Magnetic Fields Due to Currents

Lecture 14

Chapter 29
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
Chapter 29
Magnetic Fields Due to Currents

In this chapter we will explore the relationship between an electric current and the magnetic field it generates in the space around it. We will follow a two-pronged approach, depending on the symmetry of the problem.

For problems with low symmetry we will use the law of Biot-Savart in combination with the principle of superposition.

For problems with high symmetry we will introduce Ampere’s law.

Both approaches will be used to explore the magnetic field generated by currents in a variety of geometries (straight wire, wire loop, solenoid coil, toroid coil). We will also determine the force between two parallel, current-carrying conductors. We will then use this force to define the SI unit for electric current (the ampere).
The Law of Biot-Savart

This law gives the magnetic field $\mathbf{dB}$ generated by a wire segment of length $ds$ that carries a current $i$. Consider the geometry shown in the figure. Associated with the element $ds$ we define an associated vector $d\mathbf{s}$ that has magnitude equal to the length $ds$. The direction of $d\mathbf{s}$ is the same as that of the current that flows through segment $ds$.

The magnetic field $\mathbf{dB}$ generated at point $P$ by the element $d\mathbf{s}$ located at point $A$ is given by the equation $\mathbf{dB} = \frac{\mu_0 i}{4\pi} \frac{d\mathbf{s} \times \mathbf{r}}{r^3}$. Here $\mathbf{r}$ is the vector that connects point $A$ (location of element $ds$) with point $P$ at which we want to determine $\mathbf{dB}$. The constant $\mu_0 = 4\pi \times 10^{-7} \text{ T m/A} = 1.26 \times 10^{-6} \text{ T m/A}$ and is known as the "permeability constant." The magnitude of $\mathbf{dB}$ is $dB = \frac{\mu_0 i}{4\pi} \frac{ds \sin \theta}{r^2}$.

Here $\theta$ is the angle between $d\mathbf{s}$ and $\mathbf{r}$. 

$$d\mathbf{B} = \frac{\mu_0 i}{4\pi} \frac{d\mathbf{s} \times \mathbf{r}}{r^3}$$
The magnitude of the magnetic field generated by the wire at point $P$ located at a distance $R$ from the wire is given by the equation

$$B = \frac{\mu_0 i}{2\pi R}.$$  

The magnetic field lines form circles that have their centers at the wire. The magnetic field vector $\vec{B}$ is tangent to the magnetic field lines. The sense for $\vec{B}$ is given by the **right-hand rule**. We point the thumb of the right hand in the direction of the current. The direction along which the fingers of the right hand curl around the wire gives the direction of $\vec{B}$. 

---

**Magnetic Field Generated by a Long Straight Wire**

The magnitude of the magnetic field generated by the wire at point $P$ located at a distance $R$ from the wire is given by the equation

$$B = \frac{\mu_0 i}{2\pi R}.$$
Proof of the equation.
Consider the wire element of length $ds$ shown in the figure. The element generates at point $P$ a magnetic field of magnitude $dB = \frac{\mu_0i}{4\pi} \frac{ds \sin \theta}{r^2}$. Vector $d\vec{B}$ is pointing into the page. The magnetic field generated by the whole wire is found by integration:

$$B = \int_{-\infty}^{\infty} dB = 2 \int_{0}^{\infty} dB = \frac{\mu_0i}{2\pi} \int_{0}^{\infty} ds \sin \theta$$

$$r = \sqrt{s^2 + R^2} \quad \sin \theta = \sin \phi = R / r = R / \sqrt{s^2 + R^2}$$

$$B = \frac{\mu_0i}{2\pi} \int_{0}^{\infty} \left(\frac{Rds}{(s^2 + R^2)^{3/2}}\right) = \frac{\mu_0i}{2\pi R} \left[ \frac{s}{\sqrt{s^2 + R^2}} \right]_{0}^{\infty} = \frac{\mu_0i}{2\pi R}$$

$$\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{x}{a^2 \sqrt{x^2 + a^2}}$$
Magnetic Field Generated by a Circular Wire Arc of Radius $R$ at Its Center $C$

$$B = \frac{\mu_0 i \phi}{4\pi R}$$

A wire section of length $ds$ generates at the center $C$ a magnetic field $d\vec{B}$.

