ROLLING, TORQUE, and ANGULAR MOMENTUM

Lectures 18-19

Chapter 11
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
Chapter 11
Rolling, Torque, and Angular Momentum

In this chapter we will cover the following topics:

- Rolling of circular objects and its relationship with friction
- Redefinition of torque as a vector to describe rotational problems that are more complicated than the rotation of a rigid body about a fixed axis
- Angular Momentum of single particles and systems or particles
- Newton’s second law for rotational motion
- Conservation of angular momentum
- Applications of the conservation of angular momentum
Rolling as Translation and Rotation Combined

Consider an object with circular cross section that rolls along a surface without slipping. This motion, though common, is complicated. We can simplify its study by treating it as a combination of translation of the center of mass and rotation of the object about the center of mass.

Consider the two snapshots of a rolling bicycle wheel shown in the figure. An observer stationary with the ground will see the center of mass O of the wheel move forward with a speed $v_{com}$. The point P at which the wheel makes contact with the road also moves with the same speed. During the time interval $t$ between the two snapshots both O and P cover a distance $s$. $v_{com} = \frac{ds}{dt}$ (eqs.1) During $t$ the bicycle rider sees the wheel rotate by an angle $\theta$ about O so that

$s = R\theta \rightarrow \frac{ds}{dt} = R \frac{d\theta}{dt} = R\omega$ (eqs.2) If we combine equation 1 with equation 2 we get the condition for rolling without slipping. $v_{com} = R\omega$
We have seen that rolling is a combination of purely translational motion with speed \( v_{\text{com}} \) and a purely rotational motion about the center of mass with angular velocity \( \omega = \frac{v_{\text{com}}}{R} \). The velocity of each point is the vector sum of the velocities of the two motions. For the translational motion the velocity vector is the same for every point \( (\vec{v}_{\text{com}}, \text{see fig.b}) \). The rotational velocity varies from point to point. Its magnitude is equal to \( \omega r \) where \( r \) is the distance of the point from O. Its direction is tangent to the circular orbit (see fig.a). The net velocity is the vector sum of these two terms. For example the velocity of point P is always zero. The velocity of the center of mass O is \( \vec{v}_{\text{com}} \) \( (r = 0) \). Finally the velocity of the top point T is equal to \( 2\vec{v}_{\text{com}} \).
Problem 2. An automobile traveling at 80.0 km/h has tires of 75.0 cm diameter.

(a) What is the angular speed of the tires about their axes?
(b) If the car is brought to a stop uniformly in 30.0 complete turns of the tires (without skidding), what is the magnitude of the angular acceleration of the wheels?
(c) How far does the car move during the braking?

The initial speed of the car is

\[ v = (80 \text{ km/h})(1000 \text{ m/km})(1 \text{ h/3600 s}) = 22.2 \text{ m/s}. \]

The tire radius is \( R = 0.750/2 = 0.375 \text{ m}. \)

(a) The initial speed of the car is the initial speed of the center of mass of the tire, so

\[ \omega_0 = \frac{v_{\text{com0}}}{R} = \frac{22.2 \text{ m/s}}{0.375 \text{ m}} = 59.3 \text{ rad/s}. \]

(b) With \( \theta = (30.0)(2\pi) = 188 \text{ rad} \) and \( \omega = 0 \), Eq. 10-14 leads to

\[ \omega^2 = \omega_0^2 + 2\alpha \theta \implies |\alpha| = \frac{(59.3 \text{ rad/s})^2}{2(188 \text{ rad})} = 9.31 \text{ rad/s}^2. \]

(c) \( R\theta = 70.7 \text{ m} \) for the distance traveled.
Rolling as Pure Rotation

Another way of looking at rolling is shown in the figure. We consider rolling as a pure rotation about an axis of rotation that passes through the contact point P between the wheel and the road. The angular velocity of the rotation is $\omega = \frac{v_{com}}{R}$.

In order to define the velocity vector for each point we must know its magnitude as well as its direction. The direction for each point on the wheel points along the tangent to its circular orbit. For example at point A the velocity vector $\vec{v}_A$ is perpendicular to the dotted line that connects point A with point P. The speed of each point is given by: $v = \omega r$. Here $r$ is the distance between a particular point and the contact point P. For example at point T $r = 2R$. Thus $v_T = 2R\omega = 2v_{com}$. For point O $r = R$ thus $v_O = \omega R = v_{com}$. For point P $r = 0$ thus $v_P = 0$. 
The Kinetic Energy of Rolling

Consider the rolling object shown in the figure.

