Waves - I

Lectures 25-26

Chapter 16
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
In this chapter we will start the discussion on wave phenomena. We will study the following topics:

Types of waves
Amplitude, phase, frequency, period, propagation speed of a wave
Mechanical waves propagating along a stretched string
Wave equation
Principle of superposition of waves
Wave interference
Standing waves, resonance
Waves

A wave is a vibrational, trembling motion in an elastic, deformable body.

- The wave is initiated by some external force that acts on some parts of the body and deforms it.
- The elastic restoring force communicates this initial disturbance from one part of the body to the next, adjacent part.
- The disturbance gradually spreads outward through the entire elastic body.

The elastic body in which the wave propagate is called the medium.

When a wave propagates through a medium, the particles in the medium vibrate back and forth, but the medium as a whole does not perform translational motion.
Transverse Wave Motion

If we shake one end of the tightly stretched elastic string up and down with a flick of the wrist a disturbance travels along the string.

String is regarded as a row of particles connected by small, massless springs.

1 particle up => it will pull the 2nd particle up.

Transverse wave motion – the direction of the disturbance propagation is perpendicular to the back and forth vibration of the particles.
Longitudinal Wave Motion

Disturbance generated by pushing the 1\textsuperscript{st} particle towards the 2\textsuperscript{nd}

Longitudinal wave - direction of disturbance motion is parallel to the vibration of the particles.
The wave is a propagation of the disturbance through the medium without any net displacement of the medium.

Consider a horizontal string connected to the mass executing simple harmonic motion in the vertical direction.

Each particle transmits the motion to the next particle along the entire length of the string. The resulting wave propagates in the horizontal direction with a velocity “v”, while any one particle of the string executes simple harmonic motion in the vertical direction.

The particle of the string is moving perpendicular to the direction of wave propagation and is not moving in the direction of the wave.

A transverse wave is a wave in which the particles of the medium execute simple harmonic motion in a direction perpendicular to its direction of propagation.
A wave is defined as a disturbance that is self-sustained and propagates in space with a constant speed.

Waves can be classified in the following three categories:

1. **Mechanical waves.** These involve motions that are governed by Newton’s laws and can exist only within a material medium such as air, water, rock, etc. Common examples are: sound waves, seismic waves, etc.

2. **Electromagnetic waves.** These waves involve propagating disturbances in the electric and magnetic field governed by Maxwell’s equations. They do not require a material medium in which to propagate but they travel through vacuum. Common examples are: radio waves of all types, visible, infra-red, and ultra-violet light, x-rays, gamma rays. All electromagnetic waves propagate in vacuum with the same speed $c = 300,000$ km/s.

3. **Matter waves.** All microscopic particles such as electrons, protons, neutrons, atoms etc have a wave associated with them governed by Schroedinger’s equation.
Transverse and Longitudinal waves

Waves can be divided into the following two categories depending on the orientation of the disturbance with respect to the wave propagation velocity $\vec{v}$.

If the disturbance associated with a particular wave is perpendicular to the wave propagation velocity, this wave is called "transverse". An example is given in the upper figure which depicts a mechanical wave that propagates along a string. The movement of each particle on the string is along the $y$-axis; the wave itself propagates along the $x$-axis.

A wave in which the associated disturbance is parallel to the wave propagation velocity is known as a "longitudinal wave". An example of such a wave is given in the lower figure. It is produced by a piston oscillating in a tube filled with air. The resulting wave involves movement of the air molecules along the axis of the tube which is also the direction of the wave propagation velocity $\vec{v}$. 
Consider the transverse wave propagating along the string as shown in the figure. The position of any point on the string can be described by a function \( y = h(x,t) \). Further along the chapter we shall see that function \( h \) has to have a specific form to describe a wave. Once such suitable function is: \( y(x,t) = y_m \sin(kx - \omega t) \)

Such a wave which is described by a sine (or a cosine) function is known as "harmonic wave".

The various terms that appear in the expression for a harmonic wave are identified in the lower figure. Function \( y(x,t) \) depends on \( x \) and \( t \). There are two ways to visualize it. The first is to "freeze" time (i.e. set \( t = t_o \)). This is like taking a snapshot of the wave at \( t = t_o \). \( y = y(x,t_o) \) The second is to set \( x = x_o \). In this case \( y = y(x_o,t) \)
The amplitude \( y_m \) is the absolute value of the maximum displacement from the equilibrium position.

The phase is defined as the argument \( kx - \omega t \) of the sine function.

