PH 222-2C Fall 2012

Circuits

Lectures 11-12

Chapter 27
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
Chapter 27

Circuits

In this chapter we will cover the following topics:

- Electromotive force (emf)
- Ideal and real emf devices
- Kirchhoff’s loop rule
- Kirchhoff’s junction rule
- Multiloop circuits
- Resistors in series
- Resistors in parallel
- RC circuits, charging and discharging of a capacitor
In order to create a current through a resistor, a potential difference must be created across its terminals. One way of doing this is to connect the resistor to a battery. A device that can maintain a potential difference between two terminals is called a "seat of an emf" or an "emf device." Here emf stands for electromotive force. Examples of emf devices are a battery, an electric generator, a solar cell, a fuel cell, etc.

These devices act like "charge pumps" in the sense that they move positive charges from the low-potential (negative) terminal to the high-potential (positive) terminal. A mechanical analog is given in the figure below.

In this mechanical analog a water pump transfers water from the low to the high reservoir. The water returns from the high to the low reservoir through a pipe, which is the analog of the resistor. The emf (symbol $\mathcal{E}$) is defined as the potential difference between the terminals of the emf device when no current flows through it.
The polarity of an emf device is indicated by an arrow with a small circle at its tail. The arrow points from the negative to the positive terminal of the device. When the emf device is connected to a circuit its internal mechanism transports positive charges from the negative to the positive terminal and sets up a charge flow (a.k.a. current) around the circuit. In doing so the emf device does work $dW$ on a charge $dq$, which is given by the equation $dW = \mathcal{E}dq$. The required energy comes from chemical reactions in the case of a battery; in the case of a generator it comes from the mechanical force that rotates the generator shaft; in the case of a solar cell it comes from the Sun. In the circuit of the figure the energy stored in emf device B changes form: It does mechanical work on the motor. It produces thermal energy on the resistor. It gets converted into chemical energy in emf device A.
Ideal and Real Emf Devices

An emf device is said to be **ideal** if the voltage $V$ across its terminals $a$ and $b$ does **not** depend on the current $i$ that flows through the emf device: $V = \mathcal{E}$.

An emf device is said to be **real** if the voltage $V$ across its terminals $a$ and $b$ **decreases** with current $i$ according to the equation $V = \mathcal{E} - ir$.

The parameter $r$ is known as the "**internal resistance**" of the emf device.
Consider the circuit shown in the figure. We assume that the emf device is ideal and that the connecting wires have negligible resistance. A current $i$ flows through the circuit in the clockwise direction.

In a time interval $dt$ a charge $dq = idt$ passes through the circuit. The battery is doing work $dW = \mathcal{E}dq = \mathcal{E}idt$. Using energy conservation we can set this amount of work equal to the rate at which heat is generated on $R$: $\mathcal{E}idt = R i^2 dt \rightarrow \mathcal{E} = Ri \rightarrow \mathcal{E} - iR = 0$.

Kirchhoff put the equation above in the form of a rule known as Kirchhoff's loop rule (KLR for short).

**KLR**: The algebraic sum of the changes in potential encountered in a complete traversal of any loop in a circuit is equal to zero.

The rules that give us the algebraic sign of the charges in potential through a resistor and a battery are given on the next page.
**Resistance Rule:**
For a move through a resistance in the direction of the current, the change in the potential is \( \Delta V = -iR \).
For a move through a resistance in the direction opposite to that of the current, the change in the potential is \( \Delta V = +iR \).

**EMF Rule:**
For a move through an ideal emf device in the direction of the emf arrow, the change in the potential is \( \Delta V = +\mathcal{E} \).
For a move through an ideal emf device in a direction opposite to that of the emf arrow, the change in the potential is \( \Delta V = -\mathcal{E} \).
KLR example: Consider the circuit of fig. a. The battery is real with internal resistance \( r \). We apply KLR for this loop starting at point \( a \) and going counterclockwise:

\[
\mathcal{E} - ir - iR = 0 \rightarrow i = \frac{\mathcal{E}}{R + r}
\]

We note that for an ideal battery, \( r = 0 \) and \( i = \frac{\mathcal{E}}{R} \).