It is easier to calculate the kinetic energy of the rolling body by considering the motion as pure rotation about the contact point P. The rolling object has mass $M$ and radius $R$.

The kinetic energy $K$ is then given by the equation: $K = \frac{1}{2} I_p \omega^2$. Here $I_p$ is the rotational inertia of the rolling body about point P. We can determine $I_p$ using the parallel axis theorem. $I_p = I_{com} + MR^2 \rightarrow K = \frac{1}{2} \left( I_{com} + MR^2 \right) \omega^2$

$$K = \frac{1}{2} \left( I_{com} + MR^2 \right) \omega^2 = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} MR^2 \omega^2$$

The expression for the kinetic energy consists of two terms. The first term corresponds to the rotation about the center of mass O with angular velocity $\omega$. The second term is associated with the kinetic energy due to the translational motion of every point with speed $v_{com}$.

$$K = \frac{1}{2} I_{com} \omega^2 + \frac{1}{2} Mv_{com}^2$$
Problem 9. A solid cylinder of radius 10 cm and mass 12 kg starts from rest and rolls without slipping a distance \( L = 6.0 \) m down a roof that is inclined at the angle \( \theta = 30^\circ \).

(a) What is the angular speed of the cylinder about its center as it leaves the roof?

(b) The roof's edge is at height \( H = 5.0 \) m. How far horizontally from the roof's edge does the cylinder hit the level ground?

(a) We find its angular speed as it leaves the roof using conservation of energy. Its initial kinetic energy is \( K_i = 0 \) and its initial potential energy is \( U_i = Mgh \) where

\[
h = 6.0 \sin 30^\circ = 3.0 \text{ m} \quad \text{(we are using the edge of the roof as our reference level for computing } U) .
\]

Its final kinetic energy (as it leaves the roof) is

\[
K_f = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 .
\]

Here we use \( v \) to denote the speed of its center of mass and \( \omega \) is its angular speed — at the moment it leaves the roof. Since (up to that moment) the ball rolls without sliding we can set \( v = R\omega = v \) where \( R = 0.10 \) m. Using \( I = \frac{1}{2} MR^2 \) (Table 10-2(c)), conservation of energy leads to

\[
Mgh = \frac{1}{2} Mv^2 + \frac{1}{2} I\omega^2 = \frac{1}{2} MR^2 \omega^2 + \frac{1}{4} MR^2 \omega^2 = \frac{3}{4} MR^2 \omega^2 .
\]

The mass \( M \) cancels from the equation, and we obtain

\[
\omega = \frac{1}{R} \sqrt{\frac{4}{3} gh} = \frac{1}{0.10 \text{ m}} \sqrt{\frac{4}{3} (9.8 \text{ m/s}^2)(3.0 \text{ m})} = 63 \text{ rad/s} .
\]
(b) Now this becomes a projectile motion of the type examined in Chapter 4. We put the origin at the position of the center of mass when the ball leaves the track (the “initial” position for this part of the problem) and take +x leftward and +y downward. The result of part (a) implies \( v_0 = R \omega = 6.3 \text{ m/s} \), and we see from the figure that (with these positive direction choices) its components are

\[
\begin{align*}
    v_{0x} &= v_0 \cos 30^\circ = 5.4 \text{ m/s} \\
    v_{0y} &= v_0 \sin 30^\circ = 3.1 \text{ m/s}.
\end{align*}
\]

The projectile motion equations become

\[
x = v_{0x} t \quad \text{and} \quad y = v_{0y} t + \frac{1}{2} gt^2.
\]

We first find the time when \( y = H = 5.0 \text{ m} \) from the second equation (using the quadratic formula, choosing the positive root):

\[
t = \frac{-v_{0y} + \sqrt{v_{0y}^2 + 2gH}}{g} = 0.74 \text{ s}.
\]

Then we substitute this into the \( x \) equation and obtain \( x = (5.4 \text{ m/s})(0.74 \text{ s}) = 4.0 \text{ m} \).
Friction and Rolling

When an object rolls with constant speed (see top figure) it has no tendency to slide at the contact point P and thus no frictional force acts there. If a net force acts on the rolling body it results in a non-zero acceleration \( \mathbf{a}_{\text{com}} \) for the center of mass (see lower figure). If the rolling object accelerates to the right it has the tendency to slide at point P to the left. Thus a static frictional force \( \mathbf{f}_s \) opposes the tendency to slide. The motion is smooth rolling as long as \( f_s < f_{s,\text{max}} \).

