PH 222-3A Spring 2007

ELECTRIC FIELDS

Lectures 2,3

Chapter 22
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
Chapter 22
Electric Fields

In this chapter we will introduce the concept of an electric field. As long as charges are stationary, Coulomb’s law describes adequately the forces among charges. If the charges are not stationary we must use an alternative approach by introducing the electric field (symbol $\vec{E}$). In connection with the electric field, the following topics will be covered:

- Calculating the electric field generated by a point charge.
- Using the principle of superposition to determine the electric field created by a collection of point charges as well as continuous charge distributions.
- Once the electric field at a point $P$ is known, calculating the electric force on any charge placed at $P$.
- Defining the notion of an “electric dipole.” Determining the net force, the net torque, exerted on an electric dipole by a uniform electric field, as well as the dipole potential energy.
In Chapter 21 we discussed Coulomb’s law, which gives the force between two point charges. The law is written in such a way as to imply that \( q_2 \) acts on \( q_1 \) at a distance \( r \) \textit{instantaneously} ("action at a distance"): 

\[
F = \frac{1}{4\pi\varepsilon_0} \frac{|q_1||q_2|}{r^2}
\]

Electric interactions propagate in empty space with a large but \textbf{finite} speed \((c = 3 \times 10^8 \text{ m/s})\). In order to take into account correctly the finite speed at which these interactions propagate, we have to abandon the "action at a distance" point of view and still be able to explain how \( q_1 \) knows about the presence of \( q_2 \). The solution is to introduce the new concept of an \textbf{electric field} vector as follows: Point charge \( q_1 \) does not exert a force directly on \( q_2 \). Instead, \( q_1 \) creates in its vicinity an electric field that exerts a force on \( q_2 \).

\textbf{charge} \( q_1 \) \rightarrow \textbf{generates electric field} \( \vec{E} \) \rightarrow \textbf{\( \vec{E} \) exerts a force} \( \vec{F} \) \textbf{on} \( q_2 \)
Definition of the Electric Field Vector
Consider the positively charged rod shown in the figure. For every point $P$ in the vicinity of the rod we define the electric field vector $\vec{E}$ as follows:

1. We place a positive test charge $q_0$ at point $P$.
2. We measure the electrostatic force $\vec{F}$ exerted on $q_0$ by the charged rod.
3. We define the electric field vector $\vec{E}$ at point $P$ as:

$$\vec{E} = \frac{\vec{F}}{q_0}.$$  

SI Units: N/C

From the definition it follows that $\vec{E}$ is parallel to $\vec{F}$.

Note: We assume that the test charge $q_0$ is small enough so that its presence at point $P$ does not affect the charge distribution on the rod and thus alter the electric field vector $\vec{E}$ we are trying to determine.
Electric Field Generated by a Point Charge

Consider the positive charge $q$ shown in the figure. At point $P$ a distance $r$ from $q$ we place the test charge $q_0$. The force exerted on $q_0$ by $q_0$ is equal to:

$$F = \frac{1}{4\pi \varepsilon_0} \frac{|q| |q_0|}{r^2}$$

$$E = \frac{F}{q_0} = \frac{1}{4\pi \varepsilon_0} \frac{|q| |q_0|}{q_0 r^2} = \frac{1}{4\pi \varepsilon_0} \frac{|q|}{r^2}$$

The magnitude of $\vec{E}$ is a positive number. In terms of direction, $\vec{E}$ points radially outward as shown in the figure.

If $q$ were a negative charge the magnitude of $\vec{E}$ would remain the same. The direction of $\vec{E}$ would point radially inward instead.
Electric Field Generated by a Group of Point Charges. Superposition

The net electric electric field $\vec{E}$ generated by a group of point charges is equal to the vector sum of the electric field vectors generated by each charge.

In the example shown in the figure, $\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3$.

Here $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$ are the electric field vectors generated by $q_1$, $q_1$, and $q_3$, respectively.