Note: The internal resistance \( r \) of the battery is an integral part of the battery's internal mechanism. There is no way to open the battery and remove \( r \).

In fig. b we plot the potential \( V \) of every point in the loop as we start at point \( a \) and go around in the clockwise direction. The change \( \Delta V \) in the battery is positive because we go from the negative to the positive terminal. The change \( \Delta V \) across the two resistors is negative because we chose to traverse the loop in the direction of the current. The current flows from high to low potential.
Potential Difference Between Two Points:
Consider the circuit shown in the figure. We wish to calculate the potential difference $V_b - V_a$ between point $b$ and point $a$.

$$V_b - V_a = \text{sum of all potential changes } \Delta V \text{ along the path from point } a \text{ to point } b.$$ 

We choose a path in the loop that takes us from the initial point $a$ to the final point $b$.

$$V_f - V_i = \text{sum of all potential changes } \Delta V \text{ along the path.}$$

There are two possible paths: We will try them both.

Left path: $V_b - V_a = \mathcal{E} - ir$

Right path: $V_b - V_a = iR$

Note: The values of $V_b - V_a$ we get from the two paths are the same.
6. A current of 1A flows through the wire shown in Fig. What will a voltmeter read when connected from:
   (a) A to B
   (b) A to C
   (c) A to D

   ![Diagram](image.png)

   a) $V_{AB} = V_B - V_A = IR = 2V$;

   b) $V_{AC} = V_C - V_A = V_C - (V_C - 5V - 2V) = 7V$;

   c) $V_{AD} = V_D - V_A = V_D - (V_D + 10V - 3V - 5V - 2V) = V_D - V_D - 10V + 3V + 5V + 2V = 0V$
**Equivalent Resistance**

Consider the combination of resistors shown in the figure. We can substitute this combination of resistors with a single resistor $R_{eq}$ that is "electrically equivalent" to the resistor group it substitutes.

This means that if we apply the same voltage $V$ across the resistors in fig. $a$ and across $R_{eq}$, the same current $i$ is provided by the battery. Alternatively, if we pass the same current $i$ through the circuit in fig. $a$ and through the equivalent resistance $R_{eq}$, the voltage $V$ across them is identical. This can be stated in the following manner: If we place the resistor combination and the equivalent resistor in separate black boxes, by doing electrical measurements we cannot distinguish between the two.
Resistors in Series

Consider the three resistors connected in series (one after the other) as shown in fig. a. These resistors have the same current $i$ but different voltages $V_1$, $V_2$, and $V_3$. The net voltage across the combination is the sum $V_1 + V_2 + V_3$.

We will apply KLR for the loop in fig. a starting at point $a$, and going around the loop in the clockwise direction:

$$\mathcal{E} - iR_1 - iR_2 - iR_3 = 0 \rightarrow i = \frac{\mathcal{E}}{R_1 + R_2 + R_3} \quad (\text{eq. 1})$$

We will apply KLR for the loop in fig. b starting at point $a$, and going around the loop in the clockwise direction:

$$\mathcal{E} - iR_{eq} = 0 \rightarrow i = \frac{\mathcal{E}}{R_{eq}} \quad (\text{eq. 2})$$

If we compare eq. 1 with eq. 2 we get: $R_{eq} = R_1 + R_2 + R_3$.

For $n$ resistors connected in series, the equivalent resistance is:

$$R_{eq} = \sum_{i=1}^{n} R_i = R_1 + R_2 + \ldots + R_n$$
Consider the three resistors shown in the figure. "In parallel" means that the terminals of the resistors are connected together on both sides. Thus resistors in parallel have the same potential applied across them. In our circuit this potential is equal to the emf $\mathcal{E}$ of the battery. The three resistors have different currents flowing through them. The total current is the sum of the individual currents. We apply KJR at point $a$:

$$i = i_1 + i_2 + i_3 \quad i_1 = \frac{\mathcal{E}}{R_1} \quad i_2 = \frac{\mathcal{E}}{R_2} \quad i_3 = \frac{\mathcal{E}}{R_3} \rightarrow$$

$$i = \mathcal{E} \left( \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} \right) \quad \text{(eq. 1).}$$