The rolling condition results in a connection between the magnitude of the acceleration \( a_{\text{com}} \) of the center of mass and its angular acceleration \( \alpha \)

\[ v_{\text{com}} = \omega R \quad \text{We take time derivatives of both sides} \quad \rightarrow \quad a_{\text{com}} = \frac{dv_{\text{com}}}{dt} = R \frac{d\omega}{dt} = R\alpha \]

\[ a_{\text{com}} = R\alpha \]
Problem 7. A constant horizontal force $\vec{F}_{\text{app}}$ of magnitude 10N is applied to a wheel of mass 10 kg and radius 0.30m. The wheel rolls smoothly on the horizontal surface, and the acceleration of its center of mass has magnitude 0.60 m/s$^2$.

(a) In unit-vector notation, what is the frictional force on the wheel?
(b) What is the rotational inertia of the wheel about the rotation axis through its center of mass?

(a) Newton’s second law in the x direction leads to

$$F_{\text{app}} - f_s = ma \quad \Rightarrow f_s = 10 \text{N} - (10 \text{kg}) \left(0.60 \text{ m/s}^2\right) = 4.0 \text{ N}.$$  

In unit vector notation, we have $\vec{f}_s = (-4.0 \text{ N})\hat{i}$ which points leftward.

(b) With $R = 0.30 \text{ m}$, we find the magnitude of the angular acceleration to be

$$|\alpha| = |a_{\text{com}}| / R = 2.0 \text{ rad/s}^2,$$

The only force not directed towards (or away from) the center of mass is $\vec{f}_s$, and the torque it produces is clockwise:

$$|\tau| = I |\alpha| \quad \Rightarrow \quad (0.30 \text{ m})(4.0 \text{ N}) = I (2.0 \text{ rad/s}^2)$$

which yields the wheel’s rotational inertia about its center of mass: $I = 0.60 \text{ kg \cdot m}^2$. 

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Rolling Down a Ramp

Consider a round uniform body of mass $M$ and radius $R$ rolling down an inclined plane of angle $\theta$. We will calculate the acceleration $a_{com}$ of the center of mass along the x-axis using Newton's second law for the translational and rotational motion.

Newton's second law for motion along the x-axis: $f_s - Mg \sin \theta = -Ma_{com}$  \hspace{1cm} \text{(eqs.1)}

Newton's second law for rotation about the center of mass: $\tau = Rf_s = I_{com} \alpha$

$\alpha = \frac{a_{com}}{R}$ \hspace{1cm} \text{We substitute $\alpha$ in the second equation and get: $Rf_s = I_{com} \frac{a_{com}}{R}$} \rightarrow$

$f_s = I_{com} \frac{a_{com}}{R^2}$  \hspace{1cm} \text{We substitute $f_s$ from equation 2 into equation 1} \rightarrow

$I_{com} \frac{a_{com}}{R^2} - Mg \sin \theta = -Ma_{com}$

$$|a_{com}| = \frac{g \sin \theta}{1 + \frac{I_{com}}{MR^2}}$$
Cylinder

\[ I_1 = \frac{MR^2}{2} \]

\[ a_1 = \frac{g \sin \theta}{1 + I_1 / MR^2} \]

\[ a_1 = \frac{g \sin \theta}{1 + MR^2 / 2MR^2} \]

\[ a_1 = \frac{g \sin \theta}{1 + 1/2} \]

\[ a_1 = \frac{2g \sin \theta}{3} = (0.67)g \sin \theta \]

Hoop

\[ I_2 = MR^2 \]

\[ a_2 = \frac{g \sin \theta}{1 + I_2 / MR^2} \]

\[ a_2 = \frac{g \sin \theta}{1 + MR^2 / MR^2} \]

\[ a_2 = \frac{g \sin \theta}{1 + 1} \]

\[ a_2 = \frac{g \sin \theta}{2} = (0.5)g \sin \theta \]

\[ |a_{com}| = \frac{g \sin \theta}{1 + \frac{I_{com}}{MR^2}} \]
The Yo-Yo

Consider a yo-yo of mass $M$, radius $R$, and axle radius $R_o$ rolling down a string. We will calculate the acceleration $a_{com}$ of the center of its mass along the $y$-axis using Newton's second law for the translational and rotational motion as we did in the previous problem.