The wavelength \( \lambda \) is the shortest distance between two repetitions of the wave at a fixed time.

We fix \( t \) at \( t = 0 \). We have the condition: \( y(x_1, 0) = y(x_1 + \lambda, 0) \rightarrow y_m \sin(kx_1) = y_m \sin[k(x_1 + \lambda)] = y_m \sin(kx_1 + k\lambda) \)

Since the sine function is periodic with period \( 2\pi \rightarrow k\lambda = 2\pi \rightarrow k = \frac{2\pi}{\lambda} \)

A period \( T \) is the time it takes (with fixed \( x \) ) the sine function to complete one oscillation. We take \( x = 0 \rightarrow y(0, t) = y(0, t + T) \rightarrow -y_m \sin(\omega t) = -y_m \sin[\omega(t + T)] = -y_m \sin(\omega t + \omega T) \rightarrow \omega T = 2\pi \rightarrow \omega = \frac{2\pi}{T} \)
The speed of a traveling wave

In the figure we show two snapshots of a harmonic wave taken at times $t$ and $t + \Delta t$. During the time interval $\Delta t$ the wave has traveled a distance $\Delta x$. The wave speed $v = \frac{\Delta x}{\Delta t}$. One method of finding $v$ is to imagine that we move with the same speed along the $x$-axis. In this case the wave will seem to us that it does not change.

Since $y(x, t) = y_m \sin (kx - \omega t)$ this means that the argument of the sine function is constant. $kx - \omega t = \text{constant}$. We take the derivative with respect to $t$.

$$v = \frac{dx}{dt} = \frac{\omega}{k}$$

A harmonic wave that propagates along the negative $x$-axis is described by the equation: $y(x, t) = y_m \sin (kx + \omega t)$. The function $y(x, t) = h (kx - \omega t)$ describes a general wave that propagates along the positive $x$-axis. A general wave that propagates along the negative $x$-axis is described by the equation: $y(x, t) = h (kx + \omega t)$
A particular wave is given by  \( y = (0.200\text{m})\sin[(0.500\text{m}^{-1})x - (8.20\text{rad/s})t] \)

**Find:**

a) \( A \)  

b) \( K \)  

c) \( \lambda \)  

d) \( \omega \)  

e) \( f \)  

f) \( t \)  

g) \( v \)  

h) \( y \) at \( x = 10.0\text{m} \) and \( t = 0.500\text{s} \)

---

\[
y = A\sin(Kx - \omega t) \quad \text{- standard form of the wave.}
\]

\( a) \quad A = 0.200\text{m} \quad \text{(by inspection of both eq.)} \]

\( b) \quad K = 0.500\text{m}^{-1} \quad \text{- standard form of the wave.} \]

\( c) \quad \lambda = \frac{2\pi}{K} = \frac{2\pi}{0.500\text{m}^{-1}} = 12.6\text{m} \]

\( d) \quad \omega = 8.20\text{rad/s} \]

\( e) \quad f = \frac{\omega}{2\pi} = \frac{8.20\text{rad/s}}{2\pi\text{rad}} = 1.31\text{Hz} \]

\( f) \quad T = \frac{1}{f} = \frac{1}{1.31\text{Hz}} = 0.766\text{s} \]

\( g) \quad v = \frac{\lambda}{T} = \lambda f \]

\( h) \quad y = 0.200\sin[(0.500\text{m}^{-1})(10.0\text{m}) - (8.20\text{rad/s})(0.500\text{s})] = 0.200\sin(0.900\text{rad}) = 0.200\text{m}(0.783) = 0.157\text{m} \)
Problem 5. If \( y(x,t) = (6.0 \text{mm}) \sin(kx + (600 \text{rad} / s)t + \phi) \) describes a wave traveling along a string, how much time does any given point on the string take to move between displacements \( y = +2.0 \text{ mm} \) and \( y = -2.00 \text{mm} \)?