From fig. $b$ we have:

$$i = \frac{\mathcal{E}}{R_{eq}} \quad \text{(eq. 2).}$$

If we compare equations 1 and 2 we get:

$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}.$$
Problem: Four resistors, with \( R_1 = 25 \Omega \), \( R_2 = 15 \Omega \), \( R_3 = 40 \Omega \), and \( R_4 = 20 \Omega \), are connected to a 12V battery as shown. (a) Find the combined resistance. (b) The current in each resistor.

Kirchhoff's rule for this single-loop circuit with resistances \( R_1 \), \( R_2 \) and \( R_3 \) in series with an exert of 12V tells us:

\[ E = I R_2 + I R_3 + I R_4 = 0 \]

\[
I = \frac{E}{R_2 + R_3 + R_4} = \frac{12}{15 + 13.3 + 25.5} = \frac{12}{53.3} = 0.23\text{A}
\]

Current through resistors \( R_1 \), \( R_2 \) and \( R_4 \):

\[
I R_3 = \frac{V_A - V_B}{R_3} = \frac{I R_3}{R_3} = \frac{0.23 \times 13.3}{40} = 0.08\text{A}
\]

\[
I R_4 = \frac{V_A - V_B}{R_4} = \frac{I R_4}{R_4} = \frac{0.23 \times 13.3}{20} = 0.15\text{A}
\]
Multiloop Circuits
Consider the circuit shown in the figure. There are three branches in it: \(bad\), \(bcd\), and \(bd\).
We assign currents for each branch and define the current directions arbitrarily. The method is self-correcting. If we have made a mistake in the direction of a particular current, the calculation will yield a negative value and thus provide us with a warning.

We assign current \(i_1\) for branch \(bad\), current \(i_2\) for branch \(bcd\), and current \(i_3\) for branch \(bd\). Consider junction \(d\). Currents \(i_1\) and \(i_3\) arrive, while \(i_2\) leaves. Charge is conserved, thus we have: \(i_1 + i_3 = i_2\). This equation can be formulated as a more general principle known as Kirchhoff's junction rule (KJR).

**KJR:** The sum of the currents entering any junction is equal to the sum of the currents leaving the junction.
In order to determine the currents $i_1, i_2$, and $i_3$ in the circuit we need three equations. The first equation will come from KJR at point $d$:

KJR/junction $d$: $i_1 + i_3 = i_2$ (eq. 1)

The other two will come from KLR: If we traverse the left loop (bad) starting at $b$ and going in the counterclockwise direction we get:

KLR/loop bad: $\mathcal{E}_1 - i_1 R_1 + i_3 R_3 = 0$ (eq. 2). Now we go around the right loop (bcd) starting at point $b$ and going in the counterclockwise direction:

KLR/loop bcd: $-i_3 R_3 - i_2 R_2 - \mathcal{E}_2 = 0$ (eq. 3)

We have a system of three equations (eqs. 1, 2, and 3) and three unknowns $(i_1, i_2, \text{ and } i_3)$. If a numerical value for a particular current is negative, this means that the chosen direction for this current is wrong and that the current flows in the opposite direction.

We can write a fourth equation (KLR for the outer loop abcd) but this equation does not provide any new information.

KLR/loop abcd: $\mathcal{E}_1 - i_1 R_1 - i_2 R_2 - \mathcal{E}_2 = 0$ (eq. 4)
When calculating the potential change across a resistor we must take the product of the resistance and the net current through the resistor. The net current is the algebraic sum of all the currents flowing through the resistor.
Problem. Consider the circuit shown in Fig.
Given that $\varepsilon_1 = 6.0 \text{ V}$, $\varepsilon_2 = 10 \text{ V}$, and $R_1 = 2.0 \Omega$, what must be the value of the resistance $R_2$ if the current through this resistance is to be 2.0 A?

1) 2 loops

2) Let current loops $I_1, I_2$ be as shown.
   direction $I_1$: counterclockwise
   $I_2$: opposite.

