ON THE INVERSE RESONANCE PROBLEM

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I am reporting on joint work with

- Christer Bennewitz (Lund)
- Matthew Bledsoe (Birmingham, AL)
- Malcolm Brown (Cardiff)
- Ian Knowles (UAB)
- Marco Marletta (Cardiff)
- Sergey Naboko (St. Petersburg)
- Roman Shterenberg (UAB)

Introduction and History

Ambarzumian

"Über eine Frage der Eigenwerttheorie" (1928):

> *Wenn das Spektrum die Differentialgleichung wirklich vollständig definiert[e], so wäre es möglich, z. B. den Aufbau irgend eines Atomsystems praktisch aus dem Spektrum zu bestimmen, d. h. die Aufgabe zu lösen, welche sozusagen reziprok zum Schrödingerschen Problem steht.*

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- Unless mentioned otherwise, we assume below a Dirichlet condition at zero.

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- Borg (1946) showed that, in general, two sets of eigenvalues are needed to identify a potential on an interval uniquely.
- Levinson (1949) and Marchenko (1950) used different sets of data: in addition to one set of (Dirichlet) eigenvalues one needs also either Neumann data of the eigenfunctions or the norming constants of the eigenfunctions.

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- Gelfand-Levitan (1951): the spectral function ρ determines q uniquely.

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- For fixed x: $\psi(\cdot, x)$ is analytic in the open upper half plane and continuous in the closed upper half plane.
- Jost function: $\psi(\cdot, 0)$
- Marchenko (1955): eigenvalues, norming constants, and scattering phase ($2i\delta(k) = \overline{\psi(k, 0)}/\psi(k, 0)$) determine q uniquely.

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- Both are relevant/visible in spectroscopy.

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- It appears this went unnoticed for more than 40 years until Korotyaev (2000) and Zworski (2001/1988) pointed it out.

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- **Large resonances are physically insignificant.**
- Question: How can we state (and prove) this mathematically?

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Results

Stability for the inverse resonance theorem

Theorem (Marletta, Shterenberg, W.; Commun. Math. Phys. 2010)

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- *Inside a disc of radius R all resonances and eigenvalues of q are ε-close to those of $\tilde{q}$*

Then

$$\sup_{x \in [0,1]} \left| \int_x^1 (q - \tilde{q}) dx \right| \leq f(\varepsilon, R)$$

where $f(\varepsilon, R) \to 0$ as $R \to \infty$ but $\varepsilon R^{1/6} \to 0$.

Stability in the case of a compact interval

- Ryabushko (1983): Suppose q_0 and q are real and have zero average. Then

$$\|q - q_0\|_{L^2} \leq C\left(\|\lambda(q) - \lambda(q_0)\|_{\ell^2} + \|\mu(q) - \mu(q_0)\|_{\ell^2}\right)$$

where C depends on $\|q\|_2$ and $\|q_0\|_2$.

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  where C depends on $\|q\|_2$ and $\|q_0\|_2$.
- McLaughlin (1988) has a similar estimate involving one spectrum and norming constants.
- Marletta and myself (2005) gave an estimate (in terms of N and ε) on

$$\left|\int_0^x (q - q_0)dt\right| \le f(\varepsilon, N)$$

  where $f(\varepsilon, N) \to 0$ as $N \to \infty$ but $\varepsilon \log N \to 0$ provided that $2N$ eigenvalues are known up to an error ε.

Comparison of eigenvalues

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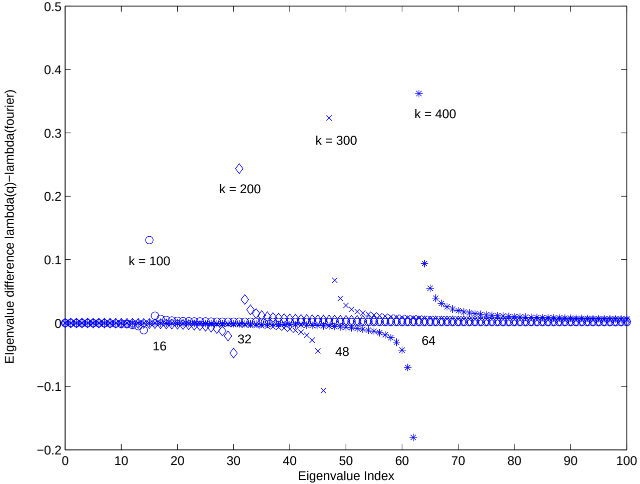

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- Marletta, Naboko, Shterenberg, W. (J. Anal. Math. 2011): Stability for several classes of Jacobi operators: Spectrum is (i) all of $\mathbb{R}$, (ii) a half-line, or (iii) one finite interval.

Full line problems

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- Bledsoe (Inverse Problems 2012): continuous case
- Eigenvalues and resonances, i.e., the poles of the reflection coefficient, do not yet determine it.
- The zeros of the reflection coefficient are also needed.