Let \( y_1 = 2.0 \text{ mm} \) (corresponding to time \( t_1 \)) and \( y_2 = -2.0 \text{ mm} \) (corresponding to time \( t_2 \)). Then we find

\[
kx + 600t_1 + \phi = \sin^{-1}(2.0/6.0)
\]

and

\[
kx + 600t_2 + \phi = \sin^{-1}(-2.0/6.0).
\]

Subtracting equations gives

\[
600(t_1 - t_2) = \sin^{-1}(2.0/6.0) - \sin^{-1}(-2.0/6.0).
\]

Thus we find \( t_1 - t_2 = 0.011 \text{ s} \) (or 1.1 ms).
Problem 9. A transverse sinusoidal wave is moving along a string in the positive direction of an x axis with a speed of 80 m/s. At t=0, the string particle at x=0 has a transverse displacement of 4.0 cm from its equilibrium position and is not moving. The maximum transverse speed of the string particle at x=0 is 16 m/s. (a) What is the frequency of the wave? (b) What is the wavelength of the wave? If the wave equation is of the form \( y(x,t) = y_m \sin(kx \pm \omega t + \phi) \) what are (c) \( y_m \), (d) \( k \), (e) \( \omega \), (f) \( \phi \), and (g) the correct choice of sign in front of \( \omega \)?

(a) Recalling from Ch. 12 the simple harmonic motion relation \( u_m = y_m \omega \), we have

\[
\omega = \frac{16}{0.040} = 400 \text{ rad/s}.
\]

Since \( \omega = 2\pi f \), we obtain \( f = 64 \text{ Hz} \).

(b) Using \( v = f\lambda \), we find \( \lambda = 80/64 = 1.26 \text{ m} \approx 1.3 \text{ m} \).

(c) The amplitude of the transverse displacement is \( y_m = 4.0 \text{ cm} = 4.0 \times 10^{-2} \text{ m} \).

(d) The wave number is \( k = 2\pi/\lambda = 5.0 \text{ rad/m} \).

(e) The angular frequency, as obtained in part (a), is \( \omega = 16/0.040 = 4.0 \times 10^2 \text{ rad/s} \).

(f) The function describing the wave can be written as

\[
y = 0.040 \sin \left(5x - 400t + \phi\right)
\]

where distances are in meters and time is in seconds. We adjust the phase constant \( \phi \) to satisfy the condition \( y = 0.040 \) at \( x = t = 0 \). Therefore, \( \sin \phi = 1 \), for which the “simplest” root is \( \phi = \pi/2 \). Consequently, the answer is

\[
y = 0.040 \sin \left(5x - 400t + \frac{\pi}{2}\right).
\]

(g) The sign in front of \( \omega \) is minus.
Problem 29. Use the wave equation to find the speed of a wave given by 
\[ y(x, t) = (2.00 \text{mm})[(20 \text{ m}^{-1})x - (4.0 \text{ s}^{-1})t]^{0.5}. \]

The wave \( y(x, t) = (2.00 \text{ mm})[(20 \text{ m}^{-1})x - (4.0 \text{ s}^{-1})t]^{1/2} \) is of the form \( h(kx - \omega t) \) with angular wave number \( k = 20 \text{ m}^{-1} \) and angular frequency \( \omega = 4.0 \text{ rad/s} \). Thus, the speed of the wave is

\[ v = \frac{\omega}{k} = \frac{(4.0 \text{ rad/s})}{(20 \text{ m}^{-1})} = 0.20 \text{ m/s}. \]
Wave speed on a stretched string

Below we will determine the speed of a wave that propagates along a string whose linear mass density is \( \mu \). The tension on the string is equal to \( \tau \).

Consider a small section of the string of length \( \Delta \ell \).

The shape of the element can be approximated to be an arc of a circle of radius \( R \) whose center is at \( O \). The net force in the direction of \( O \) is \( F = 2(\tau \sin \theta) \).

Here we assume that \( \theta \ll 1 \rightarrow \sin \theta \approx \theta = \frac{\Delta \ell}{2R} \rightarrow F = \tau \frac{\Delta \ell}{R} \) (eqs.1)

The force is also given by Newton's second law: \( F = \Delta m \frac{v^2}{R} = (\mu \Delta \ell) \frac{v^2}{R} \) (eqs.2)

If we compare equations 1 and 2 we get: \( (\mu \Delta \ell) \frac{v^2}{R} = \tau \frac{\Delta \ell}{R} \rightarrow v = \sqrt{\frac{\tau}{\mu}} \)

Note: The speed \( v \) depends on the tension \( \tau \) and the mass density \( \mu \) but not on the wave frequency \( f \).
Problem: The drawing shows a 15.0 kg ball being whirled in a circular path on the end of a string. The motion occurs on a frictionless horizontal table. The angular speed of the ball is \( \omega = 12.0 \text{ rad/s} \). The string has a mass of 0.0230 kg. How much time does it take for a wave on the string to travel from the center of the circle to the ball?
The Speed of a Wave on a String

