E})|x-x_0|} |y_0| |\sqrt{E}| + |y_0'|.$$
when \( x_0, x \in \mathbb{R} - \{0\} \) and \( |E| \leq \Delta \). In particular, the functions \( \hat{\Phi}(\cdot, y_0, y', x_0, x) \) and \( \hat{\phi}(\cdot, y_0, y', x_0, x) \) are entire and have order \( \frac{1}{2} \).

**Proof.** The following is an outline of the proof. More details can be found in [27].

Let \( \gamma : [0, 1] \to \Omega \) be a piecewise continuously differentiable simple path with initial point \( \gamma(0) = x_0 \) and such that \( x \in \gamma([0, 1]) \). If \( x \in \gamma([0, 1]) \), let \( s = \gamma^{-1}(x) \) and define \( \gamma_t : [0, 1] \to \Omega \) by \( \gamma_t(t) = \gamma(at) \). For \( n \in \mathbb{N} \) now define recursively

\[
\hat{\psi}_n(E, y_0, y', x_0, x) = -\int_{\gamma_t} \hat{\psi}_0(E, x', y, \dot{y}) \hat{\phi}_{n-1}(E, y_0, y', x_0, x') \, dx'.
\]

Since \( \hat{\psi}_0(E, \cdot, x, \dot{y}) \) and \( \hat{\phi}_n(E, \cdot, y_0, y', x_0, x) \) are analytic in \( V \) and since \( V \) is simply connected, it may be proved by induction that \( \hat{\phi}_n(E, \cdot, y_0, y', x_0, x) \) is independent of the path chosen to connect \( x_0 \) and \( x \), and analytic in \( V \) as a function of \( x \). Also, by standard arguments, we see that \( \sum_{n=0}^{\infty} \hat{\phi}_n \) converges absolutely and uniformly on compact subsets of \( C^1 \times \Omega' \). Therefore \( \hat{\phi} = \sum_{n=0}^{\infty} \hat{\phi}_n \) is a well defined (independent of the path chosen) function on \( C^1 \times \Omega' \). Uniform convergence of the series with respect to the last argument proves then that \( \hat{\phi}(E, \cdot, y_0, y', x_0, x) \) satisfies the integral equation (4) and hence Part (a) of the theorem.

The function \( \hat{\phi}(E, \cdot, y_0, y', x_0, x) \) is analytic in \( V \) as the uniform limit of analytic functions. Equation (4) shows that \( \hat{\phi}(E, y_0, y', \cdot, x_0) \) is differentiable in \( \mathbb{R} - \{0\} \) and that its derivative is given as stated in (b). This expression, in turn, is locally absolutely continuous in \( \mathbb{R} - \{0\} \), which proves Parts (b) and (c) of the theorem.

The function \( \hat{\phi}(E, y_0, y', x_0, \cdot) \) is continuously differentiable with the derivative bounded on compact subsets of \( C^1 \times \Omega' \). Hence it is locally absolutely continuous in \( \mathbb{R} - \{0\} \), and this is so uniformly with respect to \( x \). Therefore the integral equation (4) may be used to prove the absolute continuity of each function \( \hat{\phi}(E, y_0, y', \cdot, x) \). In fact, when \( \epsilon, \eta > 0 \), we obtain, for any \( \epsilon > 0 \),

\[
\sum_{k=0}^{N} |\hat{\phi}_k(E, y_0, y', t_{2k+1}, x) - \hat{\phi}_k(E, y_0, y', t_{2k+1}, x)| \leq \epsilon M^{n-1} \frac{M^{n-1}}{(n-1)!}
\]

provided \( t_0 \leq t_2 \leq \ldots \leq t_{2N+1} \) is a partition of \( [\alpha, \beta] \in \mathbb{R} - \{0\} \) and \( \sum_{k=0}^{N} (t_{2k+1} - t_{2k}) \) is suitably small. This estimate now proves the locally absolute continuity of \( \hat{\phi}(E, y_0, y', \cdot, x) \) in \( \mathbb{R} - \{0\} \). Differentiating the integral equation (4) with respect to \( x_0 \) and using the equations (1)–(3) and uniqueness of solutions of (4), gives the validity of equation (5). Equation (6) then follows immediately from Part (b). This concludes the proof of Part (d) of the theorem.

Using the notation of the appendix we obtain

\[
\begin{align*}
\hat{\nu}_0(-k^2, x_0, x) &= \text{cos}(k(x - x_0)) f_0(k, x, x_0) + \text{sin}(k(x - x_0)) g_0(k, x, x_0) \frac{1}{2}\text{sin}(k(x - x_0)), \\
\hat{\nu}_0(-k^2, x_0, x) &= \frac{k}{k} f_0(k, x, x_0) - \text{cos}(k(x - x_0)) g_0(k, x, x_0),
\end{align*}
\]

and

\[
\begin{align*}
\hat{\nu}_0(-k^2, x_0, x) &= -k \text{sin}(k(x - x_0)) \frac{1}{k} (x, x_0) + k \text{cos}(k(x - x_0)) B(x, x_0)
\end{align*}
\]
where

\[ A(kx, kx_0) = f_1(kx, kx_0) - (2kx_0f_2(kx, kx_0) + 2kxg_2(kx_0, kx) - f_1(kx, kx_0)) / (4k^2 xx_0) \]

and

\[ B(kx, kx_0) = g_1(kx, kx_0) + (2kx_0f_2(kx, kx_0) - 2kxg_2(kx_0, kx) + g_1(kx, kx_0)) / (4k^2 xx_0). \]

Let \( k = \sqrt{-E} \) have its argument in \((-\pi, 0]\), that is, \( k = \kappa - i\eta \) where \( \kappa \in \mathbb{R} \) and \( \eta \geq 0 \) and let \( r = 1/|k| \). For \( xx_0 < 0 \) and \( x_1 > 0 \) let \( \gamma : [0, 1] \to \mathbb{R} \) be defined by

\[ \gamma(t) = \begin{cases} 
  x_0 - 3t(r + xx_0) & \text{if } 0 \leq t \leq \frac{1}{3}, \\
  -r \exp(-ir(3t - 1)) & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3}, \\
  r + (3t - 2)(x_1 - r) & \text{if } \frac{2}{3} \leq t \leq 1.
\end{cases} \]

Then \( \arg(\gamma(t)) \in (-\pi, \pi] \) and \( |\gamma(t)| \geq 1 \) for all \( t \in [0, 1] \). From the boundedness of the absolute values of the functions \( f_1(kx, kx') \), \( g_1(kx, kx') \), \( f_2(kx, kx') \) and \( g_2(kx, kx') \) we obtain the existence of a constant \( C_1 \) such that for \( x, x' \in \gamma([0, 1]) \) the estimates

\[ |\hat{\gamma}_2(-k^2, x, x')|, |\hat{\gamma}_2(-k^2, x, x')|, |\hat{\gamma}_2(-k^2, x, x')| \leq C_1 |k| |\exp(|\Re x| - \Re (x'))| \]

hold. Now define

\[ f(k, x) = \exp(-i\eta(x - xx_0)) \hat{\phi}(-k^2, y_0, y_0, x_0, x). \]

Then, since \( |\hat{\gamma}_1| \) is bounded in \( V \), we obtain \( \int_0^1 |\hat{\gamma}_1(\gamma(t))| |\gamma'(x')| \, dx' = Q \) for some \( Q > 0 \). This implies that

\[ |f(k, x)| \leq C_1 (|y_0| |k| + |\gamma_0|) + C_1 F(k) Q, \]

where

\[ F(k) = \max\{|f(k, x')| : \ x' \in \gamma([0, 1])\}. \]

Since the right-hand side of the last inequality is independent of \( x \), it is, in fact, also a bound for \( F(k) \), that is,

\[ F(k) \leq 2C_1 (|y_0| |k| + |\gamma_0|) \]

provided \( |k| \) is so large that \( C_1 |\hat{\phi}||k|^{-1} \leq \frac{1}{2} \). Inserting this estimate into the integral equation (4) and its \( x \)-derivative gives the desired estimates.

3. Initial value problems

Let \( \Sigma \) be a (possibly empty) set of isolated points in \( \mathbb{R} \). For each \( \xi \in \Sigma \) choose a positive number \( b(\xi) \) such that the sets

\[ V(\xi) = \{ x \in \mathbb{C} : |x - \xi| < b(\xi), \Re(x) = \xi \Rightarrow \Im(x) > 0 \} \]

are pairwise disjoint. Let

\[ \Omega = \bigcup_{\xi \in \Sigma} V(\xi) \cup (\mathbb{R} - \Sigma). \]

We consider potentials which satisfy the following hypothesis.
HYPOTHESIS 3.1. The map \( q : \Omega \rightarrow \mathbb{C} \) has the following properties:
(a) \( q \) is locally integrable in \( \mathbb{R} - \Sigma \), and
(b) if \( \xi \in \Sigma \) then \( q \) is analytic in \( V(\xi) \) and there exists a non-zero complex number \( a(\xi) \) such that \( q(x) - a(\xi)(x - \xi)^2 \) is bounded in \( V(\xi) \).

Thus, if \( \Sigma = \emptyset \) then \( q \) is a locally integrable, complex-valued function on \( \mathbb{R} \).

For ease of notation we assume subsequently that \( \Sigma \) is unbounded below and above. However, all statements in this section have analogues for the case that \( \Sigma \) is bounded from above or below.

We may label the elements of \( \Sigma \) such that \( \Sigma = \{ \xi_n : n \in \mathbb{Z} \} \) and \( \xi_n < \xi_m \) when \( n < m \). Define \( \alpha_n = \frac{1}{2}(\xi_n + \xi_{n+1}) \) and \( \Omega_n = (\alpha_{n-1}, \alpha_n) \cup V(\xi_n) \).

If \( q \) satisfies the above hypothesis then we may write \( q = q_0 + q_1 \) where
\[
q_0(x) = \frac{a(\xi_n)}{(x - \xi_n)^2} \quad \text{for} \quad x \in \Omega_n,
\]
and \( q_1(x) = q(x) - q_0(x) \). Then \( q_1 \) is locally integrable in \( \mathbb{R} - \Sigma \) and bounded in \( V(\xi_n) \) for \( n \in \mathbb{Z} \).

When \( x_0, x \in \overline{\Omega_n} \) let \( \hat{a} = a(\xi_n) \) and \( \hat{q}_1(x) = q_1(x + \xi_n) \) in Theorem 2.1. Then define
\[
\phi_0(E, y_0, y_0, x_0, x) = \phi_0(E, y_0, y_0, x_0 - \xi_n, x - \xi_n),
\]
\[
\phi(E, y_0, y_0, x_0, x) = \phi(E, y_0, y_0, x_0 - \xi_n, x - \xi_n).
\]

In particular, for \( x_0, x \in \overline{\Omega_n} \) we have therefore defined the functions
\[
c_0(E, x_0, x) = \phi_0(E, 1, 0, x_0, x),
\]
\[
c(E, x_0, x) = \phi(E, 1, 0, x_0, x),
\]
\[
s_0(E, x_0, x) = \phi_0(E, 0, 1, x_0, x),
\]
and, last but not least,
\[
s(E, x_0, x) = \phi(E, 0, 1, x_0, x).
\]

Now assume that \( x_0 \in \overline{\Omega_n} \) and \( x \in \overline{\Omega_{n+1}} \) where \( j \geq n \). Then define recursively
\[
\phi_0(E, y_0, y_0, x_0, x) = \phi_0(E, y_0, y_0, x_0, \alpha_j)c_0(E, \alpha_j, x) + \phi(E, y_0, y_0, \alpha_j, x_0)s_0(E, \alpha_j, x),
\]
\[
\phi(E, y_0, y_0, x_0, x) = \phi(E, y_0, y_0, x_0, \alpha_j)c(E, \alpha_j, x) + \phi(E, y_0, y_0, \alpha_j, x_0)s(E, \alpha_j, x).
\]

In a similar way we may extend the definition of \( \phi_0(E, y_0, y_0, x_0, \cdot) \) and \( \phi(E, y_0, y_0, x_0, \cdot) \) to \( \Omega \) when \( j < n \).