Note: $\vec{E}_1$, $\vec{E}_2$, and $\vec{E}_3$ must be added as vectors:

$$E_x = E_{1x} + E_{2x} + E_{3x}, \quad E_y = E_{1y} + E_{2y} + E_{3y}, \quad E_z = E_{1z} + E_{2z} + E_{3z}$$
Problem (Electric field). To measure the magnitude of the horizontal electric field, an experimenter attaches a small charged cork ball to a string and suspends this device in the electric field. The electric force pushes the cork ball to one side, and the ball attains equilibrium when the string makes an angle of 35° with the vertical. The mass of the ball is 3x10⁻⁵ kg, and the charge on the ball is 4x10⁻⁷ C. What is the magnitude of the electric field?

\[
\begin{align*}
\tan \theta &= \frac{qE}{mg} \\
E &= \frac{mg \tan \theta}{q} \\
E &= \frac{3 \times 10^{-5} \times \tan 35^\circ}{4 \times 10^{-7}} \\
E &= 5.1 \times 10^2 \text{ N/C}
\end{align*}
\]
Problem: An electron moving through an electric field is observed to have an acceleration of $10^{16}$ m/s$^2$ in the x direction. What must be the magnitude and the direction of the electric field that produces this acceleration?

\[ F = qE \]

\[ \frac{F}{m} = \vec{a} = (0, 0) \]

\[ \vec{a} = \frac{\vec{E}}{m} = \frac{q\vec{E}}{m} \]

\[ \vec{E} = \frac{m\vec{a}}{q} = \left( \frac{mg}{q}, 0 \right) \]

\[ a = 1 \times 10^{16} \text{ m/s}^2 \text{ (positive)} \]

\[ \frac{mg}{q} = \left( 9.1 \times 10^{-31} \text{ kg} \right) \left( 1 \times 10^{16} \text{ m/s}^2 \right) = -5.7 \times 10^4 \text{ N/C} \]

\[ \vec{E} = (-5.7 \times 10^4 \text{ N/C}, 0) \]

The negative sign means \( \vec{E} \) points in the direction opposite to \( \vec{a} \).
Problem: The electric field in the electron gun of a TV tube is supposed to accelerate electrons uniformly from 0 to $3.3 \times 10^7$ m/s within a distance of 1.0 cm. What electric field is required?

\[ v^2 = v_0^2 + 2ax \]

\[ a = \frac{x}{t^2} - \frac{x_0}{t_0^2} = \frac{1}{2} \left( \frac{3.3 \times 10^7 \text{ m/s}}{0.01 \text{ m}} \right)^2 = 5.45 \times 10^{16} \text{ m/s}^2 \]

1) From \[ v^2 = v_0^2 + 2ax \] (if $x_0 = 0$ at $t_0 = 0$)

\[ a = \frac{1}{2} \left( \frac{v^2 - v_0^2}{x - x_0} \right) \]

2) \[ E = \frac{F}{q} = \frac{ma}{q} \]

\[ E = \left( 9.1 \times 10^{-31} \text{ kg} \right) \left( 5.45 \times 10^{16} \text{ m/s}^2 \right) \left( 1.6 \times 10^{-19} \text{ C} \right) = -3.1 \times 10^5 \text{ N/C} \]

Direction opposite to \[ F \] and \[ a \]
Problem: Three point charges \(-Q, 2Q,\) and \(-Q\) are arranged on a straight line as illustrated. What is the electric field that these charges produce at a distance \(x\) to the right of the central charge?