**Problem**  To measure the acceleration due to gravity on a distant planet, an astronaut hangs a 0.086 kg ball from the end of a wire. The wire has length of 1.5 m and a linear density of $3.1 \times 10^{-4}$ kg/m. Using electronic equipment, the astronaut measures the time for a transverse pulse to travel the length of the wire and obtains a value of 0.083 s. The mass of the wire is negligible compared to the mass of the ball. Determine the acceleration due to gravity.

\[
\text{Given:} \\
L = 3.1 \times 10^{-4} \text{ kg/m} \\
L = 1.5 \text{ m} \\
M = 0.085 \text{ kg} \\
T = 0.083 \text{ s} \\
m \ll \mu \\

\text{Find } g = ?
\]

1) Free body diagram of a hanging mass M:

\[
T - Mg = 0; \quad T = Mg \\
\frac{T}{m} = g
\]

2) For the pulse on the wire:

\[
v = \sqrt{\frac{T}{mL}} \\
T = (mL)v^2 \\
v^2 = \left(\frac{L}{T}\right)^2
\]

\[
g = \left(\frac{mL}{T^2}\right)\frac{1}{m} = (3.1 \times 10^{-4} \text{ kg/m}) \left(\frac{1.5 \text{ m}}{0.083 \text{ s}}\right)^2 \cdot \frac{1}{0.085 \text{ kg}} = \frac{1.2 \text{ m/s}^2}{18}
\]
The Speed of a Wave on a String

Problem: A horizontal wire is under a tension of 315 N and has a mass per unit length of \(6.50 \times 10^{-3}\) kg/m. A transverse wave with an amplitude of 2.50 mm and a frequency of 585 Hz is traveling on this wire. As the wave passes, a particle of the wire moves up and down in simple harmonic motion.

Obtain:

a) The speed of the wave
b) The maximum speed with which the particle moves up and down

\[
\begin{align*}
\text{Given} & \\
T & = 315 \text{ N} \\
\mu & = 6.50 \times 10^{-3} \text{ kg/m} \\
f & = 585 \text{ Hz} \\
A & = 2.50 \times 10^{-3} \text{ m}
\end{align*}
\]

\[
\begin{align*}
\text{Find} & \\
(a) & \text{ } v_{\text{wave}} = ? \\
(b) & \text{ } (v_{\text{particle}})_{\text{max}} = ?
\end{align*}
\]

(a) \[
\begin{align*}
v_{\text{wave}} & = \sqrt{\frac{T}{\mu}} \\
& = \sqrt{\frac{315 \text{ N}}{6.50 \times 10^{-3} \text{ kg/m}}} \\
& = 2.20 \times 10^{2} \text{ m/s}
\end{align*}
\]

(b) \[
\begin{align*}
v_{\text{particle}} & = -A \omega \sin \omega t \\
(v_{\text{particle}})_{\text{max}} & = A \omega \\
& = 2\pi f A \\
& = (2\pi f) A = 2\pi (585 \text{ Hz}) (2.50 \times 10^{-3} \text{ m}) \\
& = 9.19 \text{ m/s}
\end{align*}
\]
Problem 23. A 100 g wire is held under a tension of 250 N with one end at x=0 and the other at x=10.0 m. At time t=0, pulse 1 is sent along the wire from the end at x=10.0 m. At time t=30.0 ms, pulse 2 is sent along the wire from the end at x=0. At what position x do the pulses begin to meet?

The pulses have the same speed \( v \). Suppose one pulse starts from the left end of the wire at time \( t = 0 \). Its coordinate at time \( t \) is \( x_1 = vt \). The other pulse starts from the right end, at \( x = L \), where \( L \) is the length of the wire, at time \( t = 30 \) ms. If this time is denoted by \( t_0 \) then the coordinate of this wave at time \( t \) is \( x_2 = L - v(t - t_0) \). They meet when \( x_1 = x_2 \), or, what is the same, when \( vt = L - v(t - t_0) \). We solve for the time they meet: \( t = (L + vt_0)/2v \) and the coordinate of the meeting point is \( x = vt = (L + vt_0)/2 \). Now, we calculate the wave speed:

\[
v = \sqrt{\frac{\tau L}{m}} = \sqrt{\frac{(250 \text{ N})(10.0 \text{ m})}{0.100 \text{ kg}}} = 158 \text{ m/s}.
\]

Here \( \tau \) is the tension in the wire and \( m/L \) is the linear mass density of the wire. The coordinate of the meeting point is

\[
x = \frac{10.0 \text{ m} + (158 \text{ m/s})(30.0 \times 10^{-3} \text{s})}{2} = 7.37 \text{ m}.
\]

This is the distance from the left end of the wire. The distance from the right end is \( L - x = (10.0 \text{ m} - 7.37 \text{ m}) = 2.63 \text{ m} \).
Rate of energy transmission

Consider a transverse wave propagating along a string which is described by the equation:
\[ y(x,t) = y_m \sin(kt - \omega t) \]. The transverse velocity
\[ u = \frac{\partial y}{\partial t} = -\omega y_m \cos(kt - \omega t) \]
At point a both \( u \) and \( K \) are equal to zero. At point b both \( u \) and \( K \) have maxima.