DEFINITION 3.1. Let \( \mathcal{C}(\Sigma) \) be the vector space of all complex-valued functions \( y \) on \( \Omega \) which are locally integrable in \( \mathbb{R} - \Sigma \) and analytic in \( V(\xi) \) for every \( \xi \in \Sigma \). Also let \( \mathcal{D}(\Sigma) \) be the vector space of all functions \( y \in \mathcal{C}(\Sigma) \) such that \( y \) and \( y' \) are locally absolutely continuous in \( \mathbb{R} - \Sigma \).
Theorem 3.1. Suppose \( q \) satisfies Hypothesis 3.1 and the functions \( \phi_0 \) and \( \phi \) are defined as above. Then the following hold.

(a) The function \( \phi(E, y_0, y_0', x_0, \cdot) \) is a solution of the initial value problem
\[
y'' + qy = Ey, \quad y(x_0) = y_0, \quad y'(x_0) = y'_0.
\]

(b) The function \( \phi(E, y_0, y_0', x_0, \cdot) \) is unique in \( \mathcal{D}(\Sigma) \).

(c) The functions \( \phi(E, y_0, y_0', \cdot) \) and \( \phi'(E, y_0, y_0', \cdot) \) are absolutely continuous in \( \mathbb{R} - \Sigma \). Moreover,
\[
\frac{\partial \phi}{\partial x_0}(E, y_0, y_0', x_0, x) = \phi(E, -y_0', (q(x_0) - E)y_0, x_0, x),
\]
\[
\frac{\partial \phi'}{\partial x_0}(E, y_0, y_0', x_0, x) = \phi'(E, -y_0', (q(x_0) - E)y_0, x_0, x).
\]

(d) There exist positive constants \( C \) and \( \Lambda \) depending on \( x_0 \) and \( x \), but not on \( E, y_0 \) and \( y'_0 \), such that
\[
|\phi(E, y_0, y_0', x_0, x) - \phi_0(E, y_0, y_0', x_0, x)| \leq C_\epsilon \|E\| \frac{|y_0| + |y'_0|}{|E|},
\]
\[
|\phi'(E, y_0, y_0', x_0, x) - \phi'_0(E, y_0, y_0', x_0, x)| \leq C_\epsilon \|E\| \frac{|y_0| + |y'_0|}{|E|},
\]
when \( x_0, x \in \mathbb{R} - \Sigma \) and when \( |E| \geq \Lambda \). In particular, the functions \( \phi(\cdot, y_0, y_0', x_0, \cdot) \) and \( \phi'(\cdot, y_0, y_0', x_0, \cdot) \) are entire and have order \( \frac{1}{2} \).

Proof. Parts (a), (b), and (c) follow immediately from Theorem 2.1 and induction. To prove Part (d) we also have to use induction. If \( x_0 \) and \( x \) in \( \Pi \), the statement follows from Theorem 2.1. Now assume it holds for \( x_0 \in \Pi_N \) and \( x \in \Pi_j \) where \( j \geq n \). Let \( x \in \Pi_{N+1} \). Then, for \( \sqrt{E} = \kappa + i \eta \), we have
\[
\phi(E, y_0, y_0', x_0, x) = (\phi_0(E, y_0, y_0', x_0, x) + R_1(c_0(E, \alpha_0, x) + kR_1) + (\phi_0(E, y_0, y_0', x_0, x) + kR_2) + (\phi_0(E, y_0, y_0', x_0, x) + R_4)
\]
where, by the induction hypothesis,
\[
|R_l| \leq \begin{cases} C' \frac{e^{|\alpha_0|k^2}}{|k|} \frac{|y_0| + |y'_0|}{|E|} & \text{for } l = 1, 2, \\
C' \frac{e^{|\alpha_0|k^2}}{|k|} \frac{|y_0| + |y'_0|}{|E|} & \text{for } l = 3, 4,
\end{cases}
\]
for some positive constant \( C' \). Next observe that, using induction and the estimates (7), we may show the existence of a constant \( C_2 \) such that
\[
|\phi_0(-k^2, y_0, y_0', x_0, \cdot)| \leq C_2 \frac{|y_0| + |y'_0|}{|k|} \exp(|\Im(k)(x - x_0)|)
\]
and
\[
|\phi'_0(-k^2, y_0, y_0', x_0, \cdot)| \leq C_2 \frac{|y_0| + |y'_0|}{|k|} \exp(|\Im(k)(x - x_0)|).
\]
Using these estimates and the previous ones in (8) now gives the desired result for \( |\phi - \phi_0| \). A similar argument works for \( |\phi' - \phi'_0| \).
Next we consider an important special case, namely the case when \( a(x) = \frac{1}{2}g(x) \) where \( g(x) \in \mathbb{N} \). In this case \( \nu_n \), the order of the Bessel functions used to describe the solutions \( \phi_0 \) in \( \Omega_n \), is half of an odd integer, specifically \( \nu_n = g_n + \frac{1}{2} \). Hence \( \cos(\nu(n)x) = 0 \). Therefore, using properties listed in the appendix, we obtain the existence of a positive constant \( C_1 \) such that

\[
|c_0(\lambda^2, 0, 0) - \cos(k(x - x_0))| \leq \frac{C_1}{|k|} e^{\Im(k)(x-x_0)},
\]

\[
|c_0(\lambda^2, 0, 0) - \sin(k(x - x_0))| \leq \frac{C_1}{|k|^2} e^{\Im(k)(x-x_0)},
\]

\[
|c_0(\lambda^2, 0, 0) + k \sin(k(x - x_0))| \leq \frac{C_1}{|k|} e^{\Im(k)(x-x_0)}.
\]

By arguments similar to those employed above we then obtain the following.

**Theorem 3.2.** Assume that \( a \) satisfies Hypothesis 3.1 where \( a(x) = \frac{1}{2}g(x) \) with \( g(x) \in \mathbb{N} \). Let \( \phi \) be defined as above. Then there exist positive constants \( C \) and \( \Lambda \) depending on \( x_0 \) and \( x \) but not on \( E = -\lambda^2 \) \( y_0 \) and \( y_0 \) such that

\[
|\phi(E, y_0, y_0, x_0, x) - y_0 \cos(k(x - x_0))| \leq C_0 e^{\Im(k)(x-x_0)}|y_0| + |y_0|.
\]

\[
|\phi'(E, y_0, y_0, x_0, x) + y_0 k \sin(k(x - x_0)) - y_0 \cos(k(x - x_0))| \leq C_0 e^{\Im(k)(x-x_0)}|y_0| + |y_0|.
\]

when \( x_0, x \in \mathbb{R} \setminus \{0\} \) and when \( |E| \geq \Lambda \).

4. Floquet theory and the conditional set

In this section \( q \) is always an (almost everywhere) periodic function of period \( p > 0 \) which satisfies Hypothesis 3.1. However, for some results which rely on Theorem 3.2 we will use the following hypothesis.

**Hypothesis 4.1.** The function \( q \) is periodic of period \( p > 0 \) and satisfies Hypothesis 3.1, where \( a(x) = \frac{1}{2}g(x) \) with \( g(x) \in \mathbb{N} \).

Let \( L : \varphi(\Sigma) \to \psi(\Sigma) \) be defined by \( Ly = y'' + qy \). Denote the operator \( y \mapsto y(\cdot + p) \) on \( \psi(\Sigma) \) by \( S \). Then \( LE = LS \) on \( \varphi(\Sigma) \). Hence \( S(E) \), the restriction of \( S \) to the (two-dimensional) space \( W(E) \) of solutions of \( Ly =Ey \), maps \( W(E) \) to itself, that is, \( S(E) \) is a linear operator on a two-dimensional vector space. Its eigenvalues are called Floquet multipliers and its eigenfunctions are called Floquet solutions of \( Ly = Ey \). They satisfy \( y(x+p) = \rho y(x) \) for all \( x \in \mathbb{R} - \Sigma \) when \( \rho \) is the associated Floquet multiplier. Generalized eigenfunctions of \( S(E) \) are called generalized Floquet solutions of \( Ly = Ey \).

The characteristic polynomial of \( S(E) \) is given by

\[
\mathcal{F}(E, \rho) = \rho^2 - \text{tr} S(E) \rho + 1
\]
since \( \det S(E) = 1 \), as is easily seen by choosing a basis in \( W(E) \). The most convenient basis is given by \( c(E,x_0,\cdot) = \phi(E,1,0,x_0,\cdot) \) and \( s(E,x_0,\cdot) = \phi(E,0,1,x_0,\cdot) \). Using this basis we see that the matrix representing \( S(E) \), called the monodromy matrix, equals the fundamental matrix associated with \( c(E,x_0,\cdot) \) and \( s(E,x_0,\cdot) \) evaluated at \( x_0 + p \). This shows that \( \det S(E) = 1 \). Moreover, it gives an explicit expression for \( \text{tr} S(E) \), namely
\[
\text{tr} S(E) = c(E,x_0,x_0 + p) + \frac{c'(E,x_0,x_0 + p)}{E}.
\]
Hence \( \text{tr} S(E) \) is an entire function of \( E \) of order \( \frac{1}{2} \). Finally, we use the basis to express the Floquet solutions as either of the following two expressions:
\[
\begin{align*}
f_E(E,x_0,x) &= c(E,x_0,x) + \frac{\rho_+(E) - c(E,x_0,x_0 + p)}{s(E,x_0,x_0 + p)} s(E,x_0,x), \\
\hat{f}_E(E,x_0,x) &= s(E,x_0,x) + \frac{\rho_+(E) - s'(E,x_0,x_0 + p)}{c'(E,x_0,x_0 + p)} c(E,x_0,x),
\end{align*}
\]
where \( \rho_+ \) are the Floquet multipliers of \( L = Ey \).

The function \( s(E,x_0,x) \) subsequently plays a particularly important role. The first fact which distinguishes it from every other solution of \( L = Ey \) is that it is twice differentiable with respect to \( x_0 \). In addition, it satisfies the following fundamental theorem whose proof is, given Part (c) of Theorem 3.1, merely a simple computation.

**Theorem 4.1.** Suppose \( q \) is a periodic function of period \( p \) satisfying Hypothesis 3.1. Let \( s(E,x_0,x) \) be the solution of the initial value problem
\[
y'' + qy = Ey, \quad y(x_0) = 0, \quad y'(x_0) = 1.
\]
Then \( g(E,x) = s(E,x,x+p) \) satisfies the following non-linear second-order differential equation with respect to \( x \):
\[
4(E - q(x))g(E,x)^2 - 2g(E,x)g''(E,x) + g'(E,x)^2 = \text{tr} S(E)^2 - 4,
\]
where primes denote derivatives with respect to \( x \).

This differential equation already appears in Hamel’s work \([16]\) of 1913.

A Floquet multiplier is a simple eigenvalue of \( S(E) \), that is, it has algebraic multiplicity 1, if and only if \( \text{tr} S(E)^2 - 4 \neq 0 \). In particular, the points \( E \) where the Floquet multipliers are degenerate are isolated. The algebraic and geometric multiplicities of \( \rho \) as an eigenvalue of \( S(E) \) will be denoted by \( m_H(E,\rho) \) and \( m_g(E,\rho) \), respectively. Also, by \( p(E) \) we will denote the order of \( E \) as a zero of \( \text{tr} S(E)^2 - 4 \). In particular, \( p(E) = 0 \) if and only if \( \text{tr} S(E)^2 - 4 = 0 \).