3) Apply Kirchhoff’s Rule to the first loop
   
   $\varepsilon_1 - (I_1 - I_2) R_1 = 0 \quad (1)$

   for the second
   
   $\varepsilon_2 - I_2 R_2 - (I_2 - I_1) R_1 = 0 \quad (2)$

   Substitute the numerical values in (1) and (2)

   $6.0 - (I_1 - I_2) 2.0 = 0$
   
   $10.0 - 2.0 R_2 + (I_2 - I_1) 2.0 = 0$

   $16 - 2.0 R_2 = 0$;

   $R_2 = \frac{16}{2.0} = 8.0 \Omega$
2 variant

1) Label three possible values of current $I_1, I_2, I_3$ guessing at the most likely direction of positive current flow.

2) Apply Kirchhoff's first rule

$$I_1 = I_3 + I_2$$

3) Apply Kirchhoff's second rule to any two closed loops

$$\varepsilon_1 - I_3 R_1 = 0$$
$$\varepsilon_2 - I_2 R_2 + I_3 R_1 = 0$$

$$\varepsilon_1 + \varepsilon_2 - I_2 R_2 = 0$$

$$R_2 = \frac{\varepsilon_1 + \varepsilon_2}{I_2} = \frac{16}{8} = 2 \Omega$$
1) Label 3 possible directions of current
2) Apply Kirchhoff's first rule
\[ I_2 = I_1 + I_3 \]
3) Apply Kirchhoff's second rule
\[ \begin{align*}
\varepsilon_2 - I_2 R_2 - I_1 R_1 - \varepsilon_1 &= 0 \\
\varepsilon_1 + I_1 R_1 - I_3 R_3 &= 0 \\
\varepsilon_2 R_3 - I_2 R_3 - \varepsilon_1 R_2 &= 0 \\
-\varepsilon_2 R_3 &= 0
\end{align*} \]
\[ I_3 = \frac{\varepsilon_1 + I_1 R_1}{R_3} \]
\[ \varepsilon_2 - I_2 R_2 - (\varepsilon_1 + I_1 R_1) R_2 - I_1 R_1 - \varepsilon_1 = 0 \]
\[ I_1 (R_2 R_3 + R_1 R_2 + R_1 R_3) = \frac{\varepsilon_2 R_3}{R_3} \Rightarrow \frac{\varepsilon_2 - \varepsilon_1 R_2}{R_2} \]
\[ I_1 = \frac{\varepsilon_2 R_3 - \varepsilon_1 R_2}{R_2 R_3 + R_1 R_2 + R_1 R_3} = \frac{12 \times 0.5 - 6.0 \times 0.2}{0.2 \times 0.5 + 0.2 \times 0.5 + 0.2 \times 0.2} = 6.6 A \]
**Problem 2 Variant**

**Circuit Analysis**

1. Circuit consists of 2 loops
2. Let current loops be as shown
3. Apply Kirchhoff's

\[ R_1 = 0.25 \Omega; \quad R_2 = 0.20 \Omega \]
\[ R_3 = 0.50 \Omega; \quad E_1 = 6 \text{ V}; \quad E_2 = 12 \text{ V} \]

**Find** \( I_2, I_3 \)

\[ \text{1) } I_2, I_3? \]
\[ \text{2) } \text{Let current loops be as shown} \]
\[ \text{3) } \text{Apply Kirchhoff's} \]

\[ \begin{align*}
E_2 - I_1 R_2 - (I_1 - I_2) R_1 - E_1 &= 0 \quad (1) \\
E_1 - (I_2 - I_1) R_1 - I_2 R_3 &= 0 \quad (2)
\end{align*} \]

Flow:

\[ E_1 - I_2 R_1 + I_1 R_1 - I_2 R_3 = 0 \rightarrow I_1 = \frac{I_2 (R_1 + R_3)}{R_1} \]

