PH 222-2C Fall 2012

Gauss’ Law

Lectures 3-4

Chapter 23
(Halliday/Resnick/Walker, Fundamentals of Physics 8th edition)
Chapter 23
Gauss’ Law

In this chapter we will introduce the following new concepts:

The flux (symbol $\Phi$) of the electric field
Symmetry
Gauss’ law

We will then apply Gauss’ law and determine the electric field generated by:
An infinite, uniformly charged insulating plane
An infinite, uniformly charged insulating rod
A uniformly charged spherical shell
A uniform spherical charge distribution

We will also apply Gauss’ law to determine the electric field inside and outside charged conductors.
**Flux of a Vector.** Consider an airstream of velocity $\vec{v}$ that is aimed at a loop of area $A$. The velocity vector $\vec{v}$ is at angle $\theta$ with respect to the loop normal $\hat{n}$. The product $\Phi = vA \cos \theta$ is known as the flux. In this example the flux is equal to the volume flow rate through the loop (thus the name flux).

**Note 1:** $\Phi$ depends on $\theta$. It is maximum and equal to $vA$ for $\theta = 0$ ($\vec{v}$ perpendicular to the loop plane). It is minimum and equal to zero for $\theta = 90^\circ$ ($\vec{v}$ parallel to the loop plane).

**Note 2:** $vA \cos \theta = \vec{v} \cdot \vec{A}$. The vector $\vec{A}$ is parallel to the loop normal and has magnitude equal to $A$. 

![Diagram](image-url)
Flux of the Electric Field.

Consider the closed surface shown in the figure. In the vicinity of the surface assume that we have a known electric field $\vec{E}$. The flux $\Phi$ of the electric field through the surface is defined as follows:

1. Divide the surface into small "elements" of area $\Delta A$.
2. For each element calculate the term $\vec{E} \cdot \Delta \vec{A} = E \Delta A \cos \theta$.
3. Form the sum $\Phi = \sum \vec{E} \cdot \Delta \vec{A}$.
4. Take the limit of the sum as the area $\Delta A \to 0$.

The limit of the sum becomes the integral:

$$\Phi = \int \int \vec{E} \cdot d\vec{A}$$

Flux SI unit: $N \cdot m^2 / C$

Note 1: The circle on the integral sign indicates that the surface is closed. When we apply Gauss' law the surface is known as "Gaussian."

Note 2: $\Phi$ is proportional to the net number of electric field lines that pass through the surface.
Gauss' Law

Gauss' law can be formulated as follows:

The flux of $\vec{E}$ through any closed surface $\times \varepsilon_0 = \text{net charge } q_{\text{enc}}$ enclosed by the surface.

In equation form: $\varepsilon_0 \Phi = q_{\text{enc}}$  
Equivalently: $\varepsilon_0 \int \vec{E} \cdot d\vec{A} = q_{\text{enc}}$

**Note 1**: Gauss' law holds for any closed surface. Usually one particular surface makes the problem of determining the electric field very simple.

**Note 2**: When calculating the net charge inside a closed surface we take into account the algebraic sign of each charge.

**Note 3**: When applying Gauss' law for a closed surface we ignore the charges outside the surface no matter how large they are.

**Example**:

Surface $S_1$: $\varepsilon_0 \Phi_1 = +q$,  
Surface $S_2$: $\varepsilon_0 \Phi_2 = -q$  
Surface $S_3$: $\varepsilon_0 \Phi_3 = 0$,  
Surface $S_4$: $\varepsilon_0 \Phi_4 = -q + q = 0$

**Note**: We refer to $S_1$, $S_2$, $S_3$, $S_4$ as "Gaussian surfaces."
Gauss' Law and Coulomb's Law

Gauss' law and Coulomb's law are different ways of describing the relation between electric charge and electric field in static cases. One can derive Coulomb's law from Gauss' law and vice versa. Here we will derive Coulomb's law from Gauss' law.

Consider a point charge \( q \). We will use Gauss' law to determine the electric field \( \vec{E} \) generated at a point \( P \) at a distance \( r \) from \( q \). We choose a Gaussian surface that is a sphere of radius \( r \) and has its center at \( q \).

