ON THE INVERSE RESONANCE PROBLEM

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50 Congreso Nacional de la Sociedad Matemática Mexicana

October 25, 2017
I am reporting on joint work with

- Christer Bennewitz (Lund)
- Matthew Bledsoe (Birmingham, AL)
- Malcolm Brown (Cardiff)
- Ian Knowles (UAB)
- Marco Marletta (Cardiff)
- Sergey Naboko (St. Petersburg)
- Roman Shterenberg (UAB)
Introduction and History
“Über eine Frage der Eigenwerttheorie” (1928):

If indeed the spectrum [of eigenvalues] defined the differential equation completely, it would be possible, for instance, to determine practically the structure of a system of atoms from the spectrum, i.e., to solve the problem which is, so to speak, reciprocal to Schrödinger’s problem.
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\( q \) is a locally integrable function on \([0, b)\) where \( 0 < b \leq \infty \).
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  - Dirichlet condition: $y(0) = 0$
  - Neumann condition: $y'(0) = 0$
  - Robin condition: $y(0) \cos \alpha + y'(0) \sin \alpha = 0$
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Ambarzumian and Borg

- Ambarzumian treats only a (very) special case: the Schrödinger equation on a finite interval with continuous $q$.

- If the Neumann-Neumann eigenvalues are those for potential 0, then $q$ must be 0.

- Borg (1946) showed that, in general, two sets of eigenvalues are needed to identify a potential on an interval uniquely.

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$$m(\lambda) = A\lambda + B + \int_{\mathbb{R}} \left( \frac{1}{t - \lambda} - \frac{t}{1 + t^2} \right) d\rho(t)$$
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• \( \rho \) is called the spectral measure, it is uniquely determined by \( m \).
• \( m \) determines eigenvalues and continuous spectrum as those points where it seizes to be analytic.
• Gelfand-Levitan (1951): the spectral function \( \rho \) determines \( q \) uniquely.
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- Jost function: $\psi(\cdot, 0)$
- Marchenko (1955): eigenvalues, norming constants, and scattering phase $(2i\delta(k) = \psi(k, 0)/\psi(k, 0))$ determine $q$ uniquely.
Resonances
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• If $q$ decays super-exponentially, $\psi(\cdot, x)$ can be analytically continued to the lower half of the $k$-plane (and $m$ through the continuous spectrum to a second sheet of a Riemann surface).

• Recall: if $\text{Im}(k) > 0$ and $\psi(k, 0) = 0$ then $k^2$ is an eigenvalue with eigenfunction $\psi(k, \cdot)$.

• If $\text{Im}(k) \leq 0$ and $\psi(k, 0) = 0$ then $k^2$ is a resonance. In this case $\psi(k, \cdot)$ satisfies the differential equation and the boundary condition at 0 but is not square integrable.

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The inverse resonance theorem

Theorem
Suppose that \( q \) is compactly supported. Then the location of all eigenvalues and resonances determines \( q \) uniquely.

- The Jost function extends to an entire function of growth order 1 in \( k \).
- Hadamard's factorization theorem gives \( \psi(\cdot,0) \) up to a factor \( e^{ak+b} \).
- \( a \) and \( b \) are determined from asymptotics as \( k \) tends to \( \infty \) along the positive imaginary axis (\( \psi(k,0) \sim 1 \) independently of \( q \)).
- The claim follows immediately from Marchenko's scattering theorem (norming constants are \( -i \dot{\psi}(k,0)/\psi(-k,0) \), scattering phase is \( \psi(k,0)/(2i\psi(k,0)) \)).
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Asymptotic distribution of resonances

- The uniqueness theorem requires knowledge of ALL eigenvalues and resonances.

- If $q$ is supported on $[0, R]$, absolutely continuous on $[0, R]$, and has a jump discontinuity at $R$, then the resonances are asymptotic to the curve given by
  \[ \text{Im}(z) = -\frac{1}{R} \ln(|\text{Re}(z)|) + \frac{1}{2} R \ln\left(\frac{|q(R)|}{4}\right). \]

- Small changes in $R$ or $q(R)$ produce different asymptotics.

- Large resonances are physically insignificant.

Question: How can we state (and prove) this mathematically?
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Recovery from finite data (for compact intervals)

- In practice it is impossible to know infinitely many eigenvalues or to know them precisely.

Hochstadt (1973) first poses the question what can be said when finite data are given.