Substitute into \( (1) \):

\[ E_2 - \left[ \frac{E_1}{R_1} + \frac{I_2 (R_1 + R_3 - R_2)}{R_1} \right] R_1 - \left[ \frac{I_2 R_1 + I_2 R_3}{R_1} - E_2 \right] R_2 - E_3 = 0 \]

\[ \begin{align*}
E_2 R_1 - I_2 R_1 R_2 - I_2 R_2 R_3 + E_1 R_2 - I_2 R_1 - I_2 R_3 + E_2 R_2 - E_3 R_2 - E_3 R_1 &= 0 \\
I_2 (R_1 R_2 + R_2 R_3 - R_1 R_3) &= \frac{E_2 R_1 + E_2 R_2}{R_1 R_2 + R_2 R_3 - R_1 R_3} = \frac{12 \times 0.25 + 6 \times 0.25}{0.25 \times 0.2 + 0.2 \times 0.25 + 0.25 \times 0.25} = 15.3 \text{ A} \\
I_1 &= \frac{E_2 (R_1 + R_3) - E_1}{R_1} = \frac{15.3 (0.25 + 0.5) - 6}{0.25} = 21.9 \text{ A} \]

\[ I_4 = I_1 - I_2 = 21.9 - 15.3 = 6.6 \text{ A} \]
Problem

\[ E_1 = 12.0 \text{V}, \quad E_2 = 8.0 \text{V} \]
\[ R_1 = 4.0 \Omega, \quad R_2 = 4.0 \Omega \]
\[ R_3 = 2.0 \Omega \]

Find \( I_1 \), \( I_2 \)

Currents in \( R_1, R_2, R_3 \):

\[ E_1 - (I_1 + I_2)R_1 - I_1 R_2 = 0 \]
\[ E_1 - (I_1 + I_2)R_1 - E_2 - I_2 R_3 = 0 \]

\[ \begin{align*}
12 - (I_1 + I_2)4 &- 4I_1 = 0 \\
12 - (I_1 + I_2)4 &- 8 - 2I_2 = 0
\end{align*} \]

\[ \begin{align*}
12 - 4I_1 - 4I_2 - 4I_1 = 0 \\
12 - 8I_1 - 4I_2 = 0
\end{align*} \]

\[ \begin{align*}
4 - 4I_1 - 4I_2 - 2I_2 = 0 \\
4 - 4I_1 - 6I_2 = 0
\end{align*} \]

\[ \begin{align*}
3 - 2I_1 - I_2 = 0 \\
2 - 2I_1 - 3I_2 = 0
\end{align*} \]

\[ I_2 = -0.5 \text{A} \]

\[ I_1 = \frac{3 + 0.5}{2} = 1.75 \text{A} \]

Current in \( R_1 \): \( I_1 + I_2 = 1.75 - 0.5 = 1.25 \text{A} \)

Current in \( R_2 \): \( I_1 = 1.75 \text{A} \)

Current in \( R_3 \): \( I_2 = -0.5 \text{A} \)
8. (a) Find the current in the three resistors shown in Fig.
(b) Find the power delivered by the 12 V battery

\[ \begin{align*}
12 - 4I_1 - 6(I_1 + I_2) &= 0 \\
12 - 4I_2 - 6 - 6(I_1 + I_2) &= 0
\end{align*} \]

\[ \begin{align*}
6 - 2I_1 - 3I_1 - 3I_2 &= 0 \\
3 - 2I_2 - 3I_1 - 3I_2 &= 0
\end{align*} \]

\[ \Rightarrow \begin{align*}
6 - 5I_1 - 3I_2 &= 0 \\
3 - 5(I_1 - \frac{6}{3}) - 3I_1 &= 0
\end{align*} \]

\[ \Rightarrow \begin{align*}
9 - 30 + 25I_1 &= 0 \\
16I_1 &= 21
\end{align*} \]

\[ I_1 = \frac{21}{16} \text{ A} \]

Through resistor 4.0 Ω left:

\[ I_1 = \frac{6 - 5 \times \frac{21}{16}}{3} = \frac{-0.18}{3} \text{ A} \]

Through resistor 4.0 Ω upper:

\[ I_1 + I_2 = \frac{1.13}{6} \text{ A} \]

Through resistor 6.0 Ω:

\[ P = 12V \times (I_1 + I_2) = 12 \left( \frac{21}{16} - 0.18 \right) = 13.56 \text{ W} = 14 \text{ W} \]
**Ammeters and Voltmeters**

An ammeter is an instrument that measures current. In order to measure the current that flows through a conductor at a certain point we must cut the conductor at this point and connect the two ends of the conductor to the ammeter terminals so that the current can pass through the ammeter. An example is shown in the figure, where the ammeter has been inserted between points $a$ and $b$.