Of special interest are the points \( E \) such that \( Ly = Ey \) does not have two linearly independent Floquet solutions. These points are collected in the set
\[
F_1 = \{ E \in \mathbb{C} : m_H(E,\rho) = 2, m_g(E,\rho) = 1 \text{ for some } \rho \}.
\]
The set
\[
\mathcal{S}(L) = \{ E \in \mathbb{C} : -2 \leq \text{tr} S(E) \leq 2 \}
\]
will be called the conditional stability set of \( L \). The Floquet multipliers of \( Ly = Ey \) have absolute value 1 if and only if \( E \in \mathcal{S}(L) \). This shows that there exists a solution \( y \) of \( Ly = Ey \) such that the set \( \{ y(x_0 \pm mp) : m \in \mathbb{Z} \} \) is bounded for any \( x_0 \in \mathbb{R} - \mathbb{N} \) if and only if \( E \in \mathcal{S}(L) \).
THEOREM 4.2. Assume that $q$ satisfies Hypothesis 4.1. Then the conditional stability set $\mathcal{S}(L)$ consists of a countable number of regular analytic arcs, called bands. The point $E$ is a band edge if and only if $p(E)$ is an odd integer.

Proof. Assume $E_0 \in \mathcal{S}(L)$. Then $\text{tr} S(E_0) \in [-2, 2]$. Denote the order of the zero $E_0$ of $\text{tr} S(E) - \text{tr} S(E_0)$ by $\ell$. Then there exists a conformal map $\gamma$ from a neighbourhood of $E_0$ to a neighbourhood of zero such that $\text{tr} S(E) - \text{tr} S(E_0) = \gamma(E) - \gamma(E_0)$.

First assume that $\text{tr} S(E_0) \in (-2, 2)$. A number $E$ close to $E_0$ is in $\mathcal{S}(L)$ if and only if $\gamma(E)^j = \gamma(E_0)^j$ for some $j > 0$. But the interval $(-e, e)$ has $\ell$ preimages under the map $\gamma^{-1}$ in the vicinity of $E = 0$, and hence any $E \in \mathcal{S}(L)$ close to $E_0$ lies on one of $\ell$ regular analytic arcs which intersect at $E_0$. The angle between two adjacent arcs is $\pi/\ell$. Next assume that $\text{tr} S(E_0) = 0$. Then $E$ is close to $E_0$ in $\mathcal{S}(L)$ if and only if $\gamma(E)^j = \gamma(E_0)^j$ for some $j > 0$. Again any $E \in \mathcal{S}(L)$ close to $E_0$ lies on one of the $\ell$ preimages of this interval which meet at $E_0$ in such a way that the angle between two adjacent arcs is equal to $2\pi/\ell$. In particular, when $\ell$ is even we find that $\frac{\ell}{2}$ regular analytic arcs intersect at $E_0$. Otherwise $E_0$ is the endpoint of $\ell$ arcs. A similar statement is true when $\text{tr} S(E_0) = 2$.

In any case, if $E \in \mathcal{S}(L)$ then it is either an interior point of a regular analytic arc or else the endpoint of one. For $E$ to be an endpoint of such an arc it is necessary that $p(E) = \text{ord}_E(\text{tr} S(-) - 4)$ is an odd integer. This happens only at countably many places. At each of these places at most finitely many arcs end.

When $|k|$ is very large then, by Theorem 3.2, $\text{tr} S(-k^2)$ is approximately equal to $2\cos(k\pi)$. This implies that the Floquet multipliers are very close to $e^{\pm ip}$. If $-k_0 \in \mathcal{S}(L)$ then $k_0$ is almost real. Now let $k = |k_0| e^{i\theta}$ where $\theta \in (-\frac{1}{2}\pi, \frac{1}{2}\pi)$. When the circle $|k_0| e^{i\theta}$ intersects $\mathcal{S}(L)$ then $\theta$ is close to zero and the Floquet multiplier which is almost equal to $e^{ip}$ moves radially inside the unit circle, while the one close to $e^{-ip}$ leaves the unit circle at the same time. Since this can happen at most once there is, when $|k_0|$ is large, at most one intersection of the circle of radius $|k_0|$ with $\mathcal{S}(L)$, that is, at most one arc extends to infinity. Altogether this shows that the conditional stability set consists of at most countably many arcs.

Rofe-Beketov [26] has shown that $\mathcal{S}(L)$ equals the spectrum of the maximal operator associated with $L$ in $L^2(\mathbb{R})$ when $q$ is local integrable in $\mathbb{R}$.

For $\rho \in \mathbb{C} - \{0\}$ consider the operator $T_\rho$ defined by $T_\rho y = Ly$ for all

$$ y \in \mathcal{D}(T_\rho) = \{ y \in \mathcal{D}(S) \colon y(x_0 + p) = \rho y(x_0), y'(x_0 + p) = \rho y'(x_0) \}$$

where $x_0 \in \mathbb{R} - \Sigma$ (the dependence of $T_\rho$ on $x_0$ is inessential and hence will be suppressed in the notation employed). The eigenvalues of $T_\rho$ are given as zeros of $\mathcal{F}(-\rho)$. In particular, they are independent of $x_0$. The operator $T_\rho$ will be called a Floquet operator, the boundary conditions defining $T_\rho$ will be called Floquet boundary conditions and the eigenvalues of $T_\rho$ are called Floquet eigenvalues. If $\rho = \pm 1$ then the eigenvalues of $T_\rho$ are called periodic and semi-periodic eigenvalues, respectively. Any eigenfunction of $T_\rho$ associated with an eigenvalue $E$ is a Floquet solution of $Ly = Ey$ and hence an eigenfunction of $S(E)$ associated with the Floquet multiplier $\rho$. In particular, the geometric multiplicity of $E$ as an eigenvalue of $T_\rho$ coincides with the geometric multiplicity of $\rho$ as an eigenvalue of $S(E)$, that is, with $m(E, \rho)$.
The algebraic multiplicity of $E$ as an eigenvalue of $T_p$ will be denoted by $m_0(E, \rho)$. It equals the multiplicity of the zero $E$ of $\mathcal{F}(\cdot, \rho)$. For a proof of this fact see, for example, a result of Naimark [24, § 2.3] which depends only on the fact that initial value problems have unique solutions.

**Theorem 4.3.** Suppose $q$ satisfies Hypothesis 4.1. Then the operator $T_p$ has a countably infinite number of eigenvalues $E_0(\rho)$, $E_1(\rho)$, etc. If $\theta$ is the principal value of $\log \rho$, these eigenvalues have the asymptotic behaviour

$$E_{2n-1}(\rho) = -\frac{(2n\pi - \theta)^2}{p^2} + O(1),$$

$$E_{2n}(\rho) = -\frac{(2n\pi + \theta)^2}{p^2} + O(1),$$

as $n$ tends to infinity when eigenvalues are repeated according to their algebraic multiplicities.

**Proof.** Since, by Theorem 3.2, $\mathcal{F}(E, \rho)$ is asymptotically almost equal to $\rho^2 - 2\rho \cos(kp) + 1$ up to an error term which is of the same form as in the case where no singularities are present, the proof of the theorem is the same as in that case (see for example, Naimark [24]). Therefore we give only an outline.

If $q = 0$, the eigenvalues of $T_p$ are given by $\rho^2 - 2\rho \cos(kp) + 1 = 0$, that is,

$$E_{0,0} = \frac{\theta^2}{p^2}, \quad E_{0,2n-1} = \frac{(2n\pi - \theta)^2}{p^2}, \quad E_{0,2n} = \frac{(2n\pi + \theta)^2}{p^2},$$

for $n = 1, 2, \ldots$. These eigenvalues are listed according to their algebraic multiplicity. In particular, they are simple unless $\theta = 0$ or $\theta = \pi$. In these cases eigenvalues have algebraic multiplicity 2 except that $E_{0,0}$ is simple when $\theta = 0$.

Rouché’s theorem now shows that for large $n$ there is one zero of $\mathcal{F}(-k^2, \rho)$ (treated as a function of $k$) in a disk of radius of order $1/n$ about the point $(2n\pi \pm \theta)/p$ if $\theta$ is different from zero and $\pi$, and that there are two when $\theta$ is equal to zero or $\pi$. To prove that these eigenvalues are labelled in the manner indicated, one again uses Rouche’s theorem with a circle of a very large radius centered at the origin of the $E$-plane to compare the number of zeros of $\mathcal{F}(E, \rho)$ and $\rho^2 - 2\rho \cos(\sqrt{-E}p) + 1$ (treated as functions of $E$) in the interior of this circle.

5. Sturm–Liouville eigenvalue problems

Next we consider Sturm–Liouville boundary value problems. Let $L$ be the operator defined in the previous section, that is, $L = \partial^2/\partial x^2 + q$ where $q$ is a periodic function of period $p$ which satisfies Hypothesis 3.1 unless noted otherwise. Define the operator $T_{\alpha, \beta}$ by $T_{\alpha, \beta}y = Ly$ on the domain

$$\mathcal{D}(T_{\alpha, \beta}) = \{ y \in \mathcal{D}(\mathcal{L}): U_{1,\alpha,\beta}(y) = U_{2,\alpha,\beta}(y) = 0 \}$$

where

$$U_{1,\alpha,\beta}(y) = \sin(\alpha)y(x_0) - \cos(\alpha)y'(x_0) = 0,$$

$$U_{2,\alpha,\beta}(y) = \sin(\alpha)y(x_0 + p) - \cos(\alpha)y'(x_0 + p) = 0.$$
for \( x_0 \in \mathbb{R} - \Sigma \) and \( \alpha \in \mathbb{C}/\pi \). The operator \( T_{\alpha, 0} \) is called a Sturm–Liouville operator and its eigenvalues are called Sturm–Liouville eigenvalues. The most important cases are \( \alpha = \frac{\pi}{2} \) and \( \alpha = 0 \) when we get Dirichlet and Neumann operators, respectively.

The eigenvalues of \( T_{\alpha, 0} \) and their algebraic multiplicities are given as the zeros and their multiplicities of the function

\[
\mathcal{G}(E, x_0, \alpha) = \left. U_{1, x_0, \alpha}(c(E, x_0, \cdot)) \right|_{E=\alpha} \mathcal{U}_{2, x_0, \alpha}(s(E, x_0, \cdot)) - \left. U_{1, x_0, \alpha}(s(E, x_0, \cdot)) \right|_{E=\alpha} \mathcal{U}_{2, x_0, \alpha}(c(E, x_0, \cdot))
\]

In particular, Dirichlet eigenvalues are the zeros of \( s(E, x_0, x_0 + p) \) and Neumann eigenvalues are the zeros of \( c'(E, x_0, x_0 + p) \). The geometric multiplicity of a Sturm–Liouville eigenvalue is always equal to 1.

The proof of the following theorem follows the same pattern as that of Theorem 4.3.

**Theorem 5.1.** If \( q \) satisfies Hypothesis 4.1 then the operators \( T_{\alpha, x_0} \) and \( T_{\alpha, 0} \), where \( \cos \alpha \neq 0 \), each have a countably infinite number of eigenvalues which are denoted by \( \mu_1(x_0), \mu_2(x_0), \ldots \) and \( \nu_0(x_0, \alpha), \nu_1(x_0, \alpha), \ldots \), respectively. These eigenvalues have the asymptotic behaviour

\[
\begin{align*}
\mu_n(x_0) &= -\frac{(\pi n)^2}{p^2} + O(1), \\
\nu_n(x_0, \alpha) &= -\frac{(\pi n)^2}{p^2} + O(1)
\end{align*}
\]

as \( n \) tends to infinity when eigenvalues are repeated according to their algebraic multiplicities.