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Results
Stability for the inverse resonance theorem


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Then

$$\sup_{x \in [0,1]} \left| \int_x^1 (q - \tilde{q}) dx \right| \leq f(\varepsilon, R)$$

where $f(\varepsilon, R) \to 0$ as $R \to \infty$ but $\varepsilon R^{1/6} \to 0$. 
Stability in the case of a compact interval

- Ryabushko (1983): Suppose $q_0$ and $q$ are real and have zero average. Then

$$\| q - q_0 \|_{L^2} \leq C \left( \| \lambda(q) - \lambda(q_0) \|_{\ell^2} + \| \mu(q) - \mu(q_0) \|_{\ell^2} \right)$$

where $C$ depends on $\| q \|_2$ and $\| q_0 \|_2$. 

- McLaughlin (1988) has a similar estimate involving one spectrum and norming constants.

- Marletta and myself (2005) gave an estimate (in terms of $N$ and $\varepsilon$)

$$\left| \left| \int_0^x (q - q_0) \, dt \right| \right| \leq f(\varepsilon, N)$$

where $f(\varepsilon, N) \to 0$ as $N \to \infty$ but $\varepsilon \log N \to 0$ provided that $2N$ eigenvalues are known up to an error $\varepsilon$. 

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Discrete problems


- Brown, Naboko, W. (Constructive Approximation 2009): Uniqueness for Hermite operators \( \sqrt{\frac{n}{n+1}} y_n + \frac{b}{\sqrt{n+1}} y_{n+1} + \sqrt{\frac{n}{n+1}} y_{n+1} + 1 \).


- Marletta, Naboko, Shterenberg, W. (J. Anal. Math. 2011): Stability for several classes of Jacobi operators: Spectrum is (i) all of \( \mathbb{R} \), (ii) a half-line, or (iii) one finite interval.
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Discrete problems


- Brown, Naboko, W. (Constructive Approximation 2009): Uniqueness for Hermite operators \((\sqrt{n}y_{n-1} + b_n y_n + \sqrt{n+1}y_{n+1})\).


- Marletta, Naboko, Shterenberg, W. (J. Anal. Math. 2011): Stability for several classes of Jacobi operators: Spectrum is (i) all of \(\mathbb{R}\), (ii) a half-line, or (iii) one finite interval.
Full line problems

- Bledsoe (IEOT 2012): discrete case
- Eigenvalues and resonances, i.e., the poles of the reflection coefficient, do not yet determine it.
- The zeros of the reflection coefficient are also needed.
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Left-definite operators

- If $q \geq 0$ but no requirement on the sign of $w$ is made one can develop a spectral and scattering theory for

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- Underlying Hilbert space: \( \{ y \in AC_{loc} : y', \sqrt{q}y \in L^2 \} \)
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Outline of the proof
Transformation Operators

The Jost solutions associated with $q$ and $\tilde{q}$ are related by

$$\tilde{\psi}(z, x) = \psi(z, x) + \int_x^{2-x} K(x, t)\psi(z, t)dt$$

where $K$ satisfies the wave equation

We need to estimate $K(x, x) = \frac{1}{2} \int_x^1 (\tilde{q}(s) - q(s))ds$. 

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$$K_x - K_t = 0$$

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Transformation Operators

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The wave equation

\[ K = K_x - K_t = 0 \]

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The wave equation

\[ K = K_x - K_t = 0 \]

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\[ K = 0 \]
The wave equation may be solved uniquely knowing $K(0, t)$, $0 \leq t \leq 2$ and the fact that $K(x, 2 - x) = 0$.

Iteration:

$$K_0(x, t) = K(0, x + t)$$
The wave equation may be solved uniquely knowing \( K(0, t) \), \( 0 \leq t \leq 2 \) and the fact that \( K(x, 2 - x) = 0 \).

Iteration:

\[
K_0(x, t) = K(0, x + t)
\]

\[
K_{n+1}(x, t) = \int_{(t-x)/2}^{(t+x)/2} \int_{(t-x)/2}^{(t+x)/2} \left( q(\alpha + \beta) - \tilde{q}(\alpha - \beta) \right) K_n(\alpha - \beta, \alpha + \beta) d\beta d\alpha
\]

\[
K(x, t) = \sum_{n=0}^{\infty} K_n(x, t)
\]
Solving the wave equation

The wave equation may be solved uniquely knowing $K(0, t)$, $0 \leq t \leq 2$ and the fact that $K(x, 2 - x) = 0$.

