Magnetic Field

Lecture 12

Chapter 28
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
Chapter 28
Magnetic Fields

In this chapter we will cover the following topics:

Magnetic field vector, $\vec{B}$
Magnetic force on a moving charge, $\vec{F}_B$
Magnetic field lines
Motion of a moving charge particle in a uniform magnetic field
Magnetic force on a current-carrying wire
Magnetic torque on a wire loop
Magnetic dipole, magnetic dipole moment $\vec{\mu}$
Hall effect
Cyclotron particle accelerator
WhatProducesAMagneticField

One can generate a magnetic field using one of the following methods:

Pass a current through a wire and thus form what is known as an "electromagnet."

Use a "permanent" magnet.

Empirically we know that both types of magnets attract small pieces of iron. Also, if suspended so that they can rotate freely they align themselves along the north-south direction. We can thus say that these magnets create in the surrounding space a "magnetic field" $\vec{B}$, which manifests itself by exerting a magnetic force $\vec{F}_B$. We will use the magnetic force to define precisely the magnetic field vector $\vec{B}$. 
Definition of $\vec{B}$

The magnetic field vector is defined in terms of the force $\vec{F}_B$ it exerts on a charge $q$, which moves with velocity $\vec{v}$. We inject the charge $q$ in a region where we wish to determine $\vec{B}$ at random directions, trying to scan all the possible directions. There is one direction for which the force $\vec{F}_B$ on $q$ is zero. This direction is parallel with $\vec{B}$. For all other directions $\vec{F}_B$ is not zero, and its magnitude $F_B = |q|vB \sin \phi$ where $\phi$ is the angle between $\vec{v}$ and $\vec{B}$. In addition, $\vec{F}_B$ is perpendicular to the plane defined by $\vec{v}$ and $\vec{B}$. The magnetic force vector is given by the equation $\vec{F}_B = q\vec{v} \times \vec{B}$.

**SI unit of $B$:** The defining equation is $F_B = |q|vB \sin \phi$.

If we shoot a particle with charge $q = 1$ C at right angles ($\phi = 90^\circ$) to $\vec{B}$ with speed $v = 1$ m/s and the magnetic force $F_B = 1$ N, then $B = 1$ tesla. An earlier (non-SI) unit for $\vec{B}$, still in common use, is the gauss (G) $1$ tesla = $10^4$ gauss.
The Vector Product of Two Vectors

The vector product \( \vec{c} = \vec{a} \times \vec{b} \) of the vectors \( \vec{a} \) and \( \vec{b} \) is a vector \( \vec{c} \).

The magnitude of \( \vec{c} \) is given by the equation

\[ c = ab \sin \phi. \]

The direction of \( \vec{c} \) is perpendicular to the plane \( P \) defined by the vectors \( \vec{a} \) and \( \vec{b} \).

The sense of the vector \( \vec{c} \) is given by the right-hand rule:

a. Place the vectors \( \vec{a} \) and \( \vec{b} \) tail to tail.

b. Rotate \( \vec{a} \) in the plane \( P \) along the shortest angle so that it coincides with \( \vec{b} \).

c. Rotate the fingers of the right hand in the same direction.

d. The thumb of the right hand gives the sense of \( \vec{c} \).

The vector product of two vectors is also known as the "\textbf{cross}" product.
When the right hand is oriented so the fingers point along the magnetic field $\mathbf{B}$ and the thumb points along the velocity $\mathbf{v}$ of a positively charged particle, the palm faces in the direction of the magnetic force $\mathbf{F}$ to the particle.

$\cdot \mathbf{F}$ is perpendicular to the plane defined by $\mathbf{v}$ and $\mathbf{B}$.

Alternative Right hand rule
The Vector Product \( \vec{c} = \vec{a} \times \vec{b} \) in Terms of Vector Components

\[\vec{a} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k}, \quad \vec{b} = b_x \hat{i} + b_y \hat{j} + b_z \hat{k}, \quad \vec{c} = c_x \hat{i} + c_y \hat{j} + c_z \hat{k}\]

The vector components of vector \( \vec{c} \) are given by the equations

\[c_x = a_y b_z - a_z b_y, \quad c_y = a_z b_x - a_x b_z, \quad c_z = a_x b_y - a_y b_x\]

**Note:** Those familiar with the use of determinants can use the expression

\[\vec{a} \times \vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}\]

**Note:** The order of the two vectors in the cross product is important:

\[\vec{b} \times \vec{a} = -(\vec{a} \times \vec{b})\]
The motion of a charged particle in a magnetic field
**Magnetic Field Lines:** In analogy with the electric field lines we introduce the concept of magnetic field lines, which help visualize the magnetic field vector $\vec{B}$ without using equations.