Let \( y \) be an eigenfunction of \( T_{\alpha, 0} \) associated with the eigenvalue \( E \). Then \( y \) is a Floquet solution of \( Ly = Ey \) with multiplier

\[
\rho = y(x_0 + p)/y(x_0) = y'(x_0 + p)/y'(x_0)
\]

if \( \sin(2\alpha) = 0 \) then one of these fractions is undefined). Hence every Sturm–Liouville eigenfunction is a Floquet function. Conversely, let \( y \) be a Floquet solution of \( Ly = Ey \); then \( y \) is an eigenfunction of \( T_{\alpha, 0} \) where \( \tan \alpha = y'(x_0)/y(x_0) \).

**Definition 5.1.** The algebraic multiplicity of \( E \) as an eigenvalue of \( T_{\alpha, 0} \) is denoted by \( r(E, x_0, \alpha) \). The quantities

\[
\begin{align*}
\nu(E, \alpha) &= \min \{ r(E, x_0, \alpha) : x_0 \in \mathbb{R} - \Sigma \}, \\
r_{\alpha}(E, x_0, \alpha) &= r(E, x_0, \alpha) - r(E, \alpha)
\end{align*}
\]

will be called the **immoveable part** and the **moveable part** of the multiplicity \( r(E, x_0, \alpha) \), respectively.

For Dirichlet eigenvalues we also introduce as abbreviations \( d(E, x_0) = r(E, x_0, \frac{\pi}{2}) \), \( d(E) = r(E, \frac{\pi}{2}) \), and \( d_0(E, x_0) = r_{\alpha}(E, x_0, \frac{\pi}{2}) \). If \( d(E) > 0 \), the number \( E \) will be called an immovable Dirichlet eigenvalue. If \( d(E, x_0) = d_0(E, x_0) > 0 \), the number \( E \) will be called a moveable Dirichlet eigenvalue.
Recall that \( p(E) = \text{ord}_x(\text{tr}S) = 2 \). Then we have the following proposition.

**Proposition 5.1.** The following statements are true.

1. If \( d_1(E) > 0 \) then \( r_1(E, \alpha) > 0 \) for all \( \alpha \in \mathbb{C}^2 \) and \( p(E) > 0 \). Moreover, in this case, a Floquet multiplier \( \rho \) of \( Ly = E\gamma \) satisfies \( \rho^2 = 1 \) and \( m_\gamma(E, \rho) = m_\gamma(E, \rho) = 2 \).

2. If \( r_1(E, \alpha) > 0 \) then \( d_1(E) > 0 \) unless \( q \) is a constant and \( E = q + \tan(\alpha)^2 \).

3. If \( p(E) > 0 \) and if, for some \( x_0 \in \mathbb{R} - \Sigma \), both \( r(E, x_0, \alpha) > 0 \) and \( r(E, x_0, \beta) > 0 \), where \( \sin(\alpha - \beta) \neq 0 \), then \( d_1(E) > 0 \).

**Proof.** If \( E \) is an immovable Dirichlet eigenvalue then \( s(E, x_0, x_0 + p) = 0 \) and \( s(E, x_0, \cdot) \) is Floquet for every \( x_0 \in \mathbb{R} - \Sigma \). The multiplier of \( s(E, x_0, \cdot) \) is given by \( s'(E, x_0, x_0 + p) \). Since \( s'(E, x_0, x_0 + p) \) is continuous as a function of \( x_0 \) and always equal to one of the two Floquet multipliers, one may, in fact, conclude that \( s'(E, x_0, x_0 + p) \) is identically equal to some constant \( \rho \). Now assume that \( \alpha \) is such that \( \cos \alpha \neq 0 \) and let \( y \) be the unique solution of \( Ly = E\gamma \) which satisfies initial conditions \( y(x_0) = \cos \alpha \) and \( y'(x_0) = \sin \alpha \), that is, \( y = \phi(E, \cos \alpha, \sin \alpha, x_0, \cdot) \). Then \( y \) and \( s(E, x_0, \cdot) \) are linearly independent. Hence there are numbers \( \beta_1 \) and \( \beta_2 \) such that \( s(E, x_0, x) = \beta_1 s(E, x_0, x) + \beta_2 (x) \). Since both \( s(E, x_0, \cdot) \) and \( s(E, x_0, \cdot) \) are Floquet with the same multiplier \( \rho \), one finds that \( \beta_2(x) = \beta_2(y(x + p) - \rho y(x)) - \rho \beta_2(x) \). Note that \( x_0 \) can be chosen such that \( \beta_1 = s(E, x_0, x_0) \), \( \beta_2 = 0 \). Hence \( y \) is a Floquet solution with multiplier \( \rho \). This shows that \( m_\gamma(E, \rho) = m_\gamma(E, \rho) = 2 \) and that \( p(E) > 0 \). Also, \( U_{1, y, \alpha}(y) = 0 \) by our choice of initial conditions and \( U_{2, y, \alpha}(y) = 0 \), that is, \( y \) is an eigenfunction of \( T_{1, y, \alpha} \). Since \( x_0 \) was arbitrary, we have \( r_1(E, \alpha) > 0 \).

Now suppose that \( r_1(E, \alpha) > 0 \) and \( \cos \alpha \neq 0 \). Then by an argument similar to that used before we can show that

\[
\beta_2(s(E, x_0, x + p) - \rho s(E, x_0, x)) = 0
\]

where

\[
\beta_2 = \phi(E, \cos \alpha, \sin \alpha, x_0, x_0) - \phi(E, \cos \alpha, \sin \alpha, x_0, x_0) \tan \alpha.
\]

If we can find an \( x_0 \) for every \( x_0 \in \mathbb{R} - \Sigma \) such that \( \beta_2 \neq 0 \) then we have shown that \( d_1(E) > 0 \). Hence assume that \( \beta_2 = 0 \) and

\[
0 = \frac{\partial \beta_2}{\partial x_0} - \phi(E, -\sin \alpha, q(x_0) - E) \cos \alpha, x_0, x_0)
- \phi(E, -\sin \alpha, q(x_0) - E) \cos \alpha, x_0, x_0) \tan \alpha.
\]

where the last equality follows from Part (c) of Theorem 3.1. Therefore \( \phi(E, -\sin \alpha, q(x_0) - E) \cos \alpha, x_0, \cdot) \) is a multiple of \( \phi(E, \cos \alpha, \sin \alpha, x_0, \cdot) \), that is, there is a constant \( c \) such that \( \cos \alpha \neq c \cos \alpha \) and \( q(x_0) - E) \cos \alpha = c \sin \alpha \), which implies that \( q(x_0) - E = -\tan(\alpha)^2 \). Hence, either \( q \) is a constant and \( E = q + \tan(\alpha)^2 \) or we may find an \( x_0 \) such that \( \beta_2 \neq 0 \).

Finally assume that \( p(E) > 0 \), \( r(E, x_0, \alpha) > 0 \), \( r(E, x_0, \beta) > 0 \) and that \( \sin(\alpha - \beta) \neq 0 \). Then \( S(E) \) has a degenerate eigenvalue \( \rho = \pm 1 \). Also \( T_{1, y, \alpha} \) and \( T_{1, y, \beta} \) have linearly independent eigenfunctions \( y_1 \) and \( y_2 \) both of which are Floquet solutions of \( Ly = E\gamma \) with Floquet multiplier \( \rho \). Hence, every solution of \( Ly = E\gamma \) and, in
We obtain from the Hadamard factorization theorem that 

\[ s(E, x_0, x_0 + p) = F_D(E, x_0)D(E), \]

\[ c'(E, x_0, x_0 + p) = F_N(E, x_0)N(E), \]

where

\[ F_D(E, x_0) = \tau_D(x_0)E^{\sigma(0, x_0)} \prod_{\lambda \in C} (1 - E/\lambda)^{\sigma(E, \lambda)}, \]

\[ F_N(E, x_0) = \tau_N(x_0)E^{\sigma(0, x_0, 0)} \prod_{\lambda \in C} (1 - E/\lambda)^{\sigma(E, 0, \lambda)}, \]

\[ D(E) = E^{\sigma(0)} \prod_{\lambda \in C} (1 - E/\lambda)^{\gamma(\lambda)}, \]

\[ N(E) = E^{\sigma(0, 0)} \prod_{\lambda \in C} (1 - E/\lambda)^{\gamma(E, 0)}, \]

for suitable functions \( \tau_D \) and \( \tau_N \).

**Proposition 5.2.** The functions \( F_D(\cdot, x_0) \), \( F_D'(\cdot, x_0) \), \( F_D''(\cdot, x_0) \), \( F_N(\cdot, x_0) \) and \( F_N'(\cdot, x_0) \) (where primes denote derivatives with respect to \( x_0 \)) are entire for almost all values of \( x_0 \) in \( \mathbb{R} - \Sigma \).

**Proof.** Clearly \( F_D(\cdot, x_0) \) and \( F_N(\cdot, x_0) \) are entire. We present the proof that \( F_D'(\cdot, x_0) \) is entire assuming we have already shown that \( F_D''(\cdot, x_0) \) is entire. That proof and the one for \( F_N'(\cdot, x_0) \) are very similar to what follows.

Note that

\[ \frac{ds(E, x_0, x_0 + p)}{dx_0} = s'(E, x_0, x_0 + p) - c(E, x_0, x_0 + p), \]

\[ \frac{d^2s(E, x_0, x_0 + p)}{dx_0^2} = c'(E, x_0, x_0 + p) - (q(x_0) - E)s(E, x_0, x_0 + p). \]

Hence \( F_D''(\cdot, x_0) \) is meromorphic and may have poles only at the zeros of \( D \).

Assume that \( E_0 \) is indeed a pole and choose \( r \) such that \( \{ z : |z - E_0| \leq \frac{r}{2} \} \) contains no other zero of \( D \). Let \( H > 0 \) be such that \( [x_0 - H, x_0 + H] \subseteq \mathbb{R} - \Sigma \). Define

\[ G(z, h) = \frac{F_D''(z, x_0 + h) - F_D''(z, x_0)}{h} \]

on \( \{ z : |z - E_0| \leq \frac{r}{2} \} \times ([-H, H] \setminus \{ 0 \}) \). Since \( G(\cdot, h) \) is analytic, we obtain, for all \( E \) for which \( |E - E_0| \leq \frac{r}{2} \), from Cauchy’s integral formula

\[ |G(E, h)| \leq \frac{1}{r} \int_0^1 |G'(\gamma(t), h)| \frac{|\gamma(t) - E|}{|\gamma(t)|} dt \leq 2 \int_0^1 |G'(\gamma(t), h)| dt \]

by integrating along a circle \( \gamma \) of radius \( r \) around \( E_0 \).
Let $M$ be such that
$$|e^{(k)}(z,u,u+p)|, |e^{(k)}(z,u,u+p)| = M$$
for $k = 0, 1$, $|z-E_0| = r$, and $u \in [x_0-H, x_0+H]$. Then, for $h > 0$, we have
$$|G(z,h)| = \frac{1}{h} \int_{x_0}^{x_0+h} |F_D(z,u)| du \leq \frac{M}{h|D(z)|} \int_{x_0}^{x_0+h} (1 + |z| + |q(u)|) du,$$
since $F_D(z,u)$ is absolutely continuous. If $x_0$ is a Lebesgue point of $q$ then
$$|q(x_0)| = \lim_{h \to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |q(u)| du.$$ Hence $|F_D(z,u)| du$ is bounded in $(0,H)$. A similar argument works for $h < 0$. Thus there exists a constant $M$ such that $|G(z,h)| \leq M$ for all $(z,h)$ such that $|z-E_0| = r$ and $0 < |h| \leq H$. This finally shows that $|G(E,h)| \leq 2M$ for all $(E,h)$ such that $|E-E_0| = \frac{1}{2}r$ and $0 < |h| \leq H$.