Iteration:

$$K_0(x, t) = K(0, x + t)$$

$$K_{n+1}(x, t) = \int_{(t+x)/2}^{1} \int_{(t-x)/2}^{(t+x)/2} (q(\alpha + \beta) - \tilde{q}(\alpha - \beta))K_n(\alpha - \beta, \alpha + \beta)d\beta d\alpha$$

$$K(x, t) = \sum_{n=0}^{\infty} K_n(x, t)$$

We need to estimate $K(0, t)$.  

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Connecting with Jost functions I

\[ 0 \]

\[ L_q = K_q^{-1} \quad K_{\tilde{q}} \]

\[ q \rightarrow K \rightarrow \tilde{q} \]

We need to estimate \((K_{\tilde{q}} - K_q)(0, t)\).
Connecting with Jost functions

$0$

$L_q = K_q^{-1}$

$K_\tilde{q}$

$q$ \quad $K$ \quad $\tilde{q}$

\[ K(0, t) = (K_\tilde{q} - K_q)(0, t) + \int_0^t (K_\tilde{q} - K_q)(0, s)L_q(s, t)ds \]

We need to estimate $(K_\tilde{q} - K_q)(0, t)$. 

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On the inverse resonance problem 
June 1, 2017
We need to estimate \((K\bar{q} - K_q)(0, t)\).
\begin{itemize}
  \item \( \psi(z, 0) = 1 + \int_0^2 K_q(0, t)e^{izt} \, dt \)
  \item \( \tilde{\psi}(z, 0) = 1 + \int_0^2 K_{\tilde{q}}(0, t)e^{izt} \, dt \)
  \item \( \tilde{\psi}(z, 0) - \psi(z, 0) = \int_0^2 (K_{\tilde{q}} - K_q)(0, t)e^{izt} \, dt \)
  \item \( (K_{\tilde{q}} - K_q)(0, t) = \frac{1}{2\pi} \int_{\mathbb{R}} (\tilde{\psi} - \psi)(z, 0)e^{-izt} \, dz \)
\end{itemize}

We need to estimate \( (\tilde{\psi} - \psi)(z, 0) \).
• \( \psi(z, 0) = 1 + \int_0^2 K_q(0, t)e^{itz} \, dt \)

• \( \tilde{\psi}(z, 0) = 1 + \int_0^2 K_{\tilde{q}}(0, t)e^{itz} \, dt \)
• $\psi(z, 0) = 1 + \int_0^2 K_q(0, t)e^{izt} dt$

• $\tilde{\psi}(z, 0) = 1 + \int_0^2 K_{\tilde{\alpha}}(0, t)e^{izt} dt$

• $\tilde{\psi}(z, 0) - \psi(z, 0) = \int_0^2 (K_{\tilde{\alpha}} - K_q)(0, t)e^{izt} dt$
Connecting with Jost functions II

\[ \psi(z, 0) = 1 + \int_0^2 K_q(0, t)e^{izt} dt \]
\[ \tilde{\psi}(z, 0) = 1 + \int_0^2 K_{\tilde{q}}(0, t)e^{izt} dt \]
\[ \tilde{\psi}(z, 0) - \psi(z, 0) = \int_0^2 (K_{\tilde{q}} - K_q)(0, t)e^{izt} dt \]
\[ (K_{\tilde{q}} - K_q)(0, t) = \frac{1}{2\pi} \int_{\mathbb{R}} (\tilde{\psi} - \psi)(z, 0)e^{-izt} dz \]
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We need to estimate \( (\tilde{\psi} - \psi)(z, 0) \).
Hadamard’s factorization theorem

- If $f$ is entire of growth order at most one, then

$$f(z) = z^{n_0} e^{a+bz} \prod_{n=1}^{\infty} (1 - z/z_n) e^{z/z_n}.$$
Hadamard’s factorization theorem

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  f(z) = z^{n_0} e^{a+bz} \prod_{n=1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}.
  \]

- If the first \( N(R) \sim 2eR \) zeros coincide
  \[
  \frac{\psi(z,0)}{\tilde{\psi}(z,0)} = e^{(a-\tilde{a})z+\tilde{b}} \frac{\Pi(R,z)}{\tilde{\Pi}(R,z)}
  \]
  where
  \[
  \Pi(R,z) = \prod_{n=N(R)+1}^{\infty} \left(1 - \frac{z}{z_n}\right) e^{z/z_n}.
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Hadamard’s factorization theorem