The magnitude $dB = \frac{\mu_0 i}{4\pi} \frac{ds \sin 90^\circ}{R^2} = \frac{\mu_0 i}{4\pi} \frac{ds}{R^2}$. The length $ds = Rd\phi$

$$\rightarrow dB = \frac{\mu_0 i}{4\pi R} d\phi.$$ Vector $d\vec{B}$ points out of the page.

The net magnetic field $B = \int dB = \int_0^\phi \frac{\mu_0 i}{4\pi R} d\phi = \frac{\mu_0 i \phi}{4\pi R}$.

**Note:** The angle $\phi$ must be expressed in radians.

For a circular wire, $\phi = 2\pi$. In this case we get: $B_{\text{circ}} = \frac{\mu_0 i}{2R}$. 
Magnetic Field Generated by a Circular Loop
Along the Loop Axis

The wire element $ds$ generates a magnetic field $d\vec{B}$ whose magnitude $dB = \frac{\mu_0 i}{4\pi} \frac{ds \sin 90^\circ}{r^2} = \frac{\mu_0 i}{4\pi} \frac{ds}{r^2}$.

We decompose $d\vec{B}$ into two components:
One ($dB_{\parallel}$) is along the $z$-axis. The second component ($dB_{\perp}$) is in a direction perpendicular to the $z$-axis. The sum of all the $dB_{\perp}$ is equal to zero. Thus we sum only the $dB_{\parallel}$ terms:

$$r = \sqrt{R^2 + z^2}$$
$$dB_{\parallel} = dB \cos \alpha = \frac{\mu_0 i}{4\pi} \frac{ds \cos \alpha}{r^2} \quad \cos \alpha = \frac{R}{r}$$

$$\rightarrow dB_{\parallel} = \frac{\mu_0 i R}{4\pi} \frac{ds}{r^3} = \frac{\mu_0 i R}{4\pi} \left( \frac{ds}{(R^2 + z^2)^{3/2}} \right)$$

$$B = \int dB_{\parallel}$$

$$B = \frac{\mu_0 i R}{4\pi \left( R^2 + z^2 \right)^{3/2}} \int ds = \frac{\mu_0 i R}{4\pi \left( R^2 + z^2 \right)^{3/2}} \left( 2\pi R \right) = \frac{\mu_0 i R^2}{2 \left( R^2 + z^2 \right)^{3/2}}$$
Example

Find the force that each wire exerts on a segment $\Delta l$ of the other wire.

$$B_1 = \frac{\mu_0 I_1}{2\pi l}$$

$$\Delta F = \int \Delta l \, B_1$$
$$= \frac{\mu_0}{2\pi} \frac{I_1}{\Delta l} \int \Delta l$$

$$\mu_0 = 4\pi \times 10^{-7} \text{ N} \cdot \text{m} / \text{A}^2$$

$\mu_0$- permeability constant

SI unit of current (definition) is based on the force per meter of length between two parallel wires.

$F_{21} = I_2 \vec{L} \times \vec{B}_1$

A long straight wire carrying a current $I_2$ in the magnetic field of a long straight wire carrying a current $I_1$.

SI unit of charge.

The coulomb is defined as the amount of charge that a current of one ampere delivers in one second.
Superposition Principle

The net magnetic fields produced by currents flowing on wires or other current distributions, each of which produces an individual magnetic field, obeys the principle of superposition.

The net magnetic field produced by several currents is the vector sum of the individual magnetic fields of the individual currents.

Example:

\[ I_1 - I_2 = I \]

\[ B = \frac{\mu_0 I}{2\pi r} \]

Find \( B \) at a point midway between the wires.

\[ B = 2 \times \frac{\mu_0 I}{2\pi} = 2 \times \frac{126 \times 10^{-6} \text{ N} \cdot \text{A}^2}{2\pi} \times \frac{800 \text{ A}}{10 \text{ m}} = 3.2 \times 10^{-4} \text{ T} \]
Problem

\[ I = \frac{\mu_0 I}{2\pi r} \]

Find \( B = ? \)

\[ B \text{ contributed by wires 1, 3 are } \frac{\mu_0 I}{2\pi a} \text{ each in direction of dotted line.} \]

\[ \Rightarrow \text{Net } B = \frac{\mu_0 I}{2\pi} \text{ iu} \]