\[
\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 = (E_x, E_y)
\]

\[
E_{1x} = -\frac{Q}{4\pi\varepsilon_0 (d+x)^2} \\
E_{2x} = \frac{2Q}{4\pi\varepsilon_0 \cdot x^2} \\
E_{3x} = -\frac{Q}{4\pi\varepsilon_0 (x-d)^2}
\]

\[
E_x = E_{1x} + E_{2x} + E_{3x} = \frac{Q}{4\pi\varepsilon_0} \left[\frac{-1}{(d+x)^2} + \frac{2}{x^2} - \frac{1}{(x-d)^2}\right]
\]
Electric Dipole
A system of two equal charges of opposite sign \((\pm q)\) placed at a distance \(d\) apart is known as an "electric dipole." For every electric dipole we associate a vector known as "the electric dipole moment" (symbol \(\vec{p}\)) defined as follows:
The magnitude \(p = qd\)
The direction of \(\vec{p}\) is along the line that connects the two charges and points from \(-q\) to \(+q\).
Many molecules have a built-in electric dipole moment. An example is the water molecule (H₂O). The bonding between the O atom and the two H atoms involves the sharing of 10 valence electrons (8 from O and 1 from each H atom).

The 10 valence electrons have the tendency to remain closer to the O atom. Thus the O side is more negative than the H side of the H₂O molecule.
Electric Field Generated by an Electric Dipole

We will determine the electric field $\vec{E}$ generated by the electric dipole shown in the figure using the principle of superposition. The positive charge generates at $P$ an electric field whose magnitude $E_{(+)} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_+^2}$. The negative charge creates an electric field with magnitude $E_{(-)} = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_-^2}$.

The net electric field at $P$ is $E = E_{(+)} - E_{(-)}$.

$$E = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{r_+^2} - \frac{q}{r_-^2} \right) = \frac{1}{4\pi\varepsilon_0} \left( \frac{q}{(z-d/2)^2} - \frac{q}{(z+d/2)^2} \right)$$

$$E = \frac{q}{4\pi\varepsilon_0 z^2} \left[ \left(1 - \frac{d}{2z}\right)^{-2} - \left(1 + \frac{d}{2z}\right)^{-2} \right]$$

We assume: $\frac{d}{2z} \ll 1$

$$E = \frac{q}{4\pi\varepsilon_0 z^2} \left[ \left(1 + \frac{d}{z}\right) - \left(1 - \frac{d}{z}\right) \right] = \frac{qd}{2\pi\varepsilon_0 z^3} = \frac{1}{2\pi\varepsilon_0} \frac{p}{z^3}$$
Electric Field Generated by a Continuous Charge Distribution

Consider the continuous charge distribution shown in the figure. We assume that we know the volume density $\rho$ of the electric charge. This is defined as $\rho = \frac{dq}{dV}$ (Units: C/m$^3$).

Our goal is to determine the electric field $d\vec{E}$ generated by the distribution at a given point $P$. This type of problem can be solved using the principle of superposition as described below.

1. Divide the charge distribution into "elements" of volume $dV$. Each element has charge $dq = \rho dV$. We assume that point $P$ is at a distance $r$ from $dq$.

2. Determine the electric field $d\vec{E}$ generated by $dq$ at point $P$.

The magnitude $dE$ of $d\vec{E}$ is given by the equation $dE = \frac{dq}{4\pi\varepsilon_0 r^2}$.

3. Sum all the contributions: $\vec{E} = \frac{1}{4\pi\varepsilon_0} \int \frac{\rho dV \hat{r}}{r^2}$. 
**Example:** Determine the electric field \( \vec{E} \) generated at point \( P \) by a uniformly charged ring of radius \( R \) and total charge \( q \). Point \( P \) lies on the normal to the ring plane that passes through the ring center \( C \), at a distance \( z \). Consider the charge element of length \( dS \) and charge \( dq \) shown in the figure. The distance between the element and point \( P \) is \( r = \sqrt{z^2 + R^2} \).