Now, since $E_0$ is a pole of $F_D(z,E_0)$, there exists a sequence $E_m$ such that $0 < |E_m-E_0| \leq \frac{1}{2}r$ and $|F_D(E_m,x_0)| \to 0$. But for all $m \in \mathbb{N}$ there exists an $h \in [-H,H] - \{0\}$ such that
$$|F_D(E_m,x_0)| \leq |F_D(E_m,x_0) - G(E_m,h)| + |G(E_m,h)| \leq 1 + 2M.$$ This contradiction completes the proof.

**Proposition 5.3.** The branches $\mu(x)$ of $F_D(\cdot,x)$ are continuous functions of $x$.

**Proof.** Suppose that $\mu$ is a zero of order $k$ of $F_D(\cdot,x_0)$. Choose $\epsilon > 0$ and suppose $|E-\mu| = \epsilon$. Since $F_D(E,\cdot)$ is continuously differentiable, we know that there exist constants $M$ and $\delta$ such that $|F_D(E,x) - F_D(E,x_0)| \leq M|\epsilon < \delta$. Also, let
$$C = \frac{\partial}{\partial E} |F_D(\mu,x_0) \neq 0.$$
Then $k! F_D(E,x_0)|C(E-\mu)^k$ converges to 1 as $E$ tends to $\mu$. Hence, for suitably small $\epsilon$ we obtain
$$|F_D(E,x) - F_D(E,x_0)| \leq \frac{2M|x-x_0|}{|C(\epsilon)^k |} |F_D(\mu,x_0)|.$$ Therefore, by Rouché's theorem, there are precisely $k$ zeros (counting multiplicities) in a disk of radius $\epsilon$ centred at $\mu$ provided $|x-x_0| \approx \epsilon |C(\epsilon)^k |$. These represent $k$ continuous branches of a function $\mu(x)$.

Define $U(E) = (\tau S(E)^2 - 4)/D(E)^2$. Then Theorem 4.1 and Proposition 5.2 show that
$$U(E) = 4(E-q(x_0))F_D(E,x_0)^2 - 2F_D(E,x_0)F_D'(E,x_0) + F_D'(E,x_0)^2$$
and that $U$ is an entire function. This immediately gives the following important result.

**Theorem 5.2.** If $q$ is a periodic function satisfying Hypothesis 3.1 then $p(E) - 2d(E) \geq 0$ for every $E \in \mathbb{C}$. 
It was shown in Proposition 5.1 that, for given $E$, the multiplicities $r(E, \alpha)$ are either equal to zero for all $\alpha$ or else positive for all $\alpha$, unless $q$ is constant. We are now in the position to show even more.

**Theorem 5.3.** Suppose $q$ is a periodic function satisfying Hypothesis 3.1 which is not equal to a constant. Then $r(E, \alpha) = d(E)$ for all $\alpha \in \mathbb{C}/\mathbb{R}$. In particular, $D(E) = N(E)$.

**Proof.** First recall that $F_D(E, \cdot)$, $F_N(E, \cdot)$, and $F_P(E, \cdot)$ are absolutely continuous in $\mathbb{R} - \Sigma$. The functions $F_D(E, \cdot)$ and $F_N(E, \cdot)$ do not vanish identically according to their definition. Next note that $|q(u)|$ tends to infinity when $u$ tends to $\xi \in \Sigma$. Hence $q$ is not constant on any component of $\mathbb{R} - \Sigma$. By hypothesis this is also true when $\Sigma = \emptyset$.

Note that, since $\det S(E) = 1$, we have

$$\text{tr} S(E)^2 - 4 = (c(E, x_0, x_0 + p) - c'(E, x_0, x_0 + p))^2 + 4c(E, x_0, x_0 + p)c'(E, x_0, x_0 + p).$$

Using Part (c) of Theorem 3.1 we obtain next

$$D(E)(U(E) - F_D(E, x_0)^2) = 4F_D(E, x_0)F_N(E, x_0)N(E).$$

This and (11) imply that

$$D(E)(4E - q(x_0))F_D(E, x_0) - 2F_D^2(E, x_0) = 4F_N(E, x_0)N(E),$$

and hence that $d(E) = r(E, 0)$. This inequality is strict if and only if

$$4E - q(x_0) = 0$$

vanish identically in $\mathbb{R} - \Sigma$. This in turn forces $E - q(x_0) = 0$ to vanish identically in $\mathbb{R} - \Sigma$. Since $F_D(E, \cdot)$ is constant, $F_D(E, \cdot)$ vanishes at one point, at most, on a component where it is not identically equal to zero. Hence, $E - q(x_0) = 0$ on this component, which is impossible. This shows that $d(E) = r(E, 0)$ and therefore that $D(E) = N(E)$ for all $E \in \mathbb{C}$.

Since $D = N$, we now find that

$$g(E, x_0, \alpha) = (F_D(E, x_0) \sin \alpha)^2 - F_N(E, x_0) \cos \alpha)D(E).$$

Thus $r(E, \alpha) = d(E)$ and the inequality is strict if and only if

$$F_N(E, x_0) = (\tan \alpha)^2 - (\tan \alpha) \tan \alpha$$

identically.

Another application of Part (c) of Theorem 3.1 gives

$$F_P(E, x_0) = (q(x_0) - E)^2 F_D(E, x_0).$$

This, (12), and (13) show that

$$(2F_D(E, x_0) \tan \alpha - F_D(E, x_0))((\tan \alpha)^2 + q(x_0) - E) = 0$$

for all $x_0 \in \mathbb{R} - \Sigma$.

Assume that $2F_D(E, x_0) \tan \alpha - F_D(E, x_0) \neq 0$ for some $x_0$ and hence on some interval $I \subset \mathbb{R} - \Sigma$. Thus $E - q(u) = (\tan \alpha)^2$ on $I$. A combination of equations (12) and (13) shows now that $2F_D(E, u) \tan \alpha = F_D(E, u)$ and hence that
This implies that \( I \) is a component of \( \mathbb{R} - \Sigma \) and \( q \) is constant on this component. Since this is impossible, we conclude that \( 2F_D(E,\cdot)\tan\alpha - F'_{D}(E,\cdot) = 0 \) on \( \mathbb{R} - \Sigma \). Hence, on any component of \( \mathbb{R} - \Sigma \), we find that \( F_D(E,u) = F_D(E,x_0) \exp(2(u-x_0)\tan\alpha) \) vanishes identically or does not vanish at all. Choose a component where \( F_D(E,\cdot) \) does not vanish. Then \( F_D(E,u) = 2F_D(E,u)\tan\alpha \) and \( F'_{D}(E,u) = -2F_D(E,u)(\tan\alpha)^3 \). Equation (14) therefore gives \((\tan\alpha)^2 = E - q(u)\) on this component, again an impossibility.

Proposition 5.1 and Theorems 5.2 and 5.3 have also been proved in [15] (cf. Proposition 3.1 and Theorem 3.2) in the case that \( q \) is a differentiable function on \( \mathbb{R} \) (and with regard to Dirichlet and Neumann eigenvalues only). The proof of Theorem 5.3, however, is different from its equivalent in [15]. In particular, the differentiability condition in Theorem 3.2 of [15] is essential for that part of the theorem.

Recall from § 4 that

\[
F_1 = \{ E \in \mathbb{C} : m(E,\rho) = 2, m(E,\rho) = 1 \text{ for some } \rho \}.
\]

**Theorem 5.4.** The set \( F_2 = \{ E \in \mathbb{C} : p(E) - 2d(E) > 0 \} \) is isolated. Moreover, \( F_1 \) is a subset of \( F_2 \). In fact, \( F_1 = \{ E \in F_2 : d(E) = 0 \} \).

**Proof.** The set \( F_2 \) is contained in the set of zeros of the entire function \( \text{tr}(S(E))^2 - 4 \) and hence isolated. Now, if \( E \in F_1 \) then \( p(E) > 0 \) and \( m(E,\rho) = 1 \). This implies that \( d(E) = 0 \) since otherwise Proposition 5.1 asserts that \( m(E,\rho) = 2 \). Hence \( F_1 \) is contained in \( \{ E \in F_2 : d(E) = 0 \} \). Finally, if \( p(E) > 0 \), assume that \( E \notin F_1 \), that is, that \( m(E,\rho) = 2 \) for some \( \rho = (\pm,1) \). Then every solution of \( L = E \), in particular the function \( s(E,x_0,\cdot) \), is a Floquet solution with multiplier \( \rho \). But this implies that \( s(E,x_0,x_0 + p) = 0 \) for every \( x_0 \in \mathbb{R} - \Sigma \), that is, that \( d(E) > 0 \).

6. The line bundle of Floquet solutions

In this section we want to study the topological structure of the set of Floquet solutions associated with \( L = d^2/dx^2 + q \) where \( q \) is a periodic function of period \( p \) satisfying Hypothesis 3.1.

Recall that \( \delta(E,x_0,x_0 + p) - c(E,x_0,x_0 + p) = F_D'(E,x_0)D(E) \) and \( \text{tr}(S(E))^2 - 4 = U(E)(D(E))^2 \). Hence the Floquet functions given in (9) and (10) can be expressed by

\[
f_z(E,x_0,\cdot) = c(E,x_0,\cdot) + \frac{U(E)}{2F_D(E,x_0)} s(E,x_0,\cdot)
\]

and

\[
f_z(E,x_0,\cdot) = s(E,x_0,\cdot) - \frac{U(E)}{2F_D(E,x_0)} c(E,x_0,\cdot).
\]

Now let \( x_0 \) be such that \( F_D(E,x_0) \neq 0 \) in a vicinity of some point \( E_0 \). Then \( f_z(E,x_0,\cdot) \) are non-trivial Floquet solutions of \( Ly = Ey \) in this vicinity. Thus \( f_z(E_0,x_0,\cdot) \) are linearly independent if and only if \( U(E_0) \neq 0 \). Hence we have proved the following.
Theorem 6.1. We have \( p(E_0) - 2d(E_0) > 0 \) if and only if there exists an \( x_0 \in \mathbb{R} - \Sigma \) such that \( \lim_{E \to E_0} W(E, x_0) = 0 \) where \( W(E, x_0) \) denotes the Wronskian of the Floquet solutions \( f_\pm(E, x_0, \cdot) \) of \( Ly = Ey \) given by (15). In particular, \( W(E_0, x_0) \neq 0 \) if \( p(E_0) = 0 \), that is, if \( \text{tr} S(E_0)^2 - 4 \neq 0 \).

Now assume that \( \text{tr} S(E_0)^2 - 4 = 0 \) and \( m(E_0, 1) = 2 \). Then every solution of \( Ly = E_0 y \) is a Floquet solution. However, just the multiples of at most two of them appear as limits of Floquet solutions nearby. For instance, considering \( q = 0 \) as a function of period \( 2\pi \), we find that \( \sin x \) is a Floquet solution of \( Ly = -y \) but not the limit of the Floquet solutions \( ce^{ki} \) or \( ce^{-ki} \) as \( k \) tends to \( i \). In fact, if \( p(E_0) > 2d(E_0) > 0 \) and if \( x_0 \) is such that \( d_n(E_0, x_0) = 0 \) then \( f_+(E, x_0, x) \) and \( f_-(E, x_0, x) \) are both continuous as functions of \( E \) and converge to the Floquet solutions \( f_\pm(E_0, x_0, x) \). However the Wronskian of these is zero, that is, all Floquet solutions in a vicinity of \( E_0 \) converge to multiples of only one Floquet solution at \( E_0 \) even though every solution of \( Ly = E_0 y \) is Floquet. In the following we are only interested in Floquet solutions which are limits of other Floquet solutions nearby. To be more precise we now give the following definition.