It is essential that the ammeter resistance $R_A$ be much smaller than the other resistors in the circuit. In our example: $R_A \ll R_1$ and $R_A \ll R_2$.

A voltmeter is an instrument that measures the potential difference between two points in a circuit. In the example of the figure we use a voltmeter to measure the potential across $R_1$. The voltmeter terminals are connected to the two points $c$ and $d$. It is essential that the voltmeter resistance $R_V$ be much larger than the other resistors in the circuit. In our example: $R_V \gg R_1$ and $R_V \gg R_2$. 

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Consider the circuit shown in the figure. We assume that the capacitor is initially uncharged and that at \( t = 0 \) we throw the switch \( S \) from the middle position to position \( a \). The battery will charge the capacitor \( C \) through the resistor \( R \).

Our objective is to examine the charging process as a function of time.

We will write KLR starting at point \( b \) and going in the clockwise direction:

\[
\mathcal{E} - iR - \frac{q}{C} = 0.
\]

The current
\[
i = \frac{dq}{dt} \rightarrow \mathcal{E} - \frac{dq}{dt} R - \frac{q}{C} = 0.
\]

If we rearrange the terms we have:

\[
\frac{dq}{dt} R + \frac{q}{C} = \mathcal{E}.
\]

This is an inhomogeneous, first order, linear differential equation with initial condition \( q(0) = 0 \). This condition expresses the fact that at \( t = 0 \) the capacitor is uncharged.
Differential equation: \[ \frac{dq}{dt} R + \frac{q}{C} = \mathcal{E} \]

Initial condition: \( q(0) = 0 \)

Solution: \( q = C\mathcal{E}\left(1-e^{-t/\tau}\right) \)

Here: \( \tau = RC \)

The constant \( \tau \) is known as the "time constant" of the circuit. If we plot \( q \) versus \( t \) we see that \( q \) does not reach its terminal value \( C\mathcal{E} \) but instead increases from its initial value and reaches the terminal value at \( t = \infty \). Do we have to wait for an eternity to charge the capacitor? In practice, no.

\[ q(t = \tau) = (0.632) C\mathcal{E} \]
\[ q(t = 3\tau) = (0.950) C\mathcal{E} \]
\[ q(t = 5\tau) = (0.993) C\mathcal{E} \]

If we wait only a few time constants the charge, for all practical purposes, has reached its terminal value \( C\mathcal{E} \).

The current \( i = \frac{dq}{dt} = \left(\frac{\mathcal{E}}{R}\right) e^{-t/\tau} \). If we plot \( i \) versus \( t \) we get a decaying exponential (see fig. \( b \)).
Consider the circuit shown in the figure. We assume that the capacitor at $t = 0$ has charge $q_0$ and that at $t = 0$ we throw the switch S from the middle position to position $b$. The capacitor is disconnected from the battery and loses its charge through resistor $R$.

We will write KLR starting at point $b$ and going in the counterclockwise direction: $\frac{q}{C} - iR = 0$.

Taking into account that $i = \frac{dq}{dt}$ we get: $-\frac{dq}{dt} R + \frac{q}{C} = 0$.

This is a homogeneous, first order, linear differential equation with initial condition $q(0) = q_0$. The solution is: $q = q_0 e^{-t/\tau}$, where $\tau = RC$. If we plot $q$ versus $t$ we get a decaying exponential. The charge becomes zero at $t = \infty$. In practical terms we only have to wait a few time constants:

$q(\tau) = (0.368)q_0$,  
$q(3\tau) = (0.049)q_0$,  
$q(5\tau) = (0.007)q_0$.  

RC Circuits: Discharging of a Capacitor