Left-definite operators

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  - Bennewitz, Brown, W. (J. Differential Equations 2012) for the full-line case
- The inverse resonance problem for the half-line case was treated by Bledsoe, W. (J. Math. Anal. Appl., 2015)

Outline of the proof

Transformation Operators

The Jost solutions associated with q and $\tilde{q}$ are related by

$$\tilde{\psi}(z,x) = \psi(z,x) + \int_x^{2-x} K(x,t) psi(z,t) dt$$

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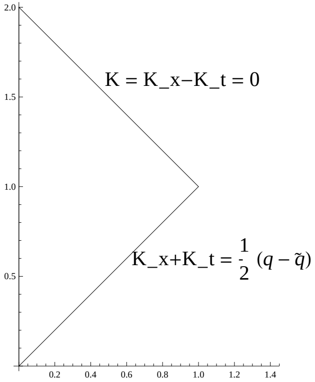

Transformation Operators

The Jost solutions associated with q and $\tilde{q}$ are related by

$$\tilde{\psi}(z,x) = \psi(z,x) + \int_x^{2-x} K(x,t) psi(z,t) dt$$

where K satisfies the wave equation

$$K_{xx}(x,t) - K_{tt}(x,t)$$
$$= (\tilde{q}(x) - q(t))K(x,t)$$

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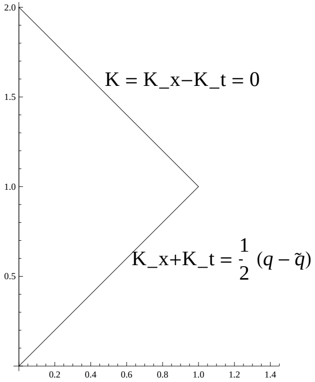
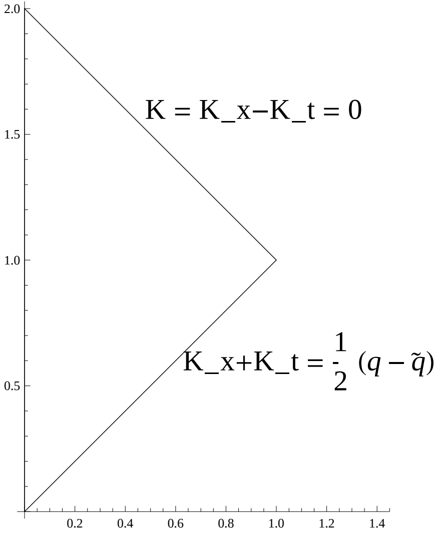


We need to estimate
$K(x,x) = \frac{1}{2}\int_x^1 (\tilde{q}(s) - q(s)) ds.$

The wave equation



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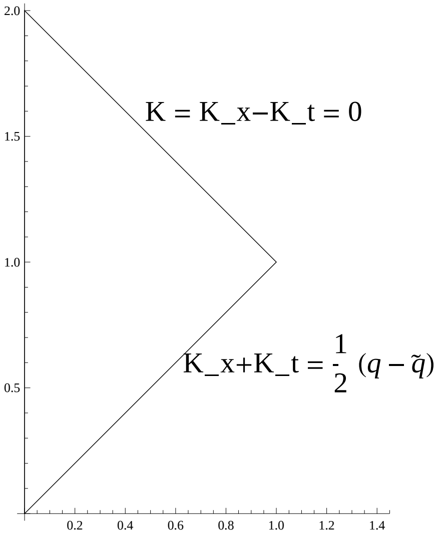

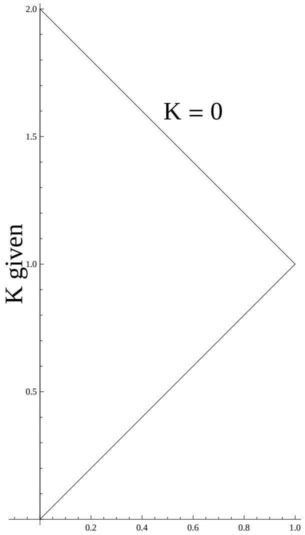

Solving the wave equation

The wave equation may be solved uniquely knowing $K(0,t)$, $0 \leq t \leq 2$ and the fact that $K(x, 2-x) = 0$.
Iteration:

$$K_0(x,t) = K(0, x+t)$$

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We need to estimate $K(0,t)$.

Connecting with Jost functions I

$$\begin{tikzcd}
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- If f is entire of growth order at most one, then

$$f(z) = z^{n_0} \mathrm{e}^{a+bz} \prod_{n=1}^{\infty} (1 - z/z_n) \mathrm{e}^{z/z_n}.$$

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- $|\Pi(R,z) - 1| \leq C|z|^2/R$ when $2|z| < R$.
- This provides an estimate for $|z| < R^{1/3}$: $\psi(z,0)/\tilde{\psi}(z,0) \approx 1$ and hence

$$|\psi(z,0) - \tilde{\psi}(z,0)| \leq CR^{-1/3}.$$

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If $|z|$ is small: $E(z) = (1-z)\mathrm{e}^z \approx 1 - z^2$, In fact, $|\log E(z)| \leq 2|z|^2$

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Estimate for $|z| \geq R^{1/6}$

- $(K_{\tilde{q}} - K_q)(0,t) = h(t) + \frac{1}{2}\int_{t/2}^1 (\tilde{q} - q)$ where h, h' is AC on $[0,2]$.

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- Here one needs the assumption that $\tilde q - q$ be in L^p.

Open problem

Two (or more) spectral bands

Suppose $q(x) = -2\wp(x+\omega)$. The spectrum of the associated Schrödinger operator has only absolutely continuous spectrum with one gap. All solutions of $-y'' + qy = \lambda y$ are explicitly known.
The inverse of the map $\wp(z) = \lambda$ maps the energy (λ) plane to parallelogram (the fundamental domain of $\wp$) in a one-to-two fashion.
Compactly supported perturbations do not change the essential spectrum but introduce eigenvalues and resonances ....

Thank you for your attention!