Definition 6.1. A Floquet solution \( f_0 \) of \( Ly = E_0 y \) is called a regular Floquet solution if there is a function \( f(E, x) \) such that \( f(E, \cdot) \) is a Floquet solution of \( Ly = E_0 y \), \( f(\cdot, x) \) is continuous in a vicinity of \( E_0 \), and \( \lim_{E \to E_0} f(E, x) = f_0(x) \).

The curve
\[
C = \{ (E, \rho) \in \mathbb{C} \times \mathbb{C}^* : \mathcal{F}(E, \rho) = 0 \}
\]
gives a twofold singular covering of the complex plane through the projection \( \pi : (E, \rho) \to E \). Desingularizing the curve \( C \) one obtains a Riemann surface \( M \). To be specific, local coordinates near a point \((E_0, \rho_0)\) are given in the following way.

1. If \( \text{tr} S(E_0)^2 - 4 \neq 0 \) then there exists a neighbourhood \( U \) of \((E_0, \rho_0)\) such that \((E, \rho) \in U \) if and only if \( \rho = f(E) \) where \( f \) is an analytic function with \( f(E_0) = \rho_0 \). This shows that \( \pi : (E, \rho) \to E \) is a local coordinate on \( U \).

2. If \( \text{tr} S(E_0)^2 - 4 = 0 \) and \( m(E_0, \rho_0) = p(E_0) \) is odd then there exists a neighbourhood \( U \) of \((E_0, \rho_0)\) such that \((E_0 + t, \rho) \in U \) if and only if \( \rho = f(t) \) where \( f \) is an analytic function with \( f(0) = \rho_0 \). Hence in this case \( (E, \rho) \to t = \sqrt{E - E_0} \) is a local coordinate on \( U \).

3. If \( \text{tr} S(E_0)^2 - 4 = 0 \) and \( m(E_0, \rho_0) = p(E_0) \) is even then there exist parametric disks \( U_1 \) and \( U_2 \) such that \( U_1 \cup U_2 \) is a neighbourhood of \((E_0, \rho_0)\) and \( U_1 \cap U_2 = \{ (E_0, \rho_0) \} \). Also \( (E, \rho) \in U_1 \) if and only if \( \rho = f_j(E) \) where \( f_j \) are analytic functions with \( f_j(E_0) = \rho_0 \) and \( \text{ord}_\rho f_j(t) - \rho_0 = \frac{1}{2} m(E_0, \rho_0) \). Hence \( \pi : (E, \rho) \to t \) serves as a local coordinate on \( U_1 \) and \( U_2 \).

Note that the projection \( (E, \rho) \to \rho \) is analytic on \( M \), that is, \( M \) is the Riemann surface of the two-valued function \( \rho(E) \).

We denote points on \( M \) by \((E, \rho, a)\) where \( a \in \{ \pm 1 \} \) determines the branch of the function \( \rho(\cdot) \). The value of \( a \) is redundant unless \( p(E) \) is even, since in this case only, a point \((E, \rho) \in C \) accounts for \( M \) on the Riemann surface \( M \).

Now let \( M_F \) be the topological space obtained from the Riemann surface \( M \) by identifying any two points lying above the same point \( E \) if \( p(E) > 2d(E) \). In other words, \( M_F \) is the topological space obtained from \( C \) by desingularizing all points
$E$ where $p(E) = 2d_i(E) > 0$. Thus $M_p$ has a branch point at $E$ when $p(E) - 2d_i(E)$ is an odd positive integer and a singular point when $p(E) - 2d_i(E)$ is an even positive integer. Let $B$ be the disjoint union of all regular Floquet solutions of $L_y = E y$ as $E$ varies in $C$, that is,

$$B = \{ (P,y) : P \in M_p, y \text{ is a regular Floquet solution of } L_y = (\pi P)y \}.$$  

A base of a topology in $B$ is given by sets of elements $(P,y)$ such that $P$ is in an open set in $M_p$ and $y(x_0), y'(x_0)$ are each in some open set of $C$ for some $x_0 \in R - \Sigma$. The projection $\hat{\pi} : B \to M_p$, $(P,y) \mapsto P$ is then continuous and the preimage of a point $P \in M_p$ under $\hat{\pi}$ is the set of all multiples of the regular Floquet solution associated with $P$ and hence a one-(complex)-dimensional vector space.

If $P_0 = (E_0, \rho_0, \sigma_0) \in M_p$ then there exists $x_0 \in R - \Sigma$ such that $F_{P_0}(E, x_0) \neq 0$ whenever $P = (E, \rho, \sigma)$ is in a vicinity $W$ of $P_0$. Define $h : \hat{\pi}^{-1}(W) \to W \times C$ by

$$(P,y) \mapsto ((E,\rho,\sigma),y(x_0)).$$

Then $h$ is a surjective, fibre-preserving homomorphism such that $h|_{\hat{\pi}^{-1}(P_0)}$ is a vector space isomorphism. Hence $(B,M_p,\hat{\pi})$ is a line bundle on $M_p$, that is, we have the following.

**Theorem 6.2.** The set of regular Floquet solutions of $L_y = E y$ for all $E \in C$, is a line bundle on $M_p$, the topological space obtained from the curve $\mathcal{F}(E,\rho) = 0$ by desingularization of all points where $p(E) = 2d_i(E) > 0$.

7. The KdV hierarchy and commuting differential expressions

Results in this section are due to Gel’fand and Dikii [11] and Al’ber [2] unless noted otherwise.

The KdV hierarchy is a sequence of certain non-linear partial differential equations usually defined in the following way. Let $\Omega$ be some open subset of $R$, $q \in C^\infty(\Omega)$, $A$ the differential operator on $C^\infty(\Omega)$ defined by $Ay = \frac{1}{4} y'' + q y'$, and $f_0 = 1$ on $\Omega$. For $k \geq 1$ define $f_k = Af_{k-1}$. Then it turns out that $f_k$ is the derivative of some polynomial of $q,q',\ldots,q^{[2k-3]}$ and integration constants $c_1,\ldots,c_k$. In fact, $f_k$ is homogeneous of weight $2k$ if $q^{[j]}$ is assigned weight $j + 2$ and $c_j$ is assigned weight $2j$. In this way one finds, for instance, that

$$f_3(c_1) = \frac{1}{4} q + c_1, \quad f_3(c_1, c_2) = \frac{1}{4} q'' + \frac{1}{2} q'^2 + \frac{1}{2} c_1 q + c_2.$$

Assume now that $q$ also depends on a real parameter $t$ in addition to its dependence on $x$, the variable in $\Omega$. We denote a derivative with respect to $t$ by a subscript $t$ and derivatives with respect to $x$ by primes. Then the collection of all equations $q_t = 2f_{k+1}$, where $g = 0, 1, 2, \ldots$, is called the KdV hierarchy. In particular, if $g = 1$ and $c_1 = c_2 = 0$, we obtain the original KdV equation

$$q_t = \frac{1}{4} q''' + \frac{1}{2} q q'.$$
Consider the differential expressions
\[ L = \frac{d^2}{dx^2} + q, \]
\[ P_{\epsilon_1, \ldots, \epsilon_q} = \sum_{j=0}^{q} \left[ -\frac{1}{2} f_j'(c_1, \ldots, c_j) + f_j(c_1, \ldots, c_j) \right] L^{e_j}. \]

Since \( Af_j = f_{j+1}' \), we obtain
\[ \left[ \frac{d^2}{dx^2} + f_j' \frac{d}{dx} \right] L = 2f_{j+1}' - 2f_j L \]
and hence \( [P_{\epsilon_1, \ldots, \epsilon_q}, L] = 2f_{j+1}'(c_1, \ldots, c_j) \). Therefore the equations of the KdV hierarchy have so-called Lax representations \( L = [P_{\epsilon_1, \ldots, \epsilon_q}, L] \) (Lax [20]).

Now define the function \( F_g(E, e) = \sum_{j=0}^{r} f_j(c_1, \ldots, c_j)E^j \) where \( e = (c_1, \ldots, c_r) \).

Since, as a function of \( E \), \( F_g \) is a monic polynomial of degree \( g \), the function
\[ R_{2g+1}(E, e) = (E - q)F_g(E, e)^2 - \frac{1}{2} F_g(E, e)F''_g(E, e) + \frac{1}{2} F'_g(E, e)^2 \]
is a monic polynomial in \( E \) of degree \( 2g + 1 \). In view of the definition of the functions \( f_j \) we find that
\[ R_{2g+1}(E, e) = 2F_g(E, e)(E F'_g(E, e) - AF_g(E, e)) = -2F_g(E, e)f_{g+1}'(c). \]

Hence, if \( q \) is a stationary solution of the equation \( a_j' = 2f_{j+1}' \), that is, if \( f_{j+1}' = 0 \), then \( R_{2g+1} \) is independent of \( x \). Moreover, when \( P = P_{\epsilon_1, \ldots, \epsilon_q} \) commutes with \( L \), we find that
\[ P^2 = \sum_{j=0}^{r} \left[ -\frac{1}{2} f_j' + f_j \frac{d}{dx} \right] P L^{e_j} \]
\[ = \sum_{j=0}^{r} \left[ -\frac{1}{2} f_j' + f_j \frac{d}{dx} \right] \left[ -\frac{1}{2} f_j + f_j \frac{d}{dx} \right] L^{2e_j - j} \]
\[ = \frac{1}{4} \sum_{j=0}^{r} \left( f_j f_j'' - 2f_j f_j'' + 4f_j f_j(L - q) \right) L^{2e_j - j} - R_{2g+1}(L). \]

Next introduce the notation \( Q_r = P_{0, \ldots, 0} \) where the number of subscripts of \( P_{0, \ldots, 0} \) is \( r \) and note that
\[ P_{\epsilon_1, \ldots, \epsilon_q} = \sum_{j=0}^{r} c_j Q_j \]
where \( c_0 = 1 \).

Now let \( P \) be a monic differential expression of order \( 2g + 1 \) which almost commutes with \( L \), that is, \([P, L] \) is multiplication with a function \( h \). Then \( S_1 = P - Q_r \) is a differential expression of order at most \( 2g \) and it satisfies \([S_1, L] = [P, L] - [Q_r, L] = h - 2f_{g+1}(0, \ldots, 0) \). Generally, if \( B = b_0 \frac{d}{dx} + \ldots + b_0 \) then
\[ [B, L] = -2b_0 \frac{d}{dx} + \frac{d}{dx} + \ldots \]
Thus, if \([B, L] \) is equal to a multiplication, we may conclude that \( b_0 \) and \( b_{g+1} \) must be constants. Hence there exist constants \( b_1 \) and \( c_1 \) and a differential expression
S_i of order at most 2g - 2 such that S_i = b_iL^k + c_iQ_{g-k} + S_{k+1}. By induction, one proves now that there exist constants b_1, ..., b_g and c_1, ..., c_g and differential expressions S_1, ..., S_{g+1} such that

\[
S_k = b_kL^{g+1-k} + c_kQ_{g-k} + S_{k+1}
\]

for k = 1, ..., g,

where S_1 has order at most 2(g + 1) - k and where

\[
[S_{k+1}, L] = h - 2 \sum_{j=0}^{g} c_{g-j} f_{j+1}(0, ..., 0).
\]

In particular, S_{g+1} is equal to a constant b_{g+1} and hence

\[
0 = [S_{g+1}, L] = h - 2 \sum_{j=0}^{g} c_{g-j} f_{j+1}(0, ..., 0) = h - 2f_{g+1}(c_1, ..., c_g).
\]

This proves that

\[
P = P_{c_1, ..., c_g} + \sum_{j=1}^{g+1} b_jL^{g+1-j} \quad \text{and} \quad [P, L] = 2f_{g+1}(c_1, ..., c_g).
\]

Hence, whenever [P, L] = 0, where P is a monic differential expression of order 2g + 1 and L = d^2/dx^2 + q, then q also satisfies [P_{c_1, ..., c_g}, L] = 0, that is, it is a stationary solution of some equation of the KdV hierarchy. Moreover, there exist polynomials F_g and R_{2g+1} with the above listed properties, in particular, P and L satisfy the algebraic relationship (P - K(L))^2 = R_{2g+1}(L) where K(L) = \sum_{j=1}^{g+1} b_jL^{g+1-j}. (Note that P - K(L) = P_{c_1, ..., c_g} is still a monic differential expression of order 2g + 1.)