Newton's second law for motion along the $y$-axis:

$$Mg - T = Ma_{com} \quad (\text{eqs.1})$$

Newton's second law for rotation about the center of mass:

$$\tau = R_o T = I_{com} \alpha. \quad \text{Angular acceleration } \alpha = \frac{a_{com}}{R_o}$$

We substitute $\alpha$ in the second equation and get:

$$T = I_{com} \frac{a_{com}}{R_o^2} \quad (\text{eqs.2}) \quad \text{We substitute } T \text{ from equation 2 into equation 1 } \rightarrow$$

$$Mg - I_{com} \frac{a_{com}}{R_o^2} = Ma_{com} \rightarrow a_{com} = \frac{g}{1 + \frac{I_{com}}{MR_o^2}}$$
Torque Revisited

In chapter 10 we defined the torque \( \tau \) of a rigid body rotating about a fixed axis with each particle in the body moving on a circular path. We now expand the definition of torque so that it can describe the motion of a particle that moves along any path relative to a fixed point. If \( \vec{r} \) is the position vector of a particle on which a force \( \vec{F} \) is acting, the torque \( \vec{\tau} \) is defined as:

\[
\vec{\tau} = \vec{r} \times \vec{F}
\]

In the example shown in the figure both \( \vec{r} \) and \( \vec{F} \) lie in the xy-plane. Using the right hand rule we can see that the direction of \( \vec{\tau} \) is along the z-axis. The magnitude of the torque vector \( \tau = rF \sin \phi \), where \( \phi \) is the angle between \( \vec{r} \) and \( \vec{F} \). From triangle OAB we have: \( r \sin \phi = r_\perp \rightarrow \tau = r_\perp F \), in agreement with the definition of chapter 10.

\[
\vec{\tau} = \vec{r} \times \vec{F}
\]
Problem 21. In unit-vector notation, what is the net torque about the origin on a flea located at coordinates (0, -4.0m, 5.0 m) when forces $\vec{F}_1 = (3.0N)\hat{k}$ and $\vec{F}_2 = (-2.0N)\hat{j}$ act on the flea?

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} = \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} \hat{i} - \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} \hat{j} + \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \hat{k} = (a_y b_z - a_z b_y)\hat{i} + (a_z b_x - a_x b_z)\hat{j} + (a_x b_y - a_y b_x)\hat{k}$$

If we write $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$, then we find $\vec{r} \times \vec{F}$ is equal to

$$\left(yF_z - zF_y \right)\hat{i} + \left(zF_x - xF_z \right)\hat{j} + \left(xF_y - yF_x \right)\hat{k}.$$  

With (using SI units) $x = 0$, $y = -4.0$, $z = 5.0$, $F_x = 0$, $F_y = -2.0$ and $F_z = 3.0$ (these latter terms being the individual forces that contribute to the net force), the expression above yields

$$\vec{\tau} = \vec{r} \times \vec{F} = (-2.0N \cdot m)\hat{i}.$$
Angular Momentum

The counterpart of linear momentum $\vec{p} = m\vec{v}$ in rotational motion is a new vector known as angular momentum. The new vector is defined as follows: $\vec{\ell} = \vec{r} \times \vec{p}$

In the example shown in the figure both $\vec{r}$ and $\vec{p}$ lie in the $xy$-plane. Using the right hand rule we can see that the direction of $\vec{\ell}$ is along the $z$-axis. The magnitude of angular momentum $\ell = rmv \sin \phi$, where $\phi$ is the angle between $\vec{r}$ and $\vec{p}$. From triangle OAB we have: $r \sin \phi = r_\perp \rightarrow \ell = r_\perp mv$

Note: Angular momentum depends on the choice of the origin O. If the origin is shifted in general we get a different value of $\vec{\ell}$

SI unit for angular momentum: kg.m$^2$/s. Sometimes the equivalent J.s is used

$\vec{\ell} = \vec{r} \times \vec{p} = m(\vec{r} \times \vec{v}) \quad \ell = r_\perp mv$
Problem 29. At one instant, force $\vec{F} = 4.0 \, \hat{j} \text{ N}$ acts on a 0.25 kg object that has position vector $\vec{r} = (2.0 \hat{i} - 2.0 \hat{k}) \text{ m}$ and velocity vector $\vec{v} = (-5.0 \hat{i} + 5.0 \hat{k}) \text{ m/s}$. About the origin and in unit vector notation, what are (a) the object's angular momentum and (b) the torque acting on the object.