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• If the first $N(R) \sim 2eR$ zeros coincide

$$\frac{\psi(z,0)}{\tilde{\psi}(z,0)} = e^{(a-\bar{a})z+b-\bar{b}} \frac{\Pi(R, z)}{\tilde{\Pi}(R, z)}$$

where

$$\Pi(R, z) = \prod_{n=N(R)+1}^{\infty} \left(1 - \frac{z}{z_n}\right)e^{z/z_n}.$$  

• $|\Pi(R, z) - 1| \leq C|z|^2/R$ when $2|z| < R.$
Hadamard’s factorization theorem

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- $|\Pi(R, z) - 1| \leq C|z|^2/R$ when $2|z| < R$.

- This provides an estimate for $|z| < R^{1/3}$: $\psi(z,0)/\tilde{\psi}(z,0) \approx 1$ and hence
  \[ |\psi(z,0) - \tilde{\psi}(z,0)| \leq CR^{-1/3}. \]
If $|z|$ is small: $E(z) = (1 - z)e^z \approx 1 - z^2$, in fact, $|\log E(z)| \leq 2|z|^2$
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$$|\log \Pi(R, z)|$$
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If $|z|$ is small: $E(z) = (1 - z)e^z \approx 1 - z^2$, in fact, $|\log E(z)| \leq 2|z|^2$

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Details

If \(|z|\) is small: \(E(z) = (1 - z)e^z \approx 1 - z^2\), In fact, \(|\log E(z)| \leq 2|z|^2\)

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\[|\Pi(R, z) - 1| \leq |\log \Pi(R, z)| \exp(|\log \Pi(R, z)|)\]

\[|\log \Pi(R, z)| \leq \sum |\log E(z/z_n)| \leq 2|z|^2 \sum |z_n|^{-2}\]

\[= 2|z|^2 \int_R^\infty \frac{dN(t)}{t^2} = 4|z|^2 \int_R^\infty \frac{N(t)}{t^3} dt = O(|z|^2/R)\]
Estimate for $|z| \geq R^{1/6}$

- $(K_{\tilde{q}} - K_q)(0, t) = h(t) + \frac{1}{2} \int_{t/2}^{1} (\tilde{q} - q)$ where $h, h'$ is AC on $[0, 2]$. 
Estimate for $|z| \geq R^{1/6}$

- $(K\bar{q} - Kq)(0, t) = h(t) + \frac{1}{2} \int_{t/2}^{1} (\bar{q} - q)$ where $h, h'$ is AC on $[0, 2]$.
- Integration by parts in $\tilde{\psi}(z, 0) - \psi(z, 0) = \int_{0}^{2} (K\bar{q} - Kq)(0, t)e^{itz} dt$ gives
  
  \[ \tilde{\psi}(z, 0) - \psi(z, 0) = \frac{i}{z} (K\bar{q} - Kq)(0, 0) - \frac{i}{4z} \hat{G}(z) \]

  where

  \[ \hat{G}(z) = \int_{0}^{2} ((\bar{q} - q)(t/2) - 4h'(t))e^{itz} dt. \]
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where
  $$\hat{G}(z) = \int_{0}^{2} ((\tilde{q} - q)(t/2) - 4h'(t))e^{izt} dt.$$  

- To deal with the last term one needs the Hausdorff-Young inequality: for $1 < p \leq 2$
  $$\|\hat{G}\|_q \leq \frac{p^{1/(2p)}}{q^{1/(2q)}} \|G\|_p.$$
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- To deal with the last term one needs the Hausdorff-Young inequality: for $1 < p \leq 2$
  \[ \| \hat{G} \|_q \leq \frac{p^{1/(2p)}}{q^{1/(2q)}} \| G \|_p. \]
- Here one needs the assumption that $\tilde{q} - q$ be in $L^p$. 
Open problem
Two (or more) spectral bands

Suppose $q(x) = -2\varphi(x + \omega)$. The spectrum of the associated Schrödinger operator has only absolutely continuous spectrum with one gap. All solutions of $-y'' + qy = \lambda y$ are explicitly known. The inverse of the map $\varphi(z) = \lambda$ maps the energy ($\lambda$) plane to a parallelogram (the fundamental domain of $\varphi$) in a one-to-two fashion. Compactly supported perturbations do not change the essential spectrum but introduce eigenvalues and resonances ....
Thank you for your attention!