In general the kinetic energy of an element of mass \( dm \) is given by:
\[ dK = \frac{1}{2} dmv^2 \]
\[ dK = \left(\frac{1}{2} \mu dx\right) \left[ -\omega y_m \cos(kt - \omega t) \right]^2 \]
The rate at which kinetic energy propagates along the string is equal to
\[ \frac{dK}{dt} = \frac{1}{2} \mu v^2 y_m^2 \cos^2(kt - \omega t) \]
The average rate
\[ \left( \frac{dK}{dt} \right)_{avg} = \frac{1}{2} \mu v^2 y_m^2 \left[ \cos^2(kt - \omega t) \right]_{avg} = \frac{1}{4} \mu v^2 y_m^2 \]
As in the case of the oscillating spring-mass system
\[ \left( \frac{dU}{dt} \right)_{avg} = \left( \frac{dK}{dt} \right)_{avg} \rightarrow P_{avg} = \left( \frac{dU}{dt} \right)_{avg} + \left( \frac{dK}{dt} \right)_{avg} = \frac{1}{2} \mu v^2 y_m^2 \]
Problem 26. A string along which waves can travel is 2.70 m long and has a mass of 260 g. The tension in the string is 36.0 N. What must be the frequency of traveling waves of amplitude 7.70 mm for the average power to be 85.0 W?

\[ P_{\text{avg}} = \frac{1}{2} \mu \nu \omega^2 y_m^2 \]

\[ v = \sqrt{\frac{\tau}{\mu}} \]

Using Eq. 16–33 for the average power and Eq. 16–26 for the speed of the wave, we solve for \( f = \frac{\omega}{2\pi} \):

\[ f = \frac{1}{2\pi y_m} \sqrt{\frac{2P_{\text{avg}}}{\mu \sqrt{\tau/\mu}}} = \frac{1}{2\pi (7.70 \times 10^{-3} \text{ m})} \sqrt{\frac{2(85.0 \text{ W})}{\sqrt{(36.0 \text{ N})(0.260 \text{ kg} / 2.70 \text{ m})}}} = 198 \text{ Hz}. \]
The wave equation
Consider a string of mass density $\mu$ and tension $\tau$
A transverse wave propagates along the string.
The transverse motion is described by $y(x,t)$
Consider an element of length $dx$ and mass $dm = \mu dx$
The forces $F_1 = F_2 = \tau$
the net force along the y-axis is given by the equation:
$F_{y\text{net}} = F_2 \sin \theta_2 - F_1 \sin \theta_1 = \tau (\sin \theta_2 - \sin \theta_1)$
Here we assume that $\theta_1 \ll 1$ and $\theta_2 \ll 1 \rightarrow \sin \theta_1 = \tan \theta_1 = \left(\frac{\partial y}{\partial x}\right)_1$
and $\sin \theta_2 = \tan \theta_2 = \left(\frac{\partial y}{\partial x}\right)_2 \rightarrow F_{y\text{net}} = \tau \left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right]$
From Newton's second law we have: $F_{y\text{net}} = dma_y = \mu dx \frac{\partial^2 y}{\partial t^2} = \tau \left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right]$

$$\tau \left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right] = \mu dx \frac{\partial^2 y}{\partial t^2} \rightarrow \left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1 = \frac{\partial^2 y}{dx^2} = \frac{\partial^2 y}{\tau \frac{\partial t^2}{v^2}} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2}$$
The Superposition of Waves

A superposition principle: when two or more waves arrive at any given point simultaneously, the resultant instantaneous deformation is the sum of the individual instantaneous deformations.

- The waves do not interact, they have no effect on one another
- Each wave propagates as though the others were not present
- The net displacement of the medium is the vector sum of the individual displacements

The superposition principle is very well satisfied for waves of low amplitude.