In the relation between the magnetic field lines and $\vec{B}$:

1. At any point $P$ the magnetic field vector $\vec{B}$ is tangent to the magnetic field lines.

2. The magnitude of the magnetic field vector $\vec{B}$ is proportional to the density of the magnetic field lines.

\[ B_P > B_Q \]
Magnetic Field Lines of a Permanent Magnet

The magnetic field lines of a permanent magnet are shown in the figure. The lines pass through the body of the magnet and form **closed** loops. This is in contrast to the electric field lines that originate and terminate on electric charges.

The closed magnetic field lines enter one point of the magnet and exit at the other end. The end of the magnet from which the lines emerge is known as the **north pole** of the magnet. The other end where the lines enter is called the **south pole** of the magnet. The two poles of the magnet cannot be separated. Together they form what is known as a "**magnetic dipole**."
Magnetic field lines

(a)

Magnetic field lines in the gap of a horseshoe magnet (c)
Like poles repel

Unlike poles attract
Magnetic Forces

The magnetic forces familiar from everyday experience:
1) Forces that permanent magnets exert on each other.

Each magnet has 2 distinct ends called North pole and South pole.

Compass needle near the South pole of a bar magnet. Needles will settle into an equilibrium configuration pointing towards the South pole of the magnet.

The magnetic effects of the earth are like those of a large permanent magnet placed within the Earth.
Magnetic field of the Earth. The axis of the magnetic field makes an angle of about 11° with the axis of rotation of the Earth.
Discovery of the Electron: A cathode ray tube is shown in the figure. Electrons are emitted from a hot filament known as the "cathode." They are accelerated by a voltage $V$ applied between the cathode and a second electrode known as the "anode." The electrons pass through a hole in the anode and they form a narrow beam. They hit the fluorescent coating of the right wall of the cathode ray tube where they produce a spot of light. J.J. Thomson in 1897 used a version of this tube to investigate the nature of the particle beam that caused the fluorescent spot. He applied constant electric and magnetic fields in the tube region to the right of the anode. With the fields oriented as shown in the figure the electric force $\vec{F}_E$ and the magnetic force $\vec{F}_B$ have opposite directions. By adjusting $B$ and $E$, Thomson was able to have a zero net force.

$$\vec{F}_E = q\vec{E}$$

$$\vec{F}_B = q\vec{v} \times \vec{B}$$
Example: A velocity selector

A device for measuring the velocity of a charged particle. Operates by applying electric and magnetic forces to the particle in such a way that these forces are balanced.

$$F_{\text{mag}} = qvB \sin \theta$$

Diagram:

- Tube
- Electric field $E$
- Magnetic field $B$
- Charged particle
- Forces $F_E = qE$, $F_B = qvB \sin \theta$

If $E$ and $B$ are adjusted properly,

$$F_E = -F_{\text{mag}}$$

$$a = 0$$

$$v = \text{const}$$

$\Rightarrow$ particle moves in a straight line at a constant speed

$\Rightarrow$ particles with different $v$ from one selected are deflected and do not exit at the right end of the tube.
Motion of a Charged Particle in a Uniform Magnetic Field
(also known as \textit{cyclotron motion})

A particle of mass $m$ and charge $q$, when injected with a speed $v$ at right angles to a uniform magnetic field $\vec{B}$, follows a circular orbit with uniform speed. The centripetal force required for such motion is provided by the magnetic force

$$\vec{F}_B = q\vec{v} \times \vec{B}$$

The circular orbit of radius $r$ for an electron is shown in the figure. The magnetic force

$$F_B = |q|vB = ma = m\frac{v^2}{r} \rightarrow r = \frac{mv}{|q|B}.$$

The period is $T = \frac{2\pi r}{v} = \frac{2\pi mv}{|q|Bv} = \frac{2\pi m}{|q|B}.$

The corresponding frequency is $f = \frac{1}{T} = \frac{|q|B}{2\pi m}$. The angular frequency is $\omega = 2\pi f = \frac{|q|B}{m}$.

\textbf{Note 1 :} The cyclotron period does not depend on the speed $v$. All particles of the same mass complete their circular orbit during the same time $T$ regardless of speed.

\textbf{Note 2 :} Fast particles move on larger-radius circular orbits, while slower particles move on smaller-radius orbits. All orbits have the same period $T$. 