Direction shown. Current from 2 contributes \( \frac{\mu_0 I}{2\pi a^2} \text{ also in the same direction.} \)

\[ \Rightarrow B = \frac{\mu_0 I}{2\pi} \frac{3}{2} \sqrt{2} \]

Problem

A wire of Cu with \( \mu = 7.9 \times 10^{-8} \text{ m} \)

\[ \text{carries current } I \]

at surface \( B = \frac{\mu_0 I}{2\pi R} \text{ with } R = 1 \text{ mm} = 0.001 \text{ m} \)

\[ B = 2 \times 10^{-7} \text{ T} \]

\[ F = qvB = 1.6 \times 10^{-19} \times 10^6 \times 4 \times 10^{-3} = 6.4 \times 10^{-16} \text{ N} \]

\[ a = \frac{F}{m} = \frac{6.4 \times 10^{-16}}{\frac{9.1 \times 10^{-31}}{4}} \text{ m/s}^2 = 7.0 \times 10^{14} \text{ m/s}^2 \]
The law of Biot-Savart combined with the principle of superposition can be used to determine $\vec{B}$ if we know the distribution of currents. In situations that have high symmetry we can use Ampere's law instead, because it is simpler to apply.

Ampere's law can be derived from the law of Biot-Savart, with which it is mathematically equivalent. Ampere's law is more suitable for advanced formulations of electromagnetism. It can be expressed as follows:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$

Ampere's law is not complete. A missing term was added by Clark Maxwell. The complete form of Ampere's law will be discussed in Chapter 32.
Implementation of Ampere's Law:

1. Determination of $\oint \vec{B} \cdot d\vec{s}$. The closed path is divided into $n$ elements $\Delta\vec{s}_1$, $\Delta\vec{s}_2$, ..., $\Delta\vec{s}_n$. We then form the sum:

$$\sum_{i=1}^{n} \vec{B}_i \cdot \Delta\vec{s}_i = \sum_{i=1}^{n} B_i \Delta s_i \cos \theta_i.$$  

Here $\vec{B}_i$ is the magnetic field in the $i$th element.

$$\oint \vec{B} \cdot d\vec{s} = \lim_{n \to \infty} \sum_{i=1}^{n} \vec{B}_i \cdot \Delta\vec{s}_i$$

2. Calculation of $i_{\text{enc}}$. We curl the fingers of the right hand in the direction in which the Amperian loop was traversed. We note the direction of the thumb.

All currents inside the loop parallel to the thumb are counted as positive.

All currents inside the loop antiparallel to the thumb are counted as negative.

All currents outside the loop are not counted.

In this example: $i_{\text{enc}} = i_1 - i_2$.  

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 i_{\text{enc}}$$
Magnetic Field Outside a Long Straight Wire

We already have seen that the magnetic field lines of the magnetic field generated by a long straight wire that carries a current \( i \) have the form of circles, which are concentric with the wire. We choose an Amperian loop that reflects the cylindrical symmetry of the problem. The loop is also a circle of radius \( r \) that has its center on the wire. The magnetic field is tangent to the loop and has a constant magnitude \( B \):

\[
\oint \vec{B} \cdot d\vec{s} = \oint B \, ds \cos 0 = B \oint ds = 2\pi r B = \mu_0 i_{\text{enc}} = \mu_0 i
\]

\[
\rightarrow B = \frac{\mu_0 i}{2\pi r}
\]

**Note:** Ampere's law holds true for any closed path. We choose to use the path that makes the calculation of \( \vec{B} \) as easy as possible.
Magnetic Field Inside a Long Straight Wire

We assume that the distribution of the current within the cross-section of the wire is uniform. The wire carries a current \( i \) and has radius \( R \). We choose an Amperian loop that is a circle of radius \( r \) \((r < R)\) with its center on the wire. The magnetic field is tangent to the loop and has a constant magnitude \( B \):

\[
\oint \vec{B} \cdot d\vec{s} = \oint B ds \cos 0 = B \oint ds = 2\pi r B = \mu_0 i_{\text{enc}}
\]

\[
i_{\text{enc}} = i \frac{\pi r^2}{\pi R^2} = i \frac{r^2}{R^2}
\]

\[
2\pi r B = \mu_0 i \frac{r^2}{R^2} \rightarrow B = \left( \frac{\mu_0 i}{2\pi R^2} \right) r
\]
The solenoid is a long, tightly wound helical wire coil in which the coil length is much larger than the coil diameter. Viewing the solenoid as a collection of single circular loops, one can see that the magnetic field inside is approximately uniform.