For waves of very large amplitude the superposition principle fails, because the first wave alters the properties of the medium and therefore affects the behavior of a second wave propagating on the same string, air, etc. (medium)
The principle of superposition for waves

The wave equation \( \frac{\partial^2 y}{\partial x^2} = \frac{\mu}{\tau} \frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2} \frac{\partial^2 y}{\partial t^2} \) even though it was derived for a transverse wave propagating along a string under tension, is true for all types of waves. This equation is "linear" which means that if \( y_1 \) and \( y_2 \) are solutions of the wave equation, the function \( c_1y_1 + c_2y_2 \) is also a solution. Here \( c_1 \) and \( c_2 \) are constants. The principle of superposition is a direct consequence of the linearity of the wave equation. This principle can be expressed as follows:

Consider two waves of the same type that overlap at some point P in space. Assume that the functions \( y_1(x,t) \) and \( y_2(x,t) \) describe the displacements if the wave arrived at P alone. The displacement at P when both waves are present is given by: \( y'(x,t) = y_1(x,t) + y_2(x,t) \)

Note: Overlapping waves do not in any way alter the travel of each other
Interference of waves

Consider two harmonic waves of the same amplitude and frequency which propagate along the x-axis. The two waves have a phase difference $\phi$. We will combine these waves using the principle of superposition. The phenomenon of combing waves is known as interference and the two waves are said to interfere. The displacement of the two waves are given by the functions: $y_1(x, t) = y_m \sin(kx - \omega t)$ and $y_2(x, t) = y_m \sin(kx - \omega t + \phi)$. $y' = y_1 + y_2$

$y'(x, t) = y_m \sin(kx - \omega t) + y_m \sin(kx - \omega t + \phi)$

$y'(x, t) = \left[2y_m \cos\frac{\phi}{2}\right] \sin\left(kx - \omega t + \frac{\phi}{2}\right)$

The resulting wave has the same frequency as the original waves, and its amplitude

$y'_m = \left|2y_m \cos\frac{\phi}{2}\right|$  

Its phase is equal to $\frac{\phi}{2}$
Constructive interference

The amplitude of two interfering waves is given by:

\[ y'_{m} = \left| 2y_{m} \cos \frac{\phi}{2} \right| \]

It has its maximum value if \( \phi = 0 \)

In this case \( y'_{m} = 2y_{m} \)

The displacement of the resulting wave is:

\[ y'(x, t) = \left[ 2y_{m} \right] \sin \left( kx - \omega t + \frac{\phi}{2} \right) \]

This phenomenon is known as *fully constructive interference*
Destructive interference

The amplitude of two interfering waves is given by:

\[ y_m' = \left| 2y_m \cos \frac{\phi}{2} \right| \]

It has its minimum value if \( \phi = \pi \)

In this case \( y_m' = 0 \)

The displacement of the resulting wave is:

\[ y'(x, t) = 0 \]

This phenomenon is known as **fully destructive interference**.
Intermediate interference

The amplitude of two interfering waves is given by:

\[ y'_m = \left| 2y_m \cos \frac{\phi}{2} \right| \]

When interference is neither fully constructive nor fully destructive it is called intermediate interference.

An example is given in the figure for \( \phi = \frac{2\pi}{3} \).

In this case \( y'_m = y_m \).

The displacement of the resulting wave is:

\[ y'(x, t) = [y_m] \sin \left( kx - \omega t + \frac{\pi}{3} \right) \]

Note: Sometimes the phase difference is expressed as a difference in wavelength \( \lambda \).

In this case remember that: \( 2\pi \text{ radians} \leftrightarrow 1\lambda \)
Problem 32. What phase difference between two identical traveling waves moving in the same
direction along a stretched string, results in the combined wave having an amplitude 1.50 times
that of the common amplitude of the two combining waves? Express your answer in
(a) degrees, (b) radians, (c) wavelengths?

\[ y_m' = \left| 2y_m \cos \frac{\phi}{2} \right| \]

(a) Let the phase difference be \( \phi \). Then from Eq. 16–52, \( 2y_m \cos(\phi/2) = 1.50y_m \), which
gives

\[ \phi = 2 \cos^{-1} \left( \frac{1.50y_m}{2y_m} \right) = 82.8^\circ. \]

(b) Converting to radians, we have \( \phi = 1.45 \text{ rad} \).