The charge \( dq \) generates at \( P \) an electric field of magnitude \( dE \) that points outward along the line \( AP \):

\[
dE = \frac{dq}{4\pi\varepsilon_0 r^2}.
\]

The \( z \)-component of \( d\vec{E} \) is given by

\[
dE_z = dE \cos \theta.
\]

From triangle PAC we have: \( \cos \theta = z / r \)

\[
\vec{dE}_z = \frac{zdq}{4\pi\varepsilon_0 r^3} = \frac{zdq}{4\pi\varepsilon_0 \left( z^2 + R^2 \right)^{3/2}}.
\]

\[
E_z = \int dE_z
\]

\[
E_z = \frac{z}{4\pi\varepsilon_0 \left( z^2 + R^2 \right)^{3/2}} \int dq = \frac{zq}{4\pi\varepsilon_0 \left( z^2 + R^2 \right)^{3/2}}
\]
**Example 2:** Determine the electric field $\vec{E}$ generated at point $P$ by a uniformly positively charged disk of radius $R$ and charge density $\sigma$. Point $P$ lies on the normal to the ring plane that passes through the disk center $C$, at a distance $z$.

Our plan is to divide the disk into concentric flat rings and then to calculate the electric field at point $P$ by integrating the contributions of all the rings. For a ring with radius $r$ and radial width $dr$, $dq = \sigma dA = \sigma(2\pi r dr)$. We have already solved the problem of the electric field due to a ring of charge:

$$dE = \frac{dq z}{4\pi \varepsilon_o (z^2 + r^2)^{3/2}} = \frac{\sigma 2\pi r dr}{4\pi \varepsilon_o (z^2 + r^2)^{3/2}} = \frac{\sigma 2r dr}{4\varepsilon_o (z^2 + r^2)^{3/2}}$$

We can now find $E$ by integrating this eq. over the surface of the disk, that is, by integrating with respect to $r$, from $r = 0$ to $r = R$, while $z = \text{const.}$
\[ E = \int dE = \frac{\sigma z}{4\varepsilon_0} \int_0^R (z^2 + r^2)^{-3/2} (2r)dr \]

To solve this integral, we cast it in the form \( \int X^m dX \) by setting
\[ X = (z^2 + r^2), \quad m = -\frac{3}{2}, \quad \text{and} \quad dX = (2r)dr \]

\[ E = \frac{\sigma z}{4\varepsilon_0} \int X^m dX = \frac{\sigma z}{4\varepsilon_0} \frac{X^{m+1}}{m+1} = \frac{\sigma z}{4\varepsilon_0} \left[ \frac{(z^2 + r^2)^{-1/2}}{-\frac{1}{2}} \right]^R_0 \]

\[ \Rightarrow E = \frac{\sigma}{2\varepsilon_0} \left( 1 - \frac{z}{\sqrt{z^2 + R^2}} \right) \]

If we let \( R \to \infty \) (infinite sheet) while keeping \( z \) finite, the second term in the parentheses approaches zero and
\[ E = \frac{\sigma}{2\varepsilon_0} \quad \text{(infinite sheet)} \]

This is the electric field produced by an infinite sheet of uniform charge located on one side of a nonconductor such as plastic.
**Electric Field Lines.** In the 19th century Michael Faraday introduced the concept of electric field lines, which help visualize the electric field vector $\vec{E}$ without using mathematics. For the relation between the electric field lines and $\vec{E}$:

1. At any point $P$ the electric field vector $\vec{E}$ is tangent to the electric field lines.

2. The magnitude of the electric field vector $\vec{E}$ is proportional to the density of the electric field lines.
Two graphical methods of graphical representation of electric field: a) electric field vectors; b) electric field lines (directed radially outward from the positive charge +q).
Electric field lines extend away from **positive charges** (where they originate) and toward **negative charges** (where they terminate).

**Example 1: Electric field lines of a negative point charge - \( q \):**

\[
E = \frac{1}{4\pi\varepsilon_0} \left| \frac{q}{r^2} \right|
\]

- The electric field lines point toward the point charge.
- The direction of the lines gives the direction of \( \vec{E} \).
- The density of the lines/unit area increases as the distance from \(-q\) decreases.