The magnetic force

$$F_B = |q|vB = ma = m\frac{v^2}{r} \rightarrow r = \frac{mv}{|q|B}.$$
We now consider the motion of a charge in a uniform magnetic field \( \vec{B} \) when its initial velocity \( \vec{v} \) forms an angle \( \phi \) with \( \vec{B} \). We decompose \( \vec{v} \) into two components.

One component \( (v_\parallel) \) is parallel to \( \vec{B} \) and the other \( (v_\perp) \) is perpendicular to \( \vec{B} \) (see fig. a):

\[
v_\parallel = v \cos \phi \quad v_\perp = v \sin \phi
\]

The particle executes two independent motions. One, the cyclotron motion, is in the plane perpendicular to \( \vec{B} \) that we have analyzed on the previous page. Its radius is \( r = \frac{mv_\perp}{|q|B} \). Its period is \( T = \frac{2\pi m}{|q|B} \).

The second motion is along the direction of \( \vec{B} \) and it is linear motion with constant speed \( v_\parallel \). The combination of the two motions results in a helical path (see fig. b).

The pitch \( p \) of the helix is given by

\[
p = T v_\parallel = \frac{2\pi m v \cos \phi}{|q|B}
\]
The mass-spectrometer

Helps to identify unknown molecules produced in chemical reaction.

\[ B \ (\text{out of paper}) \]

\[ B = \frac{eV}{m} \]

\[ r = \sqrt{\frac{2mV}{qB^2}} \]

\[ m = \left( \frac{e\tau^2}{2V} \right)B^2 \]
Magnetic Force on a Current-Carrying Wire
Consider a wire of length $L$ that carries a current $i$ as shown in the figure. A uniform magnetic field $B$ is present in the vicinity of the wire. Experimentally it was found that a force $\vec{F}_B$ is exerted by $\vec{B}$ on the wire, and that $\vec{F}_B$ is perpendicular to the wire. The magnetic force on the wire is the vector sum of all the magnetic forces exerted by $\vec{B}$ on the electrons that constitute $i$. The total charge $q$ that flows through the wire in time $t$ is given by

$$q = it = i \frac{L}{v_d}.$$ Here $v_d$ is the drift velocity of the electrons in the wire.

The magnetic force is $F_B = qv_d B \sin 90^\circ = i \frac{L}{v_d} v_d B = iLB$.

$$F_B = iLB$$
Magnetic Force on a Straight Wire in a Uniform Magnetic Field

If we assume the more general case for which the wire forms an angle $\phi$ with magnetic field $\vec{B}$, the magnetic force equation can be written in vector form as $\vec{F}_B = i\vec{L} \times \vec{B}$. Here $\vec{L}$ is a vector whose magnitude is equal to the wire length $L$ and has a direction that coincides with that of the current. The magnetic force magnitude is $F_B = iLB\sin\phi$.

Magnetic Force on a Wire of Arbitrary Shape Placed in a Nonuniform Magnetic Field

In this case we divide the wire into elements of length $dL$, which can be considered as straight. The magnetic force on each element is $d\vec{F}_B = id\vec{L} \times \vec{B}$. The net magnetic force on the wire is given by the integral $\vec{F}_B = i\int d\vec{L} \times \vec{B}$. 

\[
\vec{F}_B = i\vec{L} \times \vec{B} \\
d\vec{F}_B = id\vec{L} \times \vec{B} \\
\vec{F}_B = i\int d\vec{L} \times \vec{B}
\]
Magnetic Torque on a Current Loop

Consider the rectangular loop in fig. a with sides of lengths \( a \) and \( b \) and that carries a current \( i \). The loop is placed in a magnetic field so that the normal \( \hat{n} \) to the loop forms an angle \( \theta \) with \( \vec{B} \). The magnitude of the magnetic force on sides 1 and 3 is \( F_1 = F_3 = iaB \sin 90^\circ = iaB \). The magnetic force on sides 2 and 4 is \( F_2 = F_4 = ibB \sin(90 - \theta) = ibB \cos \theta \). These forces cancel in pairs and thus \( F_{\text{net}} = 0 \).

The torque about the loop center \( C \) of \( F_2 \) and \( F_4 \) is zero because both forces pass through point \( C \). The moment arm for \( F_1 \) and \( F_3 \) is equal to \((b / 2) \sin \theta \). The two torques tend to rotate the loop in the same (clockwise) direction and thus add up. The net torque \( \tau = \tau_1 + \tau_3 = (iabB / 2) \sin \theta + (iabB / 2) \sin \theta = iabB \sin \theta = iAB \sin \theta \).
**Magnetic Dipole Moment**

The torque of a coil that has $N$ loops exerted by a uniform magnetic field $B$ and carries a current $i$ is given by the equation $\tau = NiAB$.