(c) In terms of wavelength (the length of each cycle, where each cycle corresponds to \( 2\pi \)
rad), this is equivalent to \( 1.45 \text{ rad} / 2\pi = 0.230 \text{ wavelength} \).
Phasors
A phasor is a method for representing a wave whose displacement is: \( y_1(x, t) = y_{m1} \sin(kx - \omega t) \)

The phasor is defined as a vector with the following properties:
1. Its magnitude is equal to the wave's amplitude \( y_{m1} \)
2. The phasor has its tail at the origin O and rotates in the clockwise direction about an axis through O with angular speed \( \omega \).

Thus defined, the projection of the phasor on the y-axis (i.e. its y-component) is equal to \( y_{m1} \sin(kx - \omega t) \)

A phasor diagram can be used to represent more than one waves. (see fig.b). The displacement of the second wave is: \( y_2(x, t) = y_{m2} \sin(kx - \omega t + \phi) \) The phasor of the second wave forms an angle \( \phi \) with the phasor of the first wave indicating that it lags behind wave 1 by a phase angle \( \phi \).
Wave addition using phasors

Consider two waves that have the same frequency but different amplitudes. They also have a phase difference $\phi$. The displacements of the two waves are:

$$y_1(x, t) = y_{m1} \sin(kx - \omega t) \quad \text{and} \quad y_2(x, t) = y_{m2} \sin(kx - \omega t + \phi).$$

The superposition of the two waves yields a wave that has the same angular frequency $\omega$ and is described by:

$$y' = y'_m \sin(kx - \omega t + \beta)$$

Here $y'_m$ is the wave amplitude and $\beta$ is the phase angle.

To determine $y'_m$ and $\beta$ we add the two phasors representing the waves as vectors (see fig.c).

Note: The phasor method can be used to add vectors that have different amplitudes.
Problem 35. Two sinusoidal waves of the same frequency travel in the same direction along a string. If $y_{m1} = 3.0\, cm$, $y_{m2} = 4.0\, cm$, $\phi_1 = 0$, and $\phi_2 = \pi/2\, rad$, what is the amplitude of the resultant wave?

The phasor diagram is shown below: $y_{1m}$ and $y_{2m}$ represent the original waves and $y_m$ represents the resultant wave. The phasors corresponding to the two constituent waves make an angle of 90° with each other, so the triangle is a right triangle. The Pythagorean theorem gives

$$y_m^2 = y_{1m}^2 + y_{2m}^2 = (3.0\, cm)^2 + (4.0\, cm)^2 = (25\, cm)^2.$$

Thus $y_m = 5.0\, cm$. 

![Phasor diagram](image-url)
Standing Waves: Consider the superposition of two waves that have the same frequency and amplitude but travel in opposite directions. The displacements of two waves are: 
\[ y_1(x,t) = y_m \sin(kx - \omega t), \quad y_2(x,t) = y_m \sin(kx + \omega t) \]

The displacement of the resulting wave 
\[ y'(x,t) = y_1(x,t) + y_2(x,t) \]

\[ y'(x,t) = y_m \sin(kx - \omega t) + y_m \sin(kx + \omega t) = [2y_m \sin(kx)] \cos(\omega t) \]

This is not a traveling wave but an oscillation that has a position dependent amplitude. It is known as a **standing wave**.
The displacement of a standing wave is given by the equation:

\[ y'(x, t) = [2y_m \sin kx] \cos \omega t \]

The position dependant amplitude is equal to \( 2y_m \sin kx \)

**Nodes**: These are defined as positions where the standing wave amplitude vanishes. They occur when \( kx = n\pi \), \( n = 0, 1, 2, \ldots \)

\[ \frac{2\pi}{\lambda} x = n\pi \rightarrow x_n = n \frac{\lambda}{2} \quad n = 0, 1, 2, \ldots \]

**Antinodes**: These are defined as positions where the standing wave amplitude is maximum.

They occur when \( kx = \left( n + \frac{1}{2} \right)\pi \), \( n = 0, 1, 2, \ldots \)

\[ \frac{2\pi}{\lambda} x = \left( n + \frac{1}{2} \right)\pi \rightarrow x'_n = \left( n + \frac{1}{2} \right) \frac{\lambda}{2} \quad n = 0, 1, 2, \ldots \]

**Note 1**: The distance between adjacent nodes and antinodes is \( \lambda/2 \)

**Note 2**: The distance between a node and an adjacent antinode is \( \lambda/4 \)
Standing waves and resonance

Consider a string under tension on which is clamped at points A and B separated by a distance $L$. We send a harmonic wave traveling to the right. The wave is reflected at point B and the reflected wave travels to the left. The left going wave reflects back at point A and creates a third wave traveling to the right. Thus we have a large number of overlapping waves half of which travel to the right and the rest to the left.