(a) We use $\vec{\ell} = m\vec{r} \times \vec{v}$, where $\vec{r}$ is the position vector of the object, $\vec{v}$ is its velocity vector, and $m$ is its mass. Only the $x$ and $z$ components of the position and velocity vectors are nonzero, so Eq. 3-30 leads to $\vec{r} \times \vec{v} = (-xv_z + zv_x) \hat{j}$. Therefore,

$$\vec{\ell} = m(-xv_z + zv_x) \hat{j} = (0.25 \text{ kg})(-2.0 \text{ m})(5.0 \text{ m/s}) + (-2.0 \text{ m})(-5.0 \text{ m/s})) \hat{j} = 0.$$  

(b) If we write $\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$, then (using Eq. 3-30) we find $\vec{r} \times \vec{F}$ is equal to

$$\left(yF_z - zF_y \right) \hat{i} + \left(zF_x - xF_z \right) \hat{j} + \left(xF_y - yF_x \right) \hat{k}.$$  

With $x = 2.0$, $z = -2.0$, $F_y = 4.0$ and all other components zero (and SI units understood) the expression above yields

$$\vec{\tau} = \vec{r} \times \vec{F} = \left(8.0 \hat{i} + 8.0 \hat{k} \right) \text{ N} \cdot \text{ m}.$$  

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Newton's Second Law in Angular Form

Newton's second law for linear motion has the form: \( \vec{F}_{net} = \frac{d\vec{p}}{dt} \). Below we will derive the angular form of Newton's second law for a particle.

\[
\vec{\ell} = m(\vec{r} \times \vec{v}) \rightarrow \frac{d\vec{\ell}}{dt} = m \frac{d}{dt} (\vec{r} \times \vec{v}) = m \left( \vec{r} \times \frac{d\vec{v}}{dt} + \frac{d\vec{r}}{dt} \times \vec{v} \right) = m (\vec{r} \times \vec{a} + \vec{v} \times \vec{v})
\]

\[
\vec{v} \times \vec{v} = 0 \rightarrow \frac{d\vec{\ell}}{dt} = m(\vec{r} \times \vec{a}) = (\vec{r} \times m\vec{a}) = (\vec{r} \times \vec{F}_{net}) = \vec{\tau}_{net}
\]

Thus: \( \vec{\tau}_{net} = \frac{d\vec{\ell}}{dt} \)  Compare with: \( \vec{F}_{net} = \frac{d\vec{p}}{dt} \)
The Angular Momentum of a System of Particles

We will now explore Newton's second law in angular form for a system of \( n \) particles that have angular momentum \( \vec{\ell}_1, \vec{\ell}_2, \vec{\ell}_3, \ldots, \vec{\ell}_n \).

The angular momentum \( \vec{L} \) of the system is
\[
\vec{L} = \vec{\ell}_1 + \vec{\ell}_2 + \vec{\ell}_3 + \ldots + \vec{\ell}_n = \sum_{i=1}^{n} \vec{\ell}_i
\]

The time derivative of the angular momentum is
\[
\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \frac{d\vec{\ell}_i}{dt}
\]

The time derivative for the angular momentum of the \( i \)-th particle
\[
\frac{d\vec{\ell}_i}{dt} = \vec{\tau}_{net,i}
\]

Where \( \vec{\tau}_{net,i} \) is the net torque on the particle. This torque has contributions from external as well as internal forces between the particles of the system. Thus
\[
\frac{d\vec{L}}{dt} = \sum_{i=1}^{n} \vec{\tau}_{net,i} = \vec{\tau}_{net}
\]

Here \( \vec{\tau}_{net} \) is the net torque due to all the external forces.

By virtue of Newton's third law the vector sum of all internal torques is zero.