**Note:** In the case of a positive point charge the electric field lines have the same form but they point **outward**.
The electric field can be represented graphically by field lines:

**Direction:** The tangent to the field lines has the direction of the electric field.

**Density:** The density of lines is directly proportional to the magnitude of the electric field.

- To draw a pattern of field lines, we have to continue each line indefinitely except when it begins on a positive charge or where it ends on a negative charge.

- So, \( E \sim q; \) \( \Rightarrow \) the number (density) of field lines \( \sim q. \)

- The convention or “normalization”
  - the number of electric field lines emerging from a charge \( q \) is \( \frac{E_0}{q} \).
  - \( \Rightarrow \) if \( q = 1 \) \( \text{C} \), the number of electric field lines is \( \frac{E_0}{1} = \left( 8.85 \times 10^{-12} \right) = 1.13 \text{ x } 10^7 \) lines.

The density of field lines \( \sim E \).

Electric field lines of a positive point charge. The arrows on these lines indicate the direction of the electric field along each line.
Major properties of the field lines

1. The field lines start on positive charges and end (if they end) on negative charges—the positive charges are sources of field lines, the negative—sinks.

2. Field lines never intersect (except where they start or end on point charges).

3. The field lines are not physical objects, or not a form of matter. They represent merely mathematical approach helping in understanding of the spatial dependence of the electric fields surrounding electric charges.

4. Tangent to the field line has the direction of the electric field.

5. The density of the field lines equal to the magnitude of the electric field.
Example 2: Electric field lines of an electric field generated by an infinitely large plane uniformly charged. In the next chapter we will see that the electric field generated by such a plane has the form shown in fig. b.

1. The electric field on either side of the plane has a constant magnitude.
2. The electric field vector is perpendicular to the charge plane.
3. The electric field vector $\vec{E}$ points away from the plane.

The corresponding electric field lines are given in fig. c.

**Note:** For a negatively charged plane the electric field lines point inward.
The concept of the field lines for the computation of the electric field of a large, flat, charged sheet

Assumption: Positive charge is uniformly distributed over a very large horizontal sheet. Density of charge \( \sigma \) (C/m²)

Symmetry arguments:
1) The pattern of field lines must respect the symmetry of the charge distribution.
2) One half of the sheet + the mirror image of the pattern below it = symmetrical.
3) Consider one half of the sheet. The charge is distributed to the right of the line.
   - The line is vertical, so the charge density is to the right, neither to left, neither to right.
   - Therefore, the electric field is horizontal.

Electric field of flat sheet:
- Consider a small area \( dA \) of the sheet.
- The number of field lines intersecting is \( \frac{dA}{2\mu_0} \).

\[ E = \frac{\sigma}{2\varepsilon_0} \]
The Electric Field of the Two Charged Sheets

Individual electric fields of two sheets of opposite signs

$E = \frac{\sigma}{2\varepsilon_0} + \frac{-\sigma}{2\varepsilon_0} = \frac{0}{\varepsilon_0}$

$E$ of the two sheets are in opposite direction. Cancel each other.

In the same direction reinforce each other. Cancel.

The net electric field of the two sheets.
Problem: Electric charge is uniformly distributed over each of three large, parallel sheets of paper. The charges per unit area of the sheets $\sigma = 2 \times 10^{-6} \text{C/m}^2$, $2 \times 10^{-6} \text{C/m}^2$ and $-2 \times 10^{-6} \text{C/m}^2$, respectively. The distance between one sheet and the next is 10 cm.

Find the strength of the electric field $E$ above the sheets, below the sheets, and in space between the sheets. Find the direction of $E$ at each place.