For certain frequencies the interference produces a standing wave. Such a standing wave is said to be at resonance. The frequencies at which the standing wave occurs are known as the resonant frequencies of the system.
Resonances occur when the resulting standing wave satisfies the boundary condition of the problem. These are that the Amplitude must be zero at point A and point B and arise from the fact that the string is clamped at both points and therefore cannot move. The first resonance is shown in fig.a. The standing wave has two nodes at points A and B. Thus \( L = \frac{\lambda_1}{2} \rightarrow \lambda_1 = 2L \). The second standing wave is shown in fig.b. It has three nodes (two of them at A and B)

In this case \( L = 2 \left( \frac{\lambda}{2} \right) = \lambda \rightarrow \lambda_2 = L \)

The third standing wave is shown in fig.c. It has four nodes (two of them at A and B)

In this case \( L = 3 \left( \frac{\lambda}{2} \right) = \lambda \rightarrow \lambda_3 = \frac{2}{3}L \)  

The general expression for the resonant wavelengths is: \( \lambda_n = \frac{2L}{n} \quad n = 1, 2, 3, ... \)  

the resonant frequencies \( f_n = \frac{\nu}{\lambda_n} = n \frac{\nu}{2L} \)
Problem 43. A string fixed at both ends is 8.40 m long and has a mass of 0.120 kg. It is subjected to a tension of 96.0 N and set oscillating. (a) What is the speed of the waves on the string? (b) What is the longest possible wavelength for a standing wave? (c) Find the frequency of the wave.

(a) The wave speed is given by $v = \sqrt{\frac{\tau}{\mu}}$, where $\tau$ is the tension in the string and $\mu$ is the linear mass density of the string. Since the mass density is the mass per unit length, $\mu = \frac{M}{L}$, where $M$ is the mass of the string and $L$ is its length. Thus

$$v = \sqrt{\frac{\tau L}{M}} = \sqrt{\frac{(96.0 \text{ N})(8.40 \text{ m})}{0.120 \text{ kg}}} = 82.0 \text{ m/s}.$$ 

(b) The longest possible wavelength $\lambda$ for a standing wave is related to the length of the string by $L = \lambda/2$, so $\lambda = 2L = 2(8.40 \text{ m}) = 16.8 \text{ m}$. 

(c) The frequency is $f = \frac{v}{\lambda} = \frac{(82.0 \text{ m/s})}{(16.8 \text{ m})} = 4.88 \text{ Hz}$. 

Problem 50. A rope, under a tension of 200N and fixed at both ends, oscillates in a second harmonic standing wave pattern. The displacement of the rope is given by \( y = (0.10m)(\sin \pi x / 2)\sin 12\pi t \), where \( x=0 \) at one end of the rope, \( x \) is in meters, and \( t \) is in seconds. What are (a) the length of the rope, (b) the speed of the wave on the rope, and (c) the mass of the rope? (d) If the rope oscillates in a third harmonic standing wave pattern, what will be the period of oscillation?

Since the rope is fixed at both ends, then the phrase “second-harmonic standing wave pattern” describes the oscillation shown in Figure 16–23(b), where (see Eq. 16–65)

\[ \lambda = L \quad \text{and} \quad f = \frac{v}{L}. \]

(a) Comparing the given function with Eq. 16-60, we obtain \( k = \pi/2 \) and \( \omega = 12\pi \text{ rad/s} \). Since \( k = 2\pi/\lambda \) then

\[ \frac{2\pi}{\lambda} = \frac{\pi}{2} \quad \Rightarrow \quad \lambda = 4.0 \text{ m} \quad \Rightarrow \quad L = 4.0 \text{ m}. \]

(b) Since \( \omega = 2\pi f \) then \( 2\pi f = 12\pi \text{ rad/s} \), which yields

\[ f = 6.0 \text{ Hz} \quad \Rightarrow \quad v = f\lambda = 24 \text{ m/s}. \]

(c) Using Eq. 16–26, we have

\[ v = \sqrt{\frac{\tau}{\mu}} \quad \Rightarrow \quad 24 \text{ m/s} = \sqrt{\frac{200 \text{ N}}{m/(4.0 \text{ m})}} \]

which leads to \( m = 1.4 \text{ kg} \).

(d) With

\[ f = \frac{3v}{2L} = \frac{3(24 \text{ m/s})}{2(4.0 \text{ m})} = 9.0 \text{ Hz} \]

The period is \( T = 1/f = 0.11 \text{ s} \).