Thus Newton's second law for a system in angular form takes the form:

\[
\frac{d\vec{L}}{dt} = \vec{\tau}_{net \ ext}
\]
Angular Momentum of a Rigid Body Rotating about a Fixed Axis

We take the z-axis to be the fixed rotation axis. We will determine the z-component of the net angular momentum. The body is divided into elements of mass $\Delta m_i$ that have a position vector $\vec{r}_i$

The angular momentum $\vec{\ell}_i$ of the $i$-the element is: $\vec{\ell}_i = \vec{r}_i \times \vec{p}_i$

Its magnitude $\ell_i = r_i p_i (\sin 90^\circ) = r_i \Delta m_i v_i$. The z-component $\ell_{iz}$ of $\ell_i$ is: $\ell_{iz} = \ell_i \sin \theta = (r_i \sin \theta)(\Delta m_i v_i) = r_{\perp i} \Delta m_i v_i$

The z-component of the angular momentum $L_z$ is the sum:

$$L_z = \sum_{i=1}^{n} \ell_{iz} = \sum_{i=1}^{n} r_{\perp i} \Delta m_i v_i = \sum_{i=1}^{n} r_{\perp i} \Delta m_i (\omega r_{\perp i}) = \omega \left( \sum_{i=1}^{n} \Delta m_i r_{i\perp}^2 \right)$$

The sum $\sum_{i=1}^{n} \Delta m_i r_{i\perp}^2$ is the rotational inertia $I$ of the rigid body.

Thus: $L_z = I \omega$

$$L_z = I \omega$$
Problem 34. A particle is acted on by two torques about the origin: \( \vec{\tau}_1 \) has a magnitude of 2.0 Nm and is directed in the positive direction of the x axis, and \( \vec{\tau}_2 \) has a magnitude of 4.0 Nm and is directed in the negative direction of the y axis. In unit-vector notation, find \( \frac{d\vec{l}}{dt} \), where \( \vec{l} \) is the angular momentum of the particle about the origin.

The rate of change of the angular momentum is

\[
\frac{d\vec{l}}{dt} = \vec{\tau}_1 + \vec{\tau}_2 = (2.0 \text{ N} \cdot \text{m})\hat{i} - (4.0 \text{ N} \cdot \text{m})\hat{j}.
\]

Consequently, the vector \( \frac{d\vec{l}}{dt} \) has a magnitude \( \sqrt{(2.0 \text{ N} \cdot \text{m})^2 + (-4.0 \text{ N} \cdot \text{m})^2} = 4.5 \text{ N} \cdot \text{m} \) and is at an angle \( \theta \) (in the xy plane, or a plane parallel to it) measured from the positive x axis, where

\[
\theta = \tan^{-1}\left(\frac{-4.0 \text{ N} \cdot \text{m}}{2.0 \text{ N} \cdot \text{m}}\right) = -63^\circ,
\]

the negative sign indicating that the angle is measured clockwise as viewed “from above” (by a person on the +z axis).
Problem 42. A disk with a rotational inertia of 7.00 kgm$^2$ rotates like a merry-go-round while undergoing a torque given by $\tau = (5.00 + 2.00t) N\cdot m$. At time $t=1.00$ s, its angular momentum is 5.00 kgm$^2$/s. What is its angular momentum at $t=3.00s$?

Torque is the time derivative of the angular momentum. Thus, the change in the angular momentum is equal to the time integral of the torque. With $\tau = (5.00 + 2.00t) N\cdot m$, the angular momentum as a function of time is (in units kg$\cdot$m$^2$/s)

$$L(t) = \int \tau dt = \int (5.00 + 2.00t)dt = L_0 + 5.00t + 1.00t^2$$

Since $L = 5.00$ kg$\cdot$m$^2$/s when $t = 1.00$ s, the integration constant is $L_0 = -1$. Thus, the complete expression of the angular momentum is

$$L(t) = -1 + 5.00t + 1.00t^2.$$

At $t = 3.00$ s, we have $L(t = 3.00) = -1 + 5.00(3.00) + 1.00(3.00)^2 = 23.0$ kg$\cdot$m$^2$/s.
Conservation of Angular momentum

For any system of particles (including a rigid body) Newton's second law in angular form is:

$$\frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

If the net external torque $\vec{\tau}_{net} = 0$ then we have: $\frac{d\vec{L}}{dt} = 0 \rightarrow \vec{L} = \text{a constant}$

This result is known as the law of the conservation of angular momentum. In words:

$$\left( \begin{array}{c} \text{Net angular momentum} \\ \text{at some initial time } t_i \end{array} \right) = \left( \begin{array}{c} \text{Net angular momentum} \\ \text{at some later time } t_f \end{array} \right)$$

In equation form: $\vec{L}_i = \vec{L}_f$

Note: If the component of the external torque along a certain axis is equal to zero, then the component of the angular momentum of the system along this axis cannot change.
Example: The figure shows a student seated on a stool that can rotate freely about a vertical axis. The student who has been set into rotation at an initial angular speed \( \omega_i \), holds two dumbbells in his outstretched hands. His angular momentum vector \( \vec{L} \) lies along the rotation axis, pointing upward.