We divide the Gaussian surface into elements of area \( dA \). The flux for each element is:

\[
d\Phi = EdA \cos \theta = EdA
\]

Total flux

\[
\Phi = \int \int EdA = E \int \int dA = E \left( 4\pi r^2 \right)
\]

From Gauss' law we have:

\[
e_0 \Phi = q_{\text{enc}} = q \rightarrow 4\pi r^2 e_0 E = q \rightarrow E = \frac{q}{4\pi r^2 e_0}
\]

This is the same answer we got in Chapter 22 using Coulomb's law.
The Electric Field Inside a Conductor

We shall prove that the electric field inside a conductor vanishes. Consider the conductor shown in the figure to the left. It is an experimental fact that such an object contains negatively charged electrons, which are free to move inside the conductor. Let's assume for a moment that the electric field is not equal to zero. In such a case a nonvanishing force \( \vec{F} = -e\vec{E} \) is exerted by the field on each electron. This force would result in a nonzero velocity \( \vec{v} \), and the moving electrons would constitute an electric current. We will see in subsequent chapters that electric currents manifest themselves in a variety of ways:

(a) They heat the conductor.
(b) They generate magnetic fields around the conductor.

No such effects have ever been observed, thus the original assumption that there exists a nonzero electric field inside the conductor. We conclude that:

The electrostatic electric field \( \vec{E} \) inside a conductor is equal to zero.
A Charged Isolated Conductor

Consider the conductor shown in the figure that has a total charge $q$. In this section we will ask the question: Where is this charge located? To answer the question we will apply Gauss' law to the Gaussian surface shown in the figure, which is located just below the conductor surface. Inside the conductor the electric field $\vec{E} = 0$. Thus \[ \Phi = \oint \vec{E} \cdot \Delta \vec{A} = 0 \quad (eq. \ 1). \]

From Gauss's law we have: \[ \Phi = \frac{q_{\text{enc}}}{\varepsilon_0} \quad (eq. \ 2). \]

If we compare eq. 1 with eq. 2 we get $q_{\text{enc}} = 0$.

Thus no charge exists inside the conductor. Yet we know that the conductor has a nonzero charge $q$. Where is this charge located? There is only one place for it to be: On the surface of the conductor.

No electrostatic charges can exist inside a conductor.
All charges reside on the conductor surface.
An Isolated Charged Conductor with a Cavity

Consider the conductor shown in the figure that has a total charge $q$. This conductor differs from the one shown on the previous page in one aspect: It has a cavity. We ask the question: Can charges reside on the walls of the cavity?

As before, Gauss's law provides the answer. We will apply Gauss' law to the Gaussian surface shown in the figure, which is located just below the conductor surface. Inside the conductor the electric field $\vec{E} = 0$.

Thus $\Phi = \int \vec{E} \cdot \Delta \vec{A} = 0$ (eq. 1).

From Gauss's law we have: $\Phi = \frac{q_{\text{enc}}}{\varepsilon_0}$ (eq. 2).

If we compare eq. 1 with eq. 2 we get $q_{\text{enc}} = 0$.

**Conclusion:**

There is no charge on the cavity walls. All the excess charge $q$ remains on the outer surface of the conductor.
The Electric Field Outside a Charged Conductor

The electric field inside a conductor is zero. This is not the case for the electric field outside. The electric field vector $\vec{E}$ is perpendicular to the conductor surface. If it were not, then $\vec{E}$ would have a component parallel $E_\parallel$ to the conductor surface.

Since charges are free to move in the conductor, $E_\parallel$ would cause the free electrons to move, which is a contradiction to the assumption that we have stationary charges. We will apply Gauss' law using the cylindrical closed surface shown in the figure. The surface is further divided into three sections $S_1, S_2,$ and $S_3$ as shown in the figure. The net flux $\Phi = \Phi_1 + \Phi_2 + \Phi_3$.