Conversely, if L = d^2/dx^2 + q and there exist a monic differential expression P and polynomials K and R of orders k \leq g and 2g + 1 such that (P - K(L))^2 = R(L), then [P, L] = 0 and q is a stationary solution of some equation of the KdV hierarchy since

\[
0 = [(P - K(L))^2, L] = (P - K(L))[P, L] + [P, L](P - K(L)).
\]

Thus the following theorem, which is a special case of a result due to Burchall and Chaundy [6, 7], has been proved.

**Theorem 7.1.** Let L = d^2/dx^2 + q and suppose P is a monic differential expression of order 2g + 1. Then L and P are commutative if and only if there exist polynomials R and K of degree 2g + 1 and k \leq g, respectively, such that (P - K(L))^2 = R(L).

Because of this fundamental result, stationary solutions of equations in the KdV hierarchy are called algebro-geometric potentials (associated with the KdV hierarchy).

Another result in this circle of ideas is the following.

**Theorem 7.2.** Assume that F_g : \mathbb{C} \times \Omega \to \mathbb{C} is a monic polynomial of degree g in its first variable and that F_g^\prime, the derivative of F_g with respect to the second variable, exists and is locally absolutely continuous. If q : \Omega \to \mathbb{C} is such that

\[
(E - q(x))F_g(E, x)^2 - \frac{1}{2}F_g(E, x)F_g^\prime(E, x) + \frac{1}{2}F_g^\prime(E, x)^2 = 0
\]

(17)
is independent of \(x\) then \(q \in C^0(\Omega)\). Also the coefficients \(f_j\) of \(E^{-j}\) in \(F_k\) are in \(C^0(\Omega)\) and satisfy the recursion relations \(f_{j+1} = Af_j\) for \(j = 0, \ldots, g - 1\) and \(Af_g = 0\). Hence there exists a monic differential expression \(P_{2g+1}\) which commutes with \(L = d^2/dx^2 + q\).

**Proof.** The expression given in (17) is a monic polynomial in \(E\) of degree \(2g + 1\) whose coefficients are constants. We denote it by

\[
R_{2g+1}(E) = \sum_{j=0}^{2g+1} \hat{e}_{2g+1-j} E^j,
\]

where \(\hat{e}_0 = 1\). Comparing the coefficients of \(E^{2g+1-k}\) in (17) and (18) yields

\[
q = 2f_1 - \hat{e}_1
\]

for \(k = 1, \ldots, g + 1\). In (20) we have introduced \(g_{g+1} = 0\) for ease of notation.

By hypothesis \(f_1, \ldots, f_g \in C^1(\mathbb{R})\). Equation (19) now shows that \(q \in C^1(\Omega)\). Then the equations in (20) show recursively that \(f_k \in C^1(\mathbb{R})\) for \(k = 1, \ldots, g\). By induction, it follows that \(q\) and the functions \(f_1, \ldots, f_g\) are infinitely often differentiable. Thus we may differentiate (19) to obtain \(f_1 = Af_0\) and (20) to obtain, recursively, \(f_{j+1} = Af_j\) for \(j = 1, \ldots, g\). The final statement now follows as above.

8. Potentials with finite Floquet defect

Let \(L\) be the differential operator defined in §4. Recall that \(F_2 = \{E \in C: p(E) - 2d_j(E) > 0\}\) is the set of points \(E\) where less than two linearly independent regular Floquet solutions of \(L_y = E y\) exist and that \(F_1 = \{E \in F_2: d_j(E) = 0\}\) is the set of points \(E\) where \(L_y = E y\) has only one linearly independent Floquet solution. Also recall that \(U(E) = \langle \text{tr} S(E)^2 - 4 \rangle / D(E)^3\).

**Definition 8.1.** Let \(q\) be a periodic function satisfying Hypothesis 4.1. We will say that \(q\) or \(L = d^2/dx^2 + q\) has finite Floquet defect if the set \(F_2\) is finite. We also define the Floquet defect \(\text{def}(L)\) by

\[
\text{def}(L) = \sum_{E \in F_2} (p(E) - 2d_j(E)) = \deg(U).
\]

**Theorem 8.1.** Suppose \(q\) satisfies Hypothesis 4.1. If \(L = d^2/dx^2 + q\) has finite Floquet defect \(\text{def}(L) = 2g + 1\) then the following statements are true.

1. The defect \(\text{def}(L)\) is odd, that is, \(g\) is an integer.
2. The number of movable Dirichlet eigenvalues (with due respect to multiplicities) equals \(g\).
3. If \(\cos(\alpha) \neq 0\) and \(q\) is not constant then the number of movable eigenvalues
of \( T_{d,n} \) (accounting for multiplicities) is \( g + 1 \). (If \( q \) is constant then no eigenvalue is movable.)

4. The set \( \mathcal{S}(L) \) consists of finitely many regular analytic arcs.

5. The function \( q \in C^\infty(\mathbb{R} - \Sigma) \).

6. There exists a monic differential expression \( P \) of order \( 2g + 1 \) which commutes with \( L \) and satisfies

\[
P^2 = \prod_{E \in F_p} (L - E)^{p(E) - 2d(E)}.
\]

**Proof.** The asymptotic behaviour of Dirichlet and (semi-)periodic eigenvalues shows that \( p(E) \approx 2 \) and \( d(E, x_0) \approx 1 \) when \( |E| \) is suitably large. If \( E \) is a periodic or semi-periodic eigenvalue then \( \text{tr} \mathcal{S}(E) = \pm 2 \). Also, since \( \text{def}(L) \) is finite, we find that \( p(E) = 2d(E) = 2 \). If \( \mu \) is the Dirichlet eigenvalue close to \( E \) then, in fact, \( E = \mu \) and \( d_0(E, x_0) = 0 \). Hence, outside a large disk, there is no movable Dirichlet eigenvalue, that is, \( F_D(\cdot, x_0) \) is a polynomial. Denote its degree, the number of movable Dirichlet eigenvalues, by \( g \). By (11), \( U \) is a polynomial of degree \( 2g + 1 \).

Since \( \text{deg}(U) = \text{def}(L) \), we have proved Parts 1 and 2 of the theorem.

Part 3 now follows from (12), from \( D(E) = N(E) \), and from the fact that the movable eigenvalues of \( T_{d,n} \) are given by

\[F_D(E, x_0)(\sin \alpha)^2 - F_2(E, x_0)(\cos \alpha)^2 - F_D(E, x_0) \sin \alpha \cos \alpha,\]

a polynomial which is of degree \( g + 1 \) when \( \cos \alpha \neq 0 \).

Since asymptotically \( p(E) = 2 \), we find that \( p(E) = 1 \) or \( p(E) \geq 3 \) occurs at only finitely many places. Hence, by Theorem 4.2, there are only finitely many band edges, that is, \( \mathcal{S}(L) \) consists of finitely many arcs, which is Part 4 of the theorem.

Let \( \gamma(x_0) \) be the leading coefficient of \( F_D(\cdot, x_0) \). Equation (11) shows that \( 4\gamma(x_0)^2 \) is the leading coefficient of \( U \) and hence \( \gamma(x_0) \) is actually independent of \( x_0 \). Therefore \( F_D(\cdot, x_0) = F_D(\cdot, x_0) / 4\gamma(x_0) \) is a monic polynomial of degree \( g \) satisfying the hypotheses of Theorem 7.2 when \( \Omega = \mathbb{R} - \Sigma \). This proves that \( q \in C^\infty(\mathbb{R} - \Sigma) \) and that there exists a monic differential expression \( P \) of order \( 2g + 1 \) which commutes with \( L \). The expressions \( P \) and \( L \) satisfy \( P^2 = R_{2g+1}(L) \) where

\[R_{2g+1}(E) = U(E)(4\gamma(x_0)^2) \prod_{E \in F_p} (E - z)^{p(E) - 2d(E)},\]

which concludes Parts 5 and 6 of the theorem.

**Theorem 8.2.** Suppose \( q \) satisfies Hypothesis 4.1 and the differential equation \( Ly = E y \) has two linearly independent solutions for all but finitely many values of \( E \in C \). Then \( L \) has finite Floquet defect.

**Proof.** Assume that the Floquet defect is infinite. Therefore, and since \( p(E) \neq 2 \) eventually, there are infinitely many points \( E \) where \( p(E) > 2d(E) = 0 \), that is, \( F_1 \) is an infinite set by Theorem 5.4. This is in contradiction to the hypothesis which implies that \( F_1 \) is a finite set.

When \( q \) is continuous on \( \mathbb{R} \), Theorems 8.1 and 8.2 give Theorem 4.1 of [15].
Theorem 8.3. If $q$ is periodic, satisfies Hypothesis 3.1, and the associated Dirichlet operator has only $g$ movable eigenvalues, then $L$ has finite Floquet defect $\text{def}(L) = 2g + 1$.

Proof. If there are $g$ movable Dirichlet eigenvalues, that is, $\text{deg}(F_\mu(x, x_0)) = g$ then, by (11) $U = (\text{tr}(E)^2 - 4)/D(E)^2$ is a polynomial of degree $2g + 1$. Therefore $\text{def}(L) = \text{deg}(U)$ is finite.

Theorem 8.4. Suppose $q$ satisfies Hypothesis 4.1. If there exists a monic differential expression $P$ of order $2g + 1$, which commutes with $L = d^2y/dx^2 + q$, but none of smaller odd order, then $L$ has finite Floquet defect $\text{def}(L) = 2g + 1$.

Proof. Without loss of generality we may assume that $P = P_{\epsilon_1, \ldots, \epsilon_g}$ for suitable constants $\epsilon_j$. According to the results in § 7 there exists a monic polynomial $F_\mu(E) = \sum_{j=0}^g f_j E^j$ of order $g$ whose coefficients $f_j$ are in $C^g(\mathbb{R} - \Sigma)$ and periodic with period $p$. Hence $q \in C^g(\mathbb{R} - \Sigma)$ and $P$ is periodic with period $p$.

Denote the Dirichlet operator $T_{\mu, \epsilon_1, \ldots, \epsilon_g}$ by $T$ and let $\mu$ be a movable Dirichlet eigenvalue. Since $\mu$ is a continuous function of $x_0 \in \mathbb{R} - \Sigma$ and since it is not constant, there exists $x_0$ such that $\eta(\mu) = 0$. Suppose that for this choice of $x_0$ the eigenvalue $\mu$ has algebraic multiplicity $k$. Let $V = \ker(T - \mu)^k$ be the algebraic eigenspace of $\mu$. Then $V$ has a basis $\{y_1, \ldots, y_m\}$ such that $(T - \mu)y_j = y_{j-1}$ for $j = 1, \ldots, k$, where we agree that $y_0 = 0$. Also introduce $V_m$ the span of $\{y_1, \ldots, y_m\}$ and $V_0 = \{0\}$. First we show by induction that there exists a number $r$ such that $(S - \mu)y, (P - \nu)y \in V_{m-1}$ whenever $y \in V_m$.