We define a new vector $\vec{\mu}$ associated with the coil, which is known as the magnetic dipole moment of the coil.

The magnitude of the magnetic dipole moment is $\mu = NiA$.

Its direction is perpendicular to the plane of the coil.

The sense of $\vec{\mu}$ is defined by the right-hand rule. We curl the fingers of the right hand in the direction of the current. The thumb gives us the sense. The torque can be expressed in the form $\tau = \mu B \sin \theta$ where $\theta$ is the angle between $\vec{\mu}$ and $\vec{B}$.

In vector form: $\vec{\tau} = \vec{\mu} \times \vec{B}$.

The potential energy of the coil is: $U = -\mu B \cos \theta = -\vec{\mu} \cdot \vec{B}$.

$U$ has a minimum value of $-\mu B$ for $\theta = 0$ (position of **stable** equilibrium).

$U$ has a maximum value of $\mu B$ for $\theta = 180^\circ$ (position of **unstable** equilibrium).

**Note:** For both positions the net torque is $\tau = 0$. 

\[ \vec{\tau} = \vec{\mu} \times \vec{B} \]

\[ U = -\vec{\mu} \cdot \vec{B} \]
The torque on a current loop pivoted on an axis and placed in a strong magnetic field is exploited in electric motors.
The torque on a current loop is exploited in instruments for the measurement of current.

Current to be measured is sent into a coil of several turns pivoted on an axis. A spring provides a toroidal torque that opposes the magnetic torque.

Mechanism of a galvanometer.

Ammeter with a shunt resistor, which permits most of the current to bypass the galvanometer.

Voltmeter with a large resistor placed in series with the galvanometer.
The Hall Effect

In 1879 Edwin Hall carried out an experiment in which he was able to determine that conduction in metals is due to the motion of negative charges (electrons). He was also able to determine the concentration $n$ of the electrons.

He used a strip of copper of width $d$ and thickness $\ell$. He passed a current $i$ along the length of the strip and applied a magnetic field $B$ perpendicular to the strip as shown in the figure. In the presence of $\vec{B}$ the electrons experience a magnetic force $\vec{F}_B$ that pushes them to the right (labeled "R") side of the strip. This accumulates negative charge on the R-side and leaves the left side (labeled "L") of the strip positively charged. As a result of the accumulated charge, an electric field $\vec{E}$ is generated as shown in the figure, so that the electric force balances the magnetic force on the moving charges: $F_E = F_B \rightarrow eE = ev_d B \rightarrow E = v_d B \quad \text{(eq. 1)}$. From Chapter 26 we have: $J = nev_d \rightarrow v_d = \frac{J}{ne} = \frac{i}{A ne} = \frac{i}{\ell d ne} \quad \text{(eq. 2)}$. 
Hall measured the potential difference $V$ between the left and the right side of the metal strip: $V = Ed$ (eq. 3).

We substitute $E$ from eq. 3 and $v_d$ from eq. 2 into eq. 1 and get:

$$\frac{V}{d} = \frac{B}{\ell} \frac{i}{\ell dne} \rightarrow n = \frac{Bi}{V\ell e} \quad \text{(eq. 4)}$$

Figs. $a$ and $b$ were drawn assuming that the carriers are electrons. In this case if we define $V = V_L - V_R$ we get a **positive** value. If we assume that the current is due to the motion of positive charges (see fig. $c$) then positive charges accumulate on the R-side and negative charges on the L-side, and thus $V = V_L - V_R$ is now a **negative** number.

By determining the polarity of the voltage that develops between the left-and right-hand sides of the strip, Hall was able to prove that current was composed of moving electrons. From the value of $V$ using equation 4 he was able to determine the concentration of the negative charge carriers.
The Cyclotron Particle Accelerator

The cyclotron accelerator consists of two hollow conductors in the shape of the letter dee (these are known as the "dees" of the cyclotron). Between the two dees an oscillator of frequency $f_{osc}$ creates an oscillating electric field $E$ that exists only in the gap between the two dees. At the same time, a constant magnetic field $B$ is applied perpendicular to the plane of the dees.

In the figure we show a cyclotron accelerator for protons. The protons follow circular orbits of radius $r = \frac{mv}{eB}$ and rotate with the same frequency $f = \frac{eB}{2\pi m}$. If the cyclotron frequency matches the oscillator frequency then the protons during their trip through the gap between the dees are accelerated by the electric field that exists in the gap. The faster protons travel on increasingly larger radius orbits. Thus the electric field changes the speed of the protons while the magnetic field changes only the direction of their velocity and forces them to move on circular (cyclotron) orbits.