$\Phi_1 = EA \cos 0 = EA$ \hspace{1cm} $\Phi_2 = EA' \cos 90^\circ = 0$

$\Phi_3 = 0$ (because the electric field inside the conductor is zero). $\Phi = EA = \frac{q_{\text{enc}}}{\varepsilon_0}$ \rightarrow $E = \frac{q_{\text{enc}}}{A \varepsilon_0} \cdot 10$

The ratio $\sigma = \frac{q_{\text{enc}}}{A \varepsilon_0}$ is known as surface charge density $\rightarrow E = \frac{\sigma}{\varepsilon_0}$.
Symmetry. We say that an object is symmetric under a particular mathematical operation (e.g., rotation, translation, …) if to an observer the object looks the same before and after the operation.

Note: Symmetry is a primitive notion and as such is very powerful.

Consider a featureless beach ball that can be rotated about a vertical axis that passes through its center. The observer closes his eyes and we rotate the sphere. When the observer opens his eyes, he cannot tell whether the sphere has been rotated or not. We conclude that the sphere has rotational symmetry about the rotation axis.
A Second Example of Rotational Symmetry

Consider a featureless cylinder that can rotate about its central axis as shown in the figure. The observer closes his eyes and we rotate the cylinder. When he opens his eyes, he cannot tell whether the cylinder has been rotated or not. We conclude that the cylinder has rotational symmetry about the rotation axis.
Example of Translational Symmetry:

Consider an infinite featureless plane. An observer takes a trip on a magic carpet that flies above the plane. The observer closes his eyes and we move the carpet around. When he opens his eyes the observer cannot tell whether he has moved or not. We conclude that the plane has translational symmetry.
Recipe for Applying Gauss’ Law

1. Make a sketch of the charge distribution.

2. Identify the symmetry of the distribution and its effect on the electric field.

3. Gauss’ law is true for any closed surface \( S \). Choose one that makes the calculation of the flux \( \Phi \) as easy as possible.

4. Use Gauss’ law to determine the electric field vector:

\[
\Phi = \frac{q_{\text{enc}}}{\varepsilon_0}
\]
Electric Field Generated by a Long, Uniformly Charged Rod

Consider the long rod shown in the figure. It is uniformly charged with linear charge density \( \lambda \). Using symmetry arguments we can show that the electric field vector points radially outward and has the same magnitude for points at the same distance \( r \) from the rod. We use a Gaussian surface \( S \) that has the same symmetry. It is a cylinder of radius \( r \) and height \( h \) whose axis coincides with the charged rod.

We divide \( S \) into three sections: Top flat section \( S_1 \), middle curved section \( S_2 \), and bottom flat section \( S_3 \). The net flux through \( S \) is \( \Phi = \Phi_1 + \Phi_2 + \Phi_3 \). Fluxes \( \Phi_1 \) and \( \Phi_3 \) vanish because the electric field is at right angles with the normal to the surface:

\[
\Phi_2 = 2\pi rhE \cos \theta = 2\pi rhE \rightarrow \Phi = 2\pi rhE.
\]

From Gauss's law we have:

\[
\Phi = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\lambda h}{\varepsilon_0}.
\]

If we compare these two equations we get:

\[
2\pi rhE = \frac{\lambda h}{\varepsilon_0} \rightarrow E = \frac{\lambda}{2\pi\varepsilon_0 r}.
\]
Electric Field Generated by a Thin, Infinite, Nonconducting Uniformly Charged Sheet

We assume that the sheet has a positive charge of surface density $\sigma$. From symmetry, the electric field vector $\vec{E}$ is perpendicular to the sheet and has a constant magnitude. Furthermore, it points away from the sheet. We choose a cylindrical Gaussian surface $S$ with the caps of area $A$ on either side of the sheet as shown in the figure. We divide $S$ into three sections: $S_1$ is the cap to the right of the sheet, $S_2$ is the curved surface of the cylinder, and $S_3$ is the cap to the left of the sheet. The net flux through $S$ is $\Phi = \Phi_1 + \Phi_2 + \Phi_3$.