The student then pulls in his hands as shown in fig.\( b \). This action reduces the rotational inertia from an initial value \( I_i \) to a smaller final value \( I_f \).

No net external torque acts on the student-stool system. Thus the angular momentum of the system remains unchanged.

Angular momentum at \( t_i \): \( L_i = I_i \omega_i \)  
Angular momentum at \( t_f \): \( L_f = I_f \omega_f \)

\[
L_i = L_f \rightarrow I_i \omega_i = I_f \omega_f \rightarrow \omega_f = \frac{I_i}{I_f} \omega_i
\]

Since \( I_f < I_i \rightarrow \frac{I_i}{I_f} > 1 \rightarrow \omega_f > \omega_i \)

The rotation rate of the student in fig.\( b \) is faster
Sample Problem 11-7:

\[ I_{wh} = 1.2 \text{ kg.m}^2 \]

\[ \omega_{wh} = 2\pi \times 3.9 \text{ rad/s} \]

\[ I_b = 6.8 \text{ kg.m}^2 \]

\[ \omega_b = ? \]

\[ L_i = L_f \rightarrow L_{wh} = -L_{wh} + L_b \rightarrow L_b = 2L_{wh} \]

\[ I_b \omega_b = 2I_{wh} \omega_{wh} \rightarrow \omega_b = \frac{2I_{wh} \omega_{wh}}{I_b} = \frac{2 \times 1.2 \times 2\pi \times 3.9}{6.8} = 2\pi \times 1.4 \text{ rad/s} \]
Problem 60. A horizontal platform in the shape of a circular disk rotates on a frictionless bearing about a vertical axle through the center of the disk. The platform has a mass of 150 kg, a radius of 2.0 m, and a rotational inertia of 300 kgm$^2$ about the axis of rotation. A 60 kg student walks slowly from the rim of the platform towards the center. If the angular speed of the system is 1.5 rad/s when the student starts at the rim, what is the angular speed when she is 0.5 m from the center?

The initial rotational inertia of the system is $I_i = I_{\text{disk}} + I_{\text{student}}$, where $I_{\text{disk}} = 300 \text{ kg}\cdot\text{m}^2$ (which, incidentally, does agree with Table 10-2(c)) and $I_{\text{student}} = mR^2$ where $m = 60 \text{ kg}$ and $R = 2.0 \text{ m}$.

The rotational inertia when the student reaches $r = 0.5 \text{ m}$ is $I_f = I_{\text{disk}} + mr^2$. Angular momentum conservation leads to

$$I_i \omega_i = I_f \omega_f \Rightarrow \omega_f = \omega_i \frac{I_{\text{disk}} + mR^2}{I_{\text{disk}} + mr^2}$$

which yields, for $\omega_i = 1.5 \text{ rad/s}$, a final angular velocity of $\omega_f = 2.6 \text{ rad/s}$.
Analogies between translational and rotational Motion

Translational Motion  Rotational Motion

\[ x \leftrightarrow \theta \]
\[ v \leftrightarrow \omega \]
\[ a \leftrightarrow \alpha \]
\[ p \leftrightarrow l \]

\[ K = \frac{mv^2}{2} \leftrightarrow \quad K = \frac{I\omega^2}{2} \]
\[ m \leftrightarrow I \]
\[ F = ma \leftrightarrow \tau = I\alpha \]
\[ F \leftrightarrow \tau \]
\[ P = Fv \leftrightarrow \quad P = \tau \omega \]

\[ \vec{F}_{\text{net}} = \frac{d\vec{p}}{dt} \leftrightarrow \quad \vec{\tau}_{\text{net}} = \frac{d\vec{\ell}}{dt} \]
\[ p = mv \leftrightarrow \quad L = I\omega \]