The magnetic field inside the solenoid is parallel to the solenoid axis. The sense of $\vec{B}$ can be determined using the right-hand rule. We curl the fingers of the right hand along the direction of the current in the coil windings. The thumb of the right hand points along $\vec{B}$. The magnetic field outside the solenoid is much weaker and can be taken to be approximately zero.

**The Solenoid**

The solenoid is a long, tightly wound helical wire coil in which the coil length is much larger than the coil diameter. Viewing the solenoid as a collection of single circular loops, one can see that the magnetic field inside is approximately uniform.

The magnetic field inside the solenoid is parallel to the solenoid axis. The sense of $\vec{B}$ can be determined using the right-hand rule. We curl the fingers of the right hand along the direction of the current in the coil windings. The thumb of the right hand points along $\vec{B}$. The magnetic field outside the solenoid is much weaker and can be taken to be approximately zero.
We will use Ampere's law to determine the magnetic field inside a solenoid. We assume that the magnetic field is uniform inside the solenoid and zero outside. We assume that the solenoid has $n$ turns per unit length.

$$B = \mu_0 ni$$

We will use the Amperian loop $abcd$. It is a rectangle with its long side parallel to the solenoid axis. One long side ($ab$) is inside the solenoid, while the other ($cd$) is outside: \[ \oint \vec{B} \cdot d\vec{s} = \int_{a}^{b} \vec{B} \cdot d\vec{s} + \int_{b}^{c} \vec{B} \cdot d\vec{s} + \int_{c}^{d} \vec{B} \cdot d\vec{s} + \int_{d}^{a} \vec{B} \cdot d\vec{s} \]

\[ \int_{a}^{b} \vec{B} \cdot d\vec{s} = \int B d\vec{s} \cos 0 = B \int d\vec{s} = Bh \]

\[ \int_{c}^{d} \vec{B} \cdot d\vec{s} = \int_{b}^{c} \vec{B} \cdot d\vec{s} = \int_{d}^{a} \vec{B} \cdot d\vec{s} = 0 \]

\[ \rightarrow \oint \vec{B} \cdot d\vec{s} = Bh \]

The enclosed current $i_{enc} = nhi$.

\[ \oint \vec{B} \cdot d\vec{s} = \mu_0 i_{enc} \rightarrow Bh = \mu_0 nhi \rightarrow B = \mu_0 ni \]
Magnetic Field of a Toroid

A toroid has the shape of a doughnut (see figure). We assume that the toroid carries a current $i$ and that it has $N$ windings. The magnetic field lines inside the toroid form circles that are concentric with the toroid center. The magnetic field vector is tangent to these lines. The sense of $\vec{B}$ can be found using the right-hand rule. We curl the fingers of the right hand along the direction of the current in the coil windings. The thumb of the right hand points along $\vec{B}$. The magnetic field outside the solenoid is approximately zero.

We use an Amperian loop that is a circle of radius $r$ (orange circle in the figure):

$$\oint \vec{B} \cdot d\vec{s} = \oint Bds \cos 0 = B \oint ds = 2\pi r B.$$ The enclosed current $i_{enc} = Ni$.

Thus: $2\pi r B = \mu_0 Ni \rightarrow B = \frac{\mu_0 Ni}{2\pi r}$.

Note: The magnetic field inside a toroid is not uniform.
The Magnetic Field of a Magnetic Dipole.
Consider the magnetic field generated by a wire coil of radius $R$ that carries a current $i$. The magnetic field at a point $P$ on the $z$-axis is given by

$$B = \frac{\mu_0 i R^2}{2 \left( R^2 + z^2 \right)^{3/2}}.$$  Here $z$ is the distance between $P$ and the coil center. For points far from the loop ($z >> R$) we can use the approximation:  $B \approx \frac{\mu_0 i R^2}{2 z^3}$.  Here $\mu$ is the magnetic dipole moment of the loop. In vector form:

$$\vec{B}(z) = \frac{\mu_0}{2 \pi} \frac{\vec{\mu}}{z^3}.$$  The loop generates a magnetic field that has the same form as the field generated by a bar magnet.