$\Phi_1 = \Phi_3 = EA \cos 0 = EA$.  $\Phi_2 = 0$ \hspace{1cm} ($\theta = 90^\circ$)

$\rightarrow \Phi = 2EA$. From Gauss's law we have: $\Phi = \frac{q_{\text{enc}}}{\epsilon_0} = \frac{\sigma A}{\epsilon_0}$

$\rightarrow 2EA = \frac{\sigma A}{\epsilon_0} \rightarrow E = \frac{\sigma}{2\epsilon_0}$. 

$E = \frac{\sigma}{2\epsilon_0}$
The electric field generated by two parallel conducting infinite planes is charged with surface densities $\sigma_1$ and $-\sigma_1$. In figs. $a$ and $b$ we show the two plates isolated so that one does not influence the charge distribution of the other. The charge spreads out equally on both faces of each sheet. When the two plates are moved close to each other as shown in fig. $c$, then the charges on one plate attract those on the other. As a result the charges move on the inner faces of each plate. To find the field $E_i$ between the plates we apply Gauss' law for the cylindrical surface $S$, which has caps of area $A$.

The net flux $\Phi = E_i A = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{2\sigma_1 A}{\varepsilon_0} \rightarrow E_i = \frac{2\sigma_1}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$. To find the field $E_0$ outside the plates we apply Gauss' law for the cylindrical surface $S'$, which has caps of area $A$.

The net flux $\Phi = E_0 A = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{\sigma_1 - \sigma_1}{\varepsilon_0} = 0 \rightarrow E_0 = 0$. 

\[ E_i = \frac{2\sigma_1}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0} \] 
\[ E_0 = 0 \]
Consider a Gaussian surface that is a sphere with radius $r$ and whose center coincides with that of the charged shell.

The Electric Field Generated by a Spherical Shell of Charge $q$ and Radius $R$

**Inside the shell**: Consider a Gaussian surface $S_1$ that is a sphere with radius $r < R$ and whose center coincides with that of the charged shell.

The electric field flux $\Phi = 4\pi r^2 E_i = \frac{q_{\text{enc}}}{\varepsilon_0} = 0$.

Thus $E_i = 0$.

**Outside the shell**: Consider a Gaussian surface $S_2$ that is a sphere with radius $r > R$ and whose center coincides with that of the charged shell.

The electric field flux $\Phi = 4\pi r^2 E_0 = \frac{q_{\text{enc}}}{\varepsilon_0} = \frac{q}{\varepsilon_0}$.

Thus $E_0 = \frac{q}{4\pi \varepsilon_0 r^2}$.

**Note**: Outside the shell the electric field is the same as if all the charge of the shell were concentrated at the shell center.
Electric Field Generated by a Uniformly Charged Sphere of Radius $R$ and Charge $q$

**Outside the sphere:** Consider a Gaussian surface $S_1$ that is a sphere with radius $r > R$ and whose center coincides with that of the charged sphere. The electric field flux $\Phi = 4\pi r^2 E_0 = \frac{q_{enc}}{\varepsilon_0} = q / \varepsilon_0$

Thus $E_0 = \frac{q}{4\pi\varepsilon_0 r^2}$.

**Inside the sphere:** Consider a Gaussian surface $S_2$ that is a sphere with radius $r < R$ and whose center coincides with that of the charged sphere. The electric field flux $\Phi = 4\pi r^2 E_i = \frac{q_{enc}}{\varepsilon_0}$. Since the charge is uniform $\frac{q_{enc}}{4\pi r^3} = \frac{q}{4\pi R^3} \Rightarrow q_{enc} = \frac{r^3}{R^3} q \rightarrow 4\pi r^2 E_i = \frac{r^3}{R^3} \frac{q}{\varepsilon_0}$

Thus $E_i = \left(\frac{q}{4\pi \varepsilon_0 R^3}\right) r$. 

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Electric Field Generated by a Uniformly Charged Sphere of Radius $R$ and Charge $q$

Summary:

$$E_0 = \frac{q}{4\pi\varepsilon_0 r^2}$$

$$E_i = \left( \frac{q}{4\pi\varepsilon_0 R^3} \right) r$$