From a single sheet, the electric field is

$$E = \frac{\sigma}{2\varepsilon_0} = \frac{2 \times 10^{-6} \text{C/m}^2}{2.88 \times 10^{-12} \text{C}^2/(\text{N}\cdot\text{m}^2)} = 1.1 \times 10^5 \frac{\text{N}}{\text{C}}$$

$E_A = (1.1 \times 10^5 \text{N/C}) \cdot \frac{1}{2\varepsilon_0}$

$E_B = (1.1 \times 10^5 \text{N/C}) \sqrt{-1-1+1} = -1.1 \times 10^5 \text{N/C}$

$E_C = (6.1 \times 10^5 \text{N/C}) \sqrt{-1-1-1} = -3.3 \times 10^5 \text{N/C}$

$$E_D = (1.1 \times 10^5 \text{N/C})[-1-1+1] = -1.1 \times 10^5 \text{N/C}$$
**Example 3:**
Electric field lines generated by an electric dipole (a positive and a negative point charge of the same size but of opposite sign)

**Example 4:**
Electric field lines generated by two equal positive point charges
Forces and Torques Exerted on Electric Dipoles by a Uniform Electric Field

Consider the electric dipole shown in the figure in the presence of a uniform (constant magnitude and direction) electric field \( \vec{E} \) along the x-axis. The electric field exerts a force \( F_+ = qE \) on the positive charge and a force \( F_- = -qE \) on the negative charge. The net force on the dipole is \( F_{\text{net}} = qE - qE = 0 \).

The net torque generated by \( F_+ \) and \( F_- \) about the dipole center is

\[
\tau = \tau_+ + \tau_- = -|F_+| \frac{d}{2} \sin \theta - |F_-| \frac{d}{2} \sin \theta = -qEd \sin \theta = -pE \sin \theta
\]

In vector form: \( \vec{\tau} = \vec{p} \times \vec{E} \)

The electric dipole in a uniform electric field does not move but can rotate about its center.

\[
\vec{F}_{\text{net}} = 0 \quad \vec{\tau} = \vec{p} \times \vec{E}
\]
The dipole has its least potential energy, when it is in equilibrium orientation ($\vec{p}$ is lined up with the $\vec{E}$; $\vec{r}=\vec{p} \times \vec{E} = 0$). To rotate the dipole to any other orientation requires work by some external agent. We are free to define the zero-potential-energy configuration in an arbitrary way. The expression for the potential energy of the electric dipole in an external electric field is simplest if we choose it to be zero when angle $\theta=90^\circ$. We can find $U$ at any other value of $\theta$ by calculating the work $W$ done by the field on the dipole when the dipole is rotated to that value $\theta$ from $90^\circ$.

**Potential Energy of an Electric Dipole in a Uniform Electric Field**

$$U = -pE \cos \theta$$

$$U = -\vec{p} \cdot \vec{E}$$

At point $A$ ($\theta = 0$), $U$ has a minimum value $U_{\text{min}} = -pE$. It is a position of stable equilibrium.

At point $B$ ($\theta = 180^\circ$), $U$ has a maximum value $U_{\text{max}} = +pE$. It is a position of unstable equilibrium.

$$U = -\int_{90^\circ}^{\theta} \tau d\theta' = -\int_{90^\circ}^{\theta} pE \sin \theta d\theta'$$

$$U = -pE \int_{90^\circ}^{\theta} \sin \theta d\theta' = -pE \cos \theta = -\vec{p} \cdot \vec{E}$$
Work Done by an External Agent to Rotate an Electric Dipole in a Uniform Electric Field

Consider the electric dipole in fig. $a$. It has an electric dipole moment $\vec{p}$ and is positioned so that $\vec{p}$ is at an angle $\theta_i$ with respect to a uniform electric field $\vec{E}$.

An external agent rotates the electric dipole and brings it to its final position shown in fig. $b$. In this position $\vec{p}$ is at an angle $\theta_f$ with respect to $\vec{E}$.

The work $W$ done by the external agent on the dipole is equal to the difference between the initial and final potential energy of the dipole:

$$W = U_f - U_i = -pE \cos \theta_f - (-pE \cos \theta_i)$$

$$W = pE \left( \cos \theta_i - \cos \theta_f \right)$$