Let $m = 1$. Then $(T - \mu)y = 0$ implies $y = \alpha y_1$ for some constant $\alpha$ and hence $y$ is a Floquet solution with multiplier $\mu = \mu(y_1, x_0, x_0 + p)$, that is, $(S - \mu)y = 0$. Since $P$ commutes with both $L$ and $S$, we find that $Py$ is also a Floquet solution with multiplier $\mu$. Since $p(\mu) = 0$, we have $\eta(\mu) \neq 0$ and hence $P_{\epsilon_1, \ldots, \epsilon_g} = qy$ for a suitable constant $q$.

Now assume that the statement is true for $1 \leq m < k$. Let $y = \sum_{j=1}^{m+1} \alpha_j y_j \in V_{m+1}$. Note that $(S - \mu)y$ satisfies Dirichlet boundary conditions. Hence

$$(T - \mu)(S - \mu)y = (S - \mu) \sum_{j=1}^{m+1} \alpha_j (T - \mu)y_j = \sum_{j=1}^{m+1} \alpha_j (S - \mu)y_{j-1}$$

is an element of $V_{m-1}$, say equal to $w = \sum_{j=1}^{m+1} \beta_j y_j$. The non-homogeneous equation $(T - \mu)y = w$ has the general solution

$$w = \sum_{j=1}^{m+1} \beta_j y_{j+1} + cy_1$$

where $c$ is an arbitrary constant. Since $w$ is in $V_m$, the particular solution $(S - \mu)y$ of $(T - \mu)y = w$ is in $V_{m+1}$, too.

Since $q$ is infinitely often differentiable, so are the functions $y_1, \ldots, y_k$. Hence $(P - \nu)y$ is in the domain of definition of $L$ and, in a similar way to before,

$$(L - \mu)(P - \nu)y = \sum_{j=1}^{m+1} \alpha_j (P - \nu)y_{j-1} = \sum_{j=1}^{m+1} \gamma_j y_j$$

Therefore $\text{def}(L) = \text{deg}(U)$ is finite.
for suitable constants $\gamma_j$. Hence there are numbers $c_1$ and $c_2$ such that

$$(P - \nu)y = \sum_{j=1}^{m-1} \gamma_j y_{j+1} + c_1 \epsilon (\mu, x_0, \cdot) + c_2 g(\mu, x_0, \cdot).$$

Note that $(Py - py)(x_0) = c_1$. Let $w = (S - \rho)y$ and $v = (P - \nu)w$. Then $w \in V_m$ and $v \in V_m$. Hence

$$(Py - py)(x_0 + p) = (S(P - \nu)y)(x_0) = (P - \nu)(P - \nu)(x_0) = \rho(Py)(x_0) = \rho con.$$

On the other hand, $(Py - py)(x_0 + p) = c_1(\rho - 1/p)$ since $c(\mu, x_0, x_0 + p) = 1/p$. Hence $c_1 = 0$ and $(P - \nu)w \in V_m$

Thus we have shown that $S$ and $P$ map $V$ into itself. In particular $(Py)(x_0) = 0$ for every $y \in V$.

Next observe that the functions $y_1, \ldots, y_k$ defined above satisfy $(L - \mu)^j y_m = y_m$, where we agree that $y_m = 0$ when $m \leq 0$. Hence

$$L' y_m = \sum_{j=0}^{k-1} \left( \frac{d}{dx} \right)^j (L - \mu)^j y_m = \sum_{j=0}^{k-1} \left( \frac{d}{dx} \right)^j y_m = 0.$$

Moreover,

$$P_{y_m} = \sum_{j=0}^{k-1} \left( \frac{d}{dx} \right)^j y_m = \sum_{j=0}^{k-1} \left( \frac{d}{dx} \right)^j y_m$$

Since $(Py)(x_0) = y_j(x_0) = 0$, evaluating this at $x_0$ yields

$$0 = (Py_m)(x_0) = \sum_{\ell=0}^{g} y_{m-\ell}(x_0) \sum_{n=\ell}^{g} f_n \mu^{n-\ell} g_{n-\ell}(x_0)$$

Therefore there can be at most $g$ movable Dirichlet eigenvalues counting multiplicities. However, if there were less than $g$ movable Dirichlet eigenvalues then, by the preceding two theorems, there would exist a differential expression of odd order less than $2g + 1$ which commutes with $L$. Hence there are precisely $g$ movable Dirichlet eigenvalues and det$(L) = 2g + 1$.

The following example shows that the Floquet defect is not always finite even when $\mathcal{S}(L)$ consists of finitely many arcs. Let

$$q(x) = e^{2i\lambda x}.$$
which is a periodic function with period $p = \pi$. A fundamental system of solutions of $Ly = y'' + qy = -\nu^2 y$ is given by $y_1(x) = J_\nu(\nu^2 x)$ and $y_2(x) = Y_\nu(\nu^2 x)$.

By (24) we obtain that $y_1$ is a Floquet solution of $Ly = -\nu^2 y$ with Floquet multiplier $\rho = e^{2\pi i \nu}$. This multiplier has absolute value 1 if and only if $\nu$ is real. Hence we obtain that $\mathcal{F}(L) = (-\infty, 0]$, that is, the conditional stability set which coincides with the spectrum of the maximal operator associated with $L$ in $L^2(\mathbb{R})$ consists of one band only. In fact, $\mathcal{F}(L) = \mathcal{F}(d^2/dx^2)$. Also, $\rho = 1$ if and only if $\nu$ is an even integer and $\rho = -1$ if and only if $\nu$ is an odd integer, that is, the zeros of $\text{tr}(E)^2 - 4$ are given by $-n^2$ where $n \in \mathbb{N}_0$. In view of the asymptotic behaviour of these zeros given in Theorem 4.3, we can even conclude that $p(0) = 1$ and $p(-n^2) = 2$ when $n \in \mathbb{N}$.

Next observe that

$$s(-\nu^2, x_0, x) = \frac{1}{2\pi} \left[ (\nu^4 x^4) Y_\nu(\nu^4 x) - J_\nu(\nu^4 x) Y_\nu(\nu^4 x) \right].$$

Since $J_{-\nu}(z) = J_\nu(z) \cos(\nu \pi) + Y_\nu(z) \sin(\nu \pi)$, it turns out that

$$s(-\nu^2, x_0, x_0 + \pi) = \pi J_\nu(\nu^4 x_0) J_{\nu}(\nu^4 x_0)$$

is an entire function in $x_0$. Hence, for no value of $E$ is $s(E, \cdot + \pi)$ identically equal to zero, that is, $d_f(E) = 0$ for every $E \in \mathbb{C}$. Consequently, the Floquet defect of $d^2/dx^2 + e^{2\pi i}$ is infinite.

**Appendix. Asymptotics of Bessel functions**

Some well-known results on Bessel functions and a few immediate corollaries are listed in the following for the convenience of the reader. For a reference see, for example, Abramowitz and Stegun [1].

The Bessel functions $J_\nu$ and $Y_\nu$ and their derivatives may be expressed by

$$J_\nu(z) = \sqrt{\frac{2}{\pi z}} (P_\nu(z) \cos \chi - Q_\nu(z) \sin \chi),$$

$$J'_\nu(z) = \sqrt{\frac{2}{\pi z}} (-P_\nu(z) \sin \chi - S_\nu(z) \cos \chi),$$

$$Y_\nu(z) = \sqrt{\frac{2}{\pi z}} (P_\nu(z) \cos \chi + Q_\nu(z) \sin \chi),$$

$$Y'_\nu(z) = \sqrt{\frac{2}{\pi z}} (R_\nu(z) \cos \chi - S_\nu(z) \sin \chi),$$

where $\chi = z - \frac{1}{2} \nu \pi - \frac{1}{4} \pi$. The functions $P_\nu, R_\nu, Q_\nu$, and $S_\nu$ have the following asymptotic behaviour:

$$P_\nu(z), R_\nu(z) \sim 1 + O(z^{-2}),$$

$$Q_\nu(z) = \frac{4\nu^2 - 1}{8z} + O(z^{-3}),$$

$$S_\nu(z) = \frac{4\nu^2 + 3}{8z} + O(z^{-3}),$$

as $z$ tends to infinity if $|\arg z| < \pi$. 

When \( m \) is an integer the formulas
\[
J_n(e^{i\nu}) = e^{i\nu}J_n(w),
\]
\[
Y_n(e^{i\nu}) = e^{-i\nu}Y_n(w) + 2i\sin(m\nu)\cot(\nu)J_n(w),
\]
give analytic continuations of the Bessel functions to values of \( z \) where \(|\arg z| > \pi\).

In particular, if \( z = re^{i\phi} \), one obtains
\[
P_n(z) = P_n(w) + i\cos(\nu w)e^{-2\nu}(P_n(w) - iQ_n(w)),
\]
\[
Q_n(z) = -Q_n(w) + \cos(\nu w)e^{-2\nu}(P_n(w) - iQ_n(w)),
\]
\[
R_n(z) = R_n(w) - i\cos(\nu w)e^{-2\nu}(R_n(w) - iS_n(w)),
\]
\[
S_n(z) = -S_n(w) + \cos(\nu w)e^{-2\nu}(R_n(w) - iS_n(w)).
\]

This implies that \( P_n(z), R_n(z), Q_n(z), \) and \( S_n(z) \) are bounded as long as \( z \) is bounded away from zero and \( \arg(z) \in [-\pi, \pi] \).

Hence the functions
\[
f_1(z, z_0) = P_n(z)P_n(z_0) + Q_n(z)Q_n(z_0),
\]
\[
f_2(z, z_0) = P_n(z)R_n(z_0) + Q_n(z)S_n(z_0),
\]
\[
f_3(z, z_0) = R_n(z)R_n(z_0) + S_n(z)S_n(z_0),
\]
\[
g_1(z, z_0) = P_n(z)Q_n(z_0) - Q_n(z)P_n(z_0),
\]
\[
g_2(z, z_0) = P_n(z)S_n(z_0) - Q_n(z)R_n(z_0),
\]
\[
g_3(z, z_0) = R_n(z)S_n(z_0) - S_n(z)R_n(z_0),
\]
are bounded by some constant for all \( z, z_0 \) as long as \( z \) and \( z_0 \) are bounded away from zero and \( \arg(z), \arg(z_0) \in [-\pi, \pi] \).

If \( \cos(\nu w) = 0 \), that is, if \( 2\nu = 2n + 1 \) where \( n \) is an integer, then the spherical Bessel functions \( j_n(z) = \sqrt{\pi/(2z)}J_n(z) \) and \( y_n(z) = \sqrt{\pi/(2z)}Y_n(z) \) are elementary functions. In fact, when \( n \gg 0 \),
\[
j_n(z) = \frac{1}{z}(P_n(z)\sin(z - \frac{1}{2}i\pi) + Q_n(z)\cos(z - \frac{1}{2}i\pi)),
\]
\[
y_n(z) = \frac{1}{z}(P_n(z)\cos(z - \frac{1}{2}i\pi) + Q_n(z)\sin(z - \frac{1}{2}i\pi)),
\]
\[
j_{n+1}(z) = \frac{(-1)^n}{z}(P_n(z)\cos(z - \frac{1}{2}i\pi) - Q_n(z)\sin(z - \frac{1}{2}i\pi)),
\]
\[
y_{n+1}(z) = \frac{(-1)^n}{z}(P_n(z)\sin(z - \frac{1}{2}i\pi) + Q_n(z)\cos(z - \frac{1}{2}i\pi))
\]
and \( P_n(z) \) is an even polynomial in \( 1/z \) while \( Q_n(z) \) is an odd polynomial in \( 1/z \). It follows that the asymptotic expressions (21)–(23) hold as \( z \) tends to infinity regardless of the argument of \( z \).

Acknowledgments. Several years of highly satisfying collaboration with F. Gesztesy still have a large impact on this paper. Thanks are also due to G. Weinstein for pointing out